Projections of knots

by

Dennis Roseman (Iowa City, Iowa)

Abstract. In this paper we define and study projections of PL n-manifolds in R^{n+2} and derive an algorithm for calculating the fundamental group of the complement of the n-manifold similar to the Wirtinger presentation for knotted circles in R^3.

In this paper, we define and study projections of PL n-manifolds in R^{n+2}, and derive an algorithm for calculating the knot group (i.e., the fundamental group of the complement of the manifold in R^{n+2}). This algorithm generalizes the classical presentation of knot groups as in [2]. In Yajima [3] knot groups of orientable 2-manifolds in R^4 are studied; our results extend this further in two ways: to higher dimensions, where general position is more of a problem, and to non-orientable manifolds.

We will be concerned with closed (i.e., compact without boundary) n-dimensional PL submanifolds of R^{n+2}. If M^n is such a submanifold and i_M: $M \to R^{n+2}$ is the inclusion map, then the knot type of (R^{n+2}, M^n) is the PL ambient isotopy class of i_M. Frequently, to avoid excessive notation, we will use the letter M to also denote the image of i_M.

In order to describe the general position of a map we will use the following definitions as in Zeeman [4]. If f: $X \to Y$ is a map and r an integer, we will define $S'_r(f) = \{x \in X\colon f^{-1}f(x)$ contains at least r points$\}$. We will define $S_r(f)$ to be the closure of $S'_r(f)$, and let $\mathrm{Br}(f) = \{x \in X\colon$ no neighborhood of x is embedded by $f\}$. Then we will have $X = S_1(f) \supseteq S_2(f) \supseteq \ldots$, and $S_2(f) = S'_2(f) \cup \mathrm{Br}(f)$. If f is a PL map of polyhedra, then these subsets of X will be subcomplexes of some subdivision of X. Also, a map will be called *non-degenerate* if it embeds each simplex.

We will consider R^{n+2} as the set of all $(n+2)$-tuples of real numbers, and $R^{n+1} = \{(x_1, \ldots, x_{n+2}) \in R^{n+2}$ with $x_{n+2} = 0\}$. Then Π: $R^{n+2} \to R^{n+1}$ defined by $\Pi\big((x_1, \ldots, x_{n+1}, x_{n+2})\big) = (x_1, \ldots, x_{n+1}, 0)$, will be called the *projection* of R^{n+2} onto R^{n+1}. The map h, called the *height function*, is defined by $h(x_1, \ldots, x_{n+2}) = x_{n+2}$. Generally, if $X \subseteq R^{n+2}$, we will let X^* denote $\Pi(X)$.

PROPOSITION 1. *Given (R^{n+2}, M) there is an isotopy φ_t of R^{n+2} such that $\Pi\varphi_1|M$ is non-degenerate.*

Proof. Let $x_1, \ldots, x_m$ denote the vertices of M. Let $\Omega(x_1, \ldots, x_m)$ be the union of all k-dimensional subspaces, K, of R^{n+2} with $k < n+2$, such that K is parallel to some k-dimensional affine subspace of R^{n+2} spanned by a subset of $\{x_1, \ldots, x_m\}$. If S^{n+1} is the unit sphere of R^{n+2}, then $S^{n+1} - \Omega(x_1, \ldots, x_m)$ will be dense in S^{n+1} and, in particular, non-empty. Choose $v \in S^{n+1} - \Omega(x_1, \ldots, x_m)$ and considering v as a vector, let H be the $(n+1)$-subspace in R^{n+2} orthogonal to v. Then the orthogonal projection of R^{n+2} onto H will embed each simplex of M. If φ_t is an isotopy which takes H onto R^{n+1} by means of a rigid rotation, then φ_t is the isotopy we wish. ∎

For the definition of t-shifts, see Armstrong and Zeeman [1]; the construction of a t-shift will be contained in the proof of the following proposition, we will not need Brouwer triangulations since our embeddings are Euclidian space. This proposition will be used in applying established general position arguments to our particular situation.

PROPOSITION 2. (Shift lifting lemma). *In suitable triangulations, if σ is a t-simplex of M with g a local t-shift of $\Pi \circ i_M$ with respect to σ, then there is a local t-shift, $\tilde{g}$, of i_M with respect to σ such that $\Pi \circ \tilde{g} = g$, and such that $\tilde{g}$ is isotopic to i_M, and $\tilde{g}$ is the restriction of an isotopy of R^{n+2}.*

Proof. Let K, $(\tilde{L}, \tilde{K})$, and (L^*, K^*) be triangulations of M, (R^{n+2}, M), and (R^{n+1}, M^*) such that i_M is simplicial, $\Pi\colon R^{n+2} \to R^{n+1}$ is simplicial, and $\Pi \circ i_M$ non-degenerate. Let σ be a t-simplex of K, $\tilde{\sigma} = i_M(\sigma)$, and $\sigma^* = \Pi(\tilde{\sigma})$. Let K'', $(\tilde{L}'', \tilde{K}'')$, and $(L^{*\prime\prime}, K^{*\prime\prime})$ denote the second barycentric subdivisions of K, $(\tilde{L}, \tilde{K})$, and (L^*, K^*), respectively.

Let $\tilde{B}$ be the regular neighborhood in $\tilde{L}''$ of $\tilde{\sigma}$ modulo its boundary (i.e., $\tilde{B}$ consists of all closed simplexes of $\tilde{L}''$ which meet the interior of $\tilde{\sigma}$); $\tilde{B}$ will be an $(n+2)$-ball. Let $B^* = \Pi(\tilde{B})$; then B^* will be an $(n+1)$-ball, and in fact will be the regular neighborhood of σ^* in $L^{*\prime\prime}$ modulo its boundary.

Let $B = i_M^{-1}(\tilde{B} \cap M)$. Let v, $\tilde{v}$, and v^* denote the barycenters of σ, $\tilde{\sigma}$, and σ^*, respectively. Then $|B| = |v \circ \partial B|$, $|\tilde{B}| = |\tilde{v} \circ \partial\tilde{B}|$, and $|B^*| = |v^* \circ \partial B^*|$. Let $f_\sigma\colon B \to B^*$ denote the restriction of $\Pi \circ i_M$; then f_σ is the join of two maps: the restrictions of $\Pi \circ i_M$ to v and to ∂B.

Our t-shift will now be defined. First we need to find a point $u^* \in B^*$ near v^* such that:

(i) u^* is contained in the open star of v^* in B^*;
(ii) u^* is joinable to ∂B^*;
(iii) u^* is in general position with respect to the vertices of B^*.

We next define a homeomorphism, j, of B^* to be the join of the map which sends v^* to u^* and the map which is the identity on ∂B^*. Then j is a homeomorphism of B^* which is fixed on ∂B^*. If we then define g_σ by $g_\sigma = j \circ f_\sigma$, then g_σ is ambient isotopic to f_σ keeping ∂B fixed. The map g defined by

$$g = \begin{cases} \Pi \circ i_M & \text{on} \quad M - B, \\ g_\sigma & \text{on} \quad B \end{cases}$$

will be called a t-*shift* of $\Pi \circ i_M$ with respect to σ. (Note that g is homotopic to $\Pi \circ i_M$.)

We may finally lift this shift by choosing a point $\tilde{u}$ in $\pi^{-1}(u^*)$ such that:

(i) $\tilde{u}$ is contained in the open star of $\tilde{v}$ in $\tilde{B}$;
(ii) $\tilde{u}$ is joinable to $\partial\tilde{B}$;
(iii) $\tilde{u}$ is in general position with respect to the vertices of $(\tilde{B})$.

We then proceed as before, defining a homeomorphism, $\tilde{j}$, of $\tilde{B}$ to be the join of the map which sends $\tilde{v}$ to $\tilde{u}$ and the identity of $\partial\tilde{B}$; then defining $\tilde{g}_\sigma = \tilde{j} \circ i_\sigma$, where i_σ denotes the restriction of i_M to B. We may now define $\tilde{g}$ by

$$\tilde{g} = \begin{cases} i_M & \text{on} \quad M - B, \\ \tilde{g}_\sigma & \text{on} \quad B. \end{cases}$$

We will then have $\tilde{g}$ isotopic to i_M, and $\Pi \circ \tilde{g} = g$. ∎

PROPOSITION 3. *Given (R^{n+2}, M) we may find an isotopy ψ_t of R^{n+2} such that $\Pi \circ \psi_1 \circ i_M$ is in general position.*

Proof. By Zeeman [4] one may inductively define a sequence of arbitrarily small t-shifts, $g^{(i)}$, $i = 1, \ldots, p$ where $g^{(1)}$ is obtained by a t-shift of $\Pi \circ i_M$; $g^{(i)}$ is obtained by a t-shift of $g^{(i-1)}$ for $i = 2, \ldots, p$, and such that $g^{(p)}$ is in general position.

Corresponding to these, by Proposition 2, we may find t-shifts with associated maps $\tilde{g}^{(i)}$ such that $\Pi \circ \tilde{g}^{(i)} = g^{(i)}$ for each i. Furthermore, $\tilde{g}^{(1)}$ will be ambient isotopic to the identity by an isotopy $\varphi_t^{(1)}$, and $\tilde{g}^{(i)}$ will be isotopic to $\tilde{g}^{(i-1)}$ for $i = 2, \ldots, p$ by an isotopy $\varphi_t^{(i)}$. We will then define ψ_t to be the composition of the isotopies $\varphi_t^{(i)}$, $i = 1, \ldots, p$. ∎

We will say that M^* is *self-transverse* if for all q, if $x \in S'_q(\Pi \circ i_M) - S'_{q+1}(\Pi \circ i_M)$ (i.e., x is a point of order exactly q) then there is a neighborhood, U, of x^* in R^{n+1} such that $\Pi^{-1}(U) \cap M$ is a disjoint collection of q open n-disks, B_1, and a homeomorphism $f\colon U \to R^{n+1}$ such that $f(x) = 0$, and each $f(\Pi(B_i))$ is an n-plane containing 0, and the q n-planes, $\{f(\Pi(B_i))\}$ are in general position in R^{n+1} (i.e., the intersection of any r of them is $(n+1) - r$ dimensional.

LEMMA 4. *Given* (R^{n+2}, M), *there exists an isotopy*, ψ_t, *of* R^{n+2} *such that* $\Pi(\psi_1(M))$ *is self transverse.*

Proof. For each $i = 1, \ldots, n+1$, let $D_i = \{x \in M$ such that there exists exactly $i-1$ points, p, such that $\Pi(p) = \Pi(x)$ and $h(x) > h(p)\}$, $\bar{D}_i$ denotes the closure of D_i in M; let $\dot{D}_i = \bar{D}_i - D_i$.

Let K, (L, K), (L^*, K^*) be triangulations of M, (R^{n+2}, M), (R^{n+1}, M^*), respectively, such that i_M is simplicial, $\Pi \circ i_M$ is simplicial, non-degenerate and in general position. Let K'', (L'', K''), $(L^{*\prime\prime}, K^{*\prime\prime})$ denote the second barycentric subdivisions of these triangulations. Let N_2 be the derived neighborhood of $\bar{D}_2 \bmod \dot{D}_2$ in M. Then $\Pi|\mathrm{Int}\, N_2$ is a locally flat embedding; it is an embedding since there are no self intersections, locally flat since all non-locally flat points of M^* are images of branch points of $\Pi \circ i_M$.

Consider the simplexes of $N_2 \bmod \partial N_2$, $\sigma_1, \ldots, \sigma_p$ ordered in decreasing dimension. An in Theorem 4 of Armstrong and Zeeman [1] we may define t-shifts for these simplices and associated maps $g^{(i)}$, $i = 1, \ldots, p$ such that $g^{(i)}$ will be transimplicial at points of $\bigcup_{j \leqslant i} \mathring{\sigma}_j$; $g^{(p)}$ will agree with $\Pi \circ i_M$ of N_2. Let N_1 be the derived neighborhood of $\bar{D}_1 \bmod \dot{D}_1$ in M. Then the proof of Lemma 6 of [1] shows that $g^{(p)}(\mathrm{Int}\, N_2)$ will be transverse to $g^{(p)}(\mathrm{Int}\, N_1)$. By Proposition 2, we may find a map $\tilde{g}^{(p)}$ isotopic to i_M, by an isotopy fixed off of N_2 such that $\Pi \circ \tilde{g}^{(p)} = g^{(p)}$ and $\tilde{g}^{(p)}$ is the restriction of an isotopy of R^{n+2}. To complete the proof, we proceed similarly with $N_3, \ldots, N_{n+1}$, inductively finding isotopies $g_t^{(i)}$ of N_i in R^{n+2} such that $\Pi \circ g_1^i$ will be transimplicial, and therefore transverse to each N_j with $j < i$. ∎

If $M \subseteq R^{n+2}$ is such that $\Pi \circ i_M$ is in general position, and if M^* is self transverse, then we will say that M is in *general position with respect to projection*. In this case we need to consider the following subsets. Let $\mathring{D} = S_2(\Pi \circ i_M) - S_3(\Pi \circ i_M)$; D denote the closure of $\mathring{D}$ in M; $\mathring{D}$ will be called the *set of pure doublepoints*. Also let $\mathring{Z} = S_3(\Pi \circ i_M)$; Z denote the closure of $\mathring{Z}$ in M. Let $\{\mathring{\Sigma}_k\}$ $k = 1, \ldots, p$ be the components of $M - D$; Σ_k denote the closure of $\mathring{\Sigma}_k$ in M. Then $\Pi|\mathring{\Sigma}_k$ is an embedding for each k; however, $\Pi|\Sigma_k$ may not be an embedding since it may fail to be 1-1 on $\partial\Sigma_k$. Each Σ_k is a relative n-manifold, see Spanier [5]. We also note that each $\mathring{\Sigma}_k^*$ will be locally flat in R^{n+1}, since the non-locally flat points must correspond to branch points of the projection and thus be contained in D^*.

LEMMA 5. (Separation lemma). *Each* $\mathring{\Sigma}_k^*$ *is an open orientable* n*-manifold, and is two-sided in* R^{n+1} *(i.e.,* $\mathring{\Sigma}_k^*$ *lies on the boundary of exactly two components of* $R^{n+1} - M^*$*).*

Proof. We have already established that $\mathring{\Sigma}_k^*$ is an open locally flat n-manifold in R^{n+1}. We will need the following proposition:

PROPOSITION 6. *The map* $H^n(M^*) \to H^n(M)$ *induced by the restriction of* Π *to* M *is an onto map for any coefficients, in particular* $H^n(M^*; Z_2) \neq 0$ *since* $H^n(M; Z_2) \neq 0$. *Also, the map* $H^n(\Sigma_k^*, \partial\Sigma_k^*) \to H^n(\Sigma_k, \partial\Sigma_k)$ *induced by the restriction of* Π *to* Σ_k *is an isomorphism.*

Proof. Let $C^n(X)$ denote the j-dimensional simplicial cochains on X, δ^j denote the j-dimensional coboundary operator and let $\Pi^\#\colon C^j(M^*) \to C^j(M)$ be the cochain map induced by $\Pi|M\colon M \to M^*$. Since Π is a 1-1 mapping of the n-simplexes of M to the n-simplexes of M^*, $\Pi^\#$ is an isomorphism and its restriction gives an isomorphism of the cycle groups $Z^n(M^*) \to Z^n(M)$. Now $\Pi^\#\colon H^n(M^*) \to H^n(M)$ will be onto iff $\Pi^\#(B^n(M^*)) \subseteq B^n(M)$, where B^n refers to the coboundary groups; but this follows since $\Pi^\# \delta^{n-1} = \delta^{n-1}\Pi^\#$.

Similarly, since Σ_k^* can be considered as Σ_k with identifications on $\partial\Sigma_k$, then the cochain map induced by $\Pi|\Sigma_k$ from the $C^j(\Sigma_k^*, \partial\Sigma_k^*)$ to $C^j(\Sigma_k, \partial\Sigma_k)$ is an isomorphism for any coefficients and for all j, and thus $H^n(\Sigma_k^*, \partial\Sigma_k^*)$ is isomorphic to $H^n(\Sigma_k, \partial\Sigma_k)$. ∎

We will now show that $\mathring{\Sigma}_k^*$ is two-sided. Let $\mathcal{A} = \{X \subseteq M^*$ such that $X = \bigcup_j \Sigma_{ij}^*$ where $\{\Sigma_{ij}^*\}$ is a subcollection of $\{\Sigma_i^*\}\}$. $\mathcal{A}$ is partially ordered by inclusion; let $<$ denote strict inclusion. For each k, let $\mathcal{A}_k = \{X \in \mathcal{A}$ such that $\Sigma_k^* \subseteq X$ and such that for any $\Sigma_j^* \subseteq X$, the map $H^n(X; Z_2) \to H^n(X - \Sigma_j^*; Z_2)$, induced by inclusion, is not an isomorphism$\}$.

Next we show $\mathcal{A}_k \neq \emptyset$ by showing that $M^* \in \mathcal{A}_k$. Let $N = M - \mathring{\Sigma}_j$; then $N^* = M^* - \mathring{\Sigma}_j^*$. Consider the diagram below, which is part of the homomorphism of the cohomology sequence of the pair (M^*, N^*) induced by the restriction of Π to M, (coefficients are to be taken in Z_2).

$$\begin{tikzcd}
H^n(M^*, N^*) \arrow[r, "j^*"] \arrow[d, "\Pi^\#"] & H^n(M^*) \arrow[r, "i^*"] \arrow[d, "\Pi^\#"] & H^n(N^*) \arrow[r] \arrow[d, "\Pi^\#"] & 0 \\
H^n(M, N) \arrow[r, "j"] & H^n(M) \arrow[r, "i"] & H^n(N) \arrow[r] & 0
\end{tikzcd}$$

Now $H^n(M)$, $H^n(M, N)$ and $H^n(M^*, N^*)$ are isomorphic to Z_2 by the Lefschetz duality theorem (Spanier [5]) since each is compact relative manifold orientable over Z_2; the map j is an isomorphism by exactness of the bottom row since $H^n(N) = 0$ (N collapses to an $(n-1)$-dimensional subcomplex); $\Pi^\#\colon H^n(M^*, N^*) \to H^n(M, N)$ is an isomorphism by Proposition 6 since, by excision, this map is equivalent to the map $H^n(\Sigma_j^*, \partial\Sigma_j^*) \to H^n(\Sigma_j, \partial\Sigma_j)$; thus by commutativity of the left hand square, j^* is not a zero map; therefore, by the exactness of the top row, the map i^*, although onto, is not 1-1.

Let W_k be a minimal element of $\mathcal{A}_k$ (i.e., $W_k \in \mathcal{A}_k$ and if $Y \in \mathcal{A}$ with $Y < W_k$, then $Y \notin \mathcal{A}_k$). Such a minimal element will be called a Σ_k*-cycle*. We will show that $H^n(W_k; Z_2)$ is isomorphic to Z_2. First we show:

PROPOSITION 7. *If $X \in \mathcal{A}$, $Y \subseteq X$ with $X - Y = \mathring{\Sigma}_j^*$ and $H^n(Y; Z_2) = 0$, then $H^n(X; Z_2)$ is isomorphic to either 0 or Z_2.*

Proof. By excision, $H^n(X, Y; Z_2) \approx H^n(\Sigma_j^*, \partial\Sigma_j^*; Z_2) \approx Z_2$, and thus around dimension n, the cohomology sequence of the pair (X, Y) is $Z_2 \xrightarrow{j} H^n(X) \to 0$. Thus $H^n(X)$ is the image under j of Z_2; therefore $H^n(X)$ is 0 or Z_2. ∎

Now if $W_k = \Sigma_k^*$, then it follows from the above proposition with $X = \Sigma_k^*$, and $Y = \partial\Sigma_k^*$ that $H^n(W_k; Z_2) \approx Z_2$, otherwise $H^n(W_k; Z_2)$ and $H^n(W_k - \mathring{\Sigma}_k^*; Z_2)$ would be both zero and thus isomorphic contradicting the assumption that $W_k \in \mathcal{A}_k$. Next suppose that $W_k \neq \Sigma_k^*$. Then we must have $H^n(\Sigma_k^*; Z_2) = 0$. We may suppose that $W_k = \bigcup_{j=0}^{p} \Sigma_{i_j}^*$ where $\Sigma_{i_0}^* = \Sigma_k^*$. Let $V_q = \bigcup_{j=0}^{q} \Sigma_{i_j}^*$. By the minimality of W_k, for all $q < p$, we have, with Z_2 coefficients, $H^n(V_{q-1})$ isomorphic to $H^n(V_q - \mathring{\Sigma}_{i_q}^*)$ and since $V_0 = \Sigma_k^*$ it follows that for all $q < p$, $H^n(V_q; Z_2) = 0$, in particular, $H^n(V_{p-1}; Z_2) = 0$. Applying Proposition 7 with $X = W_k$ and $Y = V_{p-1}$, we conclude that $H^n(W_k; Z_2)$ or 0; but it cannot be 0 since then W_k could not be in $\mathcal{A}_k$; therefore $H^n(W_k; Z_2) \approx Z_2$. ∎

By Alexander duality, $H^n(W_k)$ is isomorphic to $\tilde{H}_0(R^{n+1} - W_k)$ for any coefficients, where $\tilde{H}_0$ denotes reduced zero-th homology. The number of components of $R^{n+1} - W_k$ is one more than the rank of $\tilde{H}_0(R^{n+1} - W_k)$, thus the number of components is two.

In order to complete the proof of Lemma 5 we need:

PROPOSITION 8. *Let K be a component of $R^{n+1} - M^*$. If $\mathring{\Sigma}_i^* \cap \bar{K} \neq \emptyset$, then $\Sigma_i^* \subseteq \bar{K}$.*

Proof. Using the fact that each $\mathring{\Sigma}_i^*$ is locally flat and connected, one can show that the set of points of $\mathring{\Sigma}_i^* \cap \bar{K}$ is both open and closed in $\mathring{\Sigma}_i^*$; thus $\mathring{\Sigma}_i^* \subseteq \bar{K}$ and it follows that $\Sigma_i^* \subseteq \bar{K}$. ∎

Now let the components of $R^{n+1} - W_k$ be C and D. If B is a component of $R^{n+1} - M^*$, then either $B \subseteq C$ or $B \subseteq D$. Let C' be the component of $R^{n+1} - M^*$ contained in C such that $\Sigma_k^* \subseteq \bar{C}'$, such exist since some component contained in C must meet Σ_k^* and thus by the above proposition its closure must contain Σ_k^*. Similarly let D' be the component of $R^{n+1} - M^*$ contained in D such that $\Sigma_k^* \subset \bar{D}'$. Now $\mathring{\Sigma}_k^*$ is contained in the boundary of these two components and only these components of $R^{n+1} - M^*$.

It remains to be shown that $\mathring{\Sigma}_k^*$ is orientable. If $\mathring{\Sigma}_k^*$ were non-orientable then Σ_k would be non-orientable and $H^n(\Sigma_k, \partial\Sigma_k; Z) = 0$. Let $N = W_k - \mathring{\Sigma}_k^*$; since $H^n(W_k; Z_2) \approx Z_2$. It follows from the definition of W_k that $H^n(N; Z_2) = 0$, thus, by Alexander duality, N does not separate R^{n+1}, and $H^n(N, Z) = 0$. By excision, $H^n(W_k, N) \approx H^n(\Sigma_k^*, \partial\Sigma_k^*) \approx 0$. Thus in dimension n, the cohomology exact sequence of (W_k, N) with Z coefficients becomes $0 \to H^n(W_k, Z) \to 0$, and $H^n(W_k; Z) = 0$, thus W_k does not separate R^{n+1}. But we have already established that it does separate; thus $\mathring{\Sigma}_k^*$ could not have been non-orientable. This concludes the proof of Lemma 5. ∎

Let H be an $(n+1)$-plane in R^{n+2} parallel to R^{n+1}, we may assume that M is between H and R^{n+1}. If $X \subset R^{n+2}$, then $\hat{X}$ will be $\Pi^{-1}(\Pi(X)) \cap H$. For our basepoint in the calculation of the knot group, we will take a point $x_0 \in H$ such that $x_0^* \notin M^*$. If $x \in R^{n+2}$, then x^+ will be the line segment between x and $\hat{x}$, x^- will be the line segment between x and x^*. If $X \subseteq R^{n+2}$, then $X^+ = \bigcup_{x \in X} x^+$, $X^- = \bigcup_{x \in X} x^-$.

A directed path is a path with a particular ordering of the points; we will denote this ordering by $<$. If $\{x_1, \ldots, x_k\}$ are points of R^{n+2}, by the path $(x_1, \ldots, x_k)$ we will mean the directed path from x_1 to x_k consisting of straight line segments joining x_i to x_{i+1}; at times, to conserve notation, this symbol will also refer to the unordered path. If L is a directed polygonal path, $-L$ will denote the path L directed in the opposite direction. Suppose L is a polygonal path in $R^{n+2} - M$ with endpoints a and b, directed from a to b, and such that a^* and b^* are not points of M^*, then $\varrho(L)$ will denote the loop in $R^{n+2} - M$ given by the polygonal path $(x_0, \hat{a}, a) * L * (b, \hat{b}, x_0)$, where the operation $*$ is path composition, and we note that $-\varrho(L) = \varrho(-L)$.

For each region $\mathring{\Sigma}_i^*$ choose a polygonal path L_i in R^{n+1} with endpoints a_i and b_i such that $L_i \cap M^*$ consists of a single point of $\mathring{\Sigma}_i^*$ between a_i and b_i, that intersection being transverse. Let σ_1 be the homotopy class of the loop $\varrho(L_i)$, then $\varrho(-L_i) \in \sigma_i^{-1}$. (In the case that M is orientable, we might wish to choose the direction of L_i to be consistent with the orientation of M as in the classical case ([2], Chapter VI), or as in Yajima [3].)

PROPOSITION 9. *Let L be a path with endpoints a and b in $R^{n+2} - M$ directed from a to b, such that $L^* \cap M^*$ consists of a single point $p_k^* \in \mathring{\Sigma}_k^*$, and so does $\Pi^{-1}(p_k^*) \cap L$ and such that L^* is transverse to M^* at p_k^*. Then $\varrho(L)$ represents $\sigma_k^{\varepsilon(p^*)}$ where $\varepsilon(p^*)$ is defined as follows. Let $p = \Pi^{-1}(p^*) \cap L$, $q = \Pi^{-1}(p^*) \cap M$, then $\varepsilon(p^*) = 0$ if $h(p) > h(q)$; $\varepsilon(p^*) = 1$ if $h(p) < h(q)$ and if a^* is in the same component, X, of $R^{n+1} - M^*$ as a_k^*; $\varepsilon(p^*) = -1$ otherwise (i.e., $h(p) < h(q)$ and $b^* \in X$).*

Proof. If $\varepsilon(p^*) = 0$, then $L^+ \cap M = \emptyset$ and $\varrho(L)$ is nulhomotopic in $R^{n+2} - M$, the nulhomotopy carried by $L^+ \cup C_{x_0}(\hat{L})$, where $C_x[Y]$ denotes the cone on Y from x, that is, this set is the image of a map of $I \times I$ into $R^{n+2} - M$ which gives the nulhomotopy.

Let Y be the component of $R^{n+1} - M^*$ which contains b^*. If $\varepsilon(p^*) = 1$, let Q be a path in X from a^* to a_k^*, R a path in Y from b_k^* to b^*. Let B be the disk, perhaps singular, in R^{n+1} whose boundary is the path $Q * L_k *$

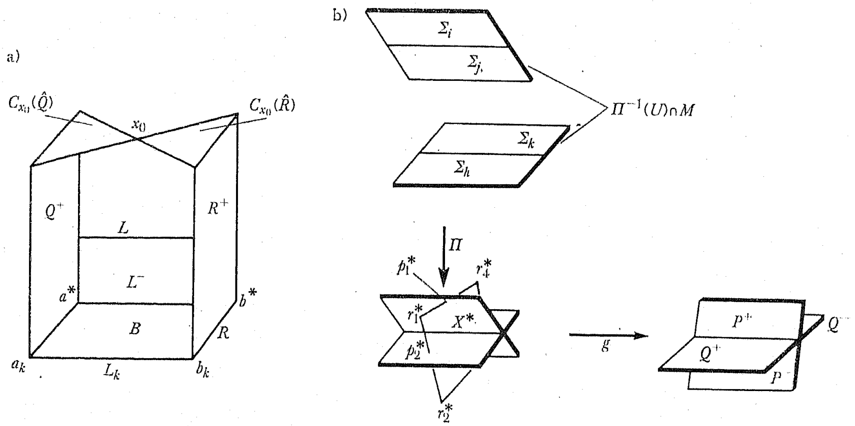

Fig. 1

$*R*(L^{-1})^*$. Then there is a homotopy between $\varrho(L)$ and $\varrho(L_k)$, carried by the set $C_{x_0}(\hat{Q})\cup Q^+\cup L^-\cup B\cup L_k\cup R^+\cup C_{x_0}(\hat{R})$, see Fig. 1a). In the case $\varepsilon(p^*)=-1$, we let Q be a path in X from b^* to a_k^*; R a path in X from b_k^* to a^*, and similarly proceed to show that $\varrho(L)$ is homotopic to $\varrho(-L_k)$.

In R^{n+1}, let $P=\{(x_1,\ldots,x_{n+1})$ with $x_n=0\}$, $P_+=\{x\in P$ with $x_1\geqslant 0\}$, $P_-=\{x\in P$ with $x_1\leqslant 0\}$, $Q=\{(x_1,\ldots,x_{n+1})$ with $x_1=0\}$, $Q_+=\{x\in Q$ with $x_n\geqslant 0\}$, $Q_-=\{x\in Q$ with $x_n\leqslant 0\}$, $G=P\cap Q$. By our self-transversality, if $x^*\in\mathring{D}^*$ then there exists a neighborhood, U, of x^* in R^{n+1} and a homeomorphism $g\colon U\to R^{n+1}$ such that $g(x^*)=(0,\ldots,0)$, and $g(U\cap M^*)=P\cap Q$. We next define the following subsets of M. Let $P'=(g\Pi)^{-1}(P)$, $P'_+=(g\Pi)^{-1}(P_+)$, $P'_-=(g\Pi)^{-1}(P_-)$, similarly define Q', Q'_+, and Q'_-. Then we may write $P'_+\subseteq\Sigma_i$, $P'_-\subseteq\Sigma_j$, $Q'_+\subseteq\Sigma_h$, and $Q'_-\subseteq\Sigma_k$, where Σ_i, Σ_j, Σ_h, Σ_k are not necessarily all distinct, see Fig. 1 b).

Let γ be the component of $\mathring{D}^*$ which contains x^*. Let $x_p=\Pi^{-1}(x^*)\cap\cap P'$, $x_q=\Pi^{-1}(x^*)\cap Q'$. If $h(x_p)>h(x_q)$, we will say that $\Sigma_i\cup\Sigma_j$ is the over surface at x^*. We will show that this definition is independent of choice of $x^*\in\gamma$. Let $V=\{x^*\in\gamma$ such that $\Sigma_i\cup\Sigma_j$ is the over surface at $x^*\}$, then V is open in γ since $\Sigma_i\cup\Sigma_j$ is the over surface for every point in $g^{-1}(G)$; similarly, $\gamma-V$ is open in γ and by connectedness of γ, $V=\gamma$.

In R^{n+1}, let S be the square $S=\{x\in R^{n+1}$, with $|x_1|\leqslant 1$, $|x_{n+1}|\leqslant 1$, $x_2=\ldots=x_n=0\}$; we wish to consider the following points in U corresponding to points of ∂S. Let

$$r_1^*=g^{-1}\big((1,0,\ldots,0,1)\big),\qquad r_2^*=g^{-1}\big((-1,0,\ldots,0,1)\big),$$
$$r_3^*=g^{-1}\big((-1,0,\ldots,0,-1)\big),\qquad r_4^*=g^{-1}\big((1,0,\ldots,0,-1)\big);$$

$$p_1^*=g^{-1}(\partial S\cap P_+),\qquad p_2^*=g^{-1}(\partial S\cap Q_+),$$
$$p_3^*=g^{-1}(\partial S\cap P_-),\qquad p_4^*=g^{-1}(\partial S\cap Q_-).$$

See figure 1b).

THEOREM 10. *$\Pi_1(R^{n+2}-M)$ has the following presentation. There is one generator, σ_k, for each component, $\mathring{\Sigma}_k^*$ of M^*-D^*. For each component, γ, of $\mathring{D}^*$ there are two relations as follows: choose an $x^*\in\gamma$,*

I) $\sigma_i=\sigma_j$ if $\Sigma_i\cup\Sigma_j$ is the over surface at x^*,

II) $\sigma_i^{\varepsilon(p_1^*)}\sigma_h^{\varepsilon(p_2^*)}\sigma_j^{\varepsilon(p_3^*)}\sigma_k^{\varepsilon(p_4^*)}=1$,

where $\varepsilon(p_i^*)$ is defined as in Proposition 9 via the path (r_{i-1}^*,r_i^*) for $i\neq 1$, $\varepsilon(p_i^*)$ determined via (r_4^*,r_1^*).

Proof. We first remark that we have shown that relations of type I do not depend on the choice of $x^*\in\gamma$; similarly one can show that relation II does not depend on the choice of $x^*\in\gamma$. We also remark that we may describe relation II as obtained as follows: let S^* be a small oriented simple closed curve about γ which meets $\mathring{\Sigma}_i^*$, $\mathring{\Sigma}_j^*$, $\mathring{\Sigma}_h^*$, $\mathring{\Sigma}_k^*$ in one point, transversely, and let $s^*\in S^*-M^*$, then by considering S^* as a directed path from s^* back to s^*, relation II is obtained by setting $w[\varrho(S^*)]=1$ where $w[\varrho(S^*)]$ is defined below. We can see that this relation does not depend on the choice of s^* or the orientation of S^*.

Let F denote the free group on the symbols $\{\sigma_i\}$, let R be the smallest normal subgroup of F containing the words $\sigma_i\sigma_j^{-1}$ and $\sigma_i^{\varepsilon(p_1^*)}\sigma_h^{\varepsilon(p_2^*)}\sigma_j^{\varepsilon(p_3^*)}\sigma_k^{\varepsilon(p_4^*)}$; we will show that $\Pi_1(R^{n+2}-M)$ is isomorphic to the quotient group F/R by defining a homomorphism $f\colon F\to\Pi_1(R^{n+2}-M)$ which is onto, such that the kernel of f is R. The map which sends each generator, σ_i of F to the element representing the loop σ_i in $\Pi_1(R^{n+2}-M)$ extends to a unique homomorphism of F to $\Pi_1(R^{n+2}-M)$, this will be our map f. If $a\in F$, and we represent a by the word $w=\sigma_{i_1}^{\varepsilon_1}\cdot\sigma_{i_2}^{\varepsilon_2}\cdot\ldots\cdot\sigma_{i_m}^{\varepsilon_m}$, $\varepsilon_i=\pm 1$ then $f(a)$ is represented by the loop $\sigma_{i_1}^{\varepsilon_1}\cdot\sigma_{i_2}^{\varepsilon_2}\cdot\ldots\cdot\sigma_{i_m}^{\varepsilon_m}$; conversely we note that if $a=\sigma_{i_1}^{\varepsilon_1}\cdot\sigma_{i_2}^{\varepsilon_2}\cdot\ldots\cdot\sigma_{i_m}^{\varepsilon_m}$ represents an element of $\Pi_1(R^{n+2}-M)$ and $f(a)=a$ then a is represented by the word $\sigma_{i_1}^{\varepsilon_1}\cdot\sigma_{i_2}^{\varepsilon_2}\cdot\ldots\cdot\sigma_{i_m}^{\varepsilon_m}$. We will let this word be denoted by $\omega(a)$. Next we show that f is onto. If $a\in\Pi_1(R^{n+2}-M)$ we may represent it by a directed polygonal path, L, in $R^{n+2}-M$ with the following properties:

1) L^* contains no points of D^* (this can be done since D^* has codimension 2 in R^{n+1}),

2) L^* intersects M^*-D^* transversely in a finite number of points $p_1^*,\ldots,p_m^*$ where $p_1^*<p_2^*<\ldots<p_m^*$. Let c_j be points of L such that $x_0^*=c_0^*<p_1^*<c_1^*<p_2^*<c_2^*<\ldots<c_{m-1}^*<p_m^*<c_m^*=x_0^*$. We define $\varepsilon(p_j^*)$ via the path (c_{j-1},c_j) as in Proposition 9, and then by that proposition we will have L homotopic to the loop $\varrho\big((c_0,c_1)\big)*\varrho(c_1,c_2))*\ldots*\varrho\big((c_{m-1},c_m)\big)$

and thus represented by the word $\sigma_{i_1}^{\varepsilon(p_1^*)}\sigma_{i_2}^{\varepsilon(p_2^*)}\ldots\sigma_{i_m}^{\varepsilon(p_m^*)}$, where $p_j^* \in \mathring{\Sigma}_{i_j}^*$: we will denote this word by $w[\varrho(L^*)]$.

We next will show that the kernel of f is R. Suppose that $a \in F$ with $f(a) = 1$; then the loop $f(a)$ is nulhomotopic in $R^{n+2}-M$. Let $g\colon D^2 \to R^{n+2}-M$ give this homotopy as follows: let z_0 be the point of D^2 corresponding to $(1, 0)$, then if we consider ∂D^2 to be the directed path from z_0 to z_0 counterclockwise about ∂D^2, then $g|\partial D^2$ represents $f(a)$. Let $B = g(D^2)$. By choosing g appropriately, we may assume that

1) g is piecewise-linear,

2) $\Pi \circ g$ is non-degenerate,

3) $B^* \cap Z^* = \emptyset$ (since Z^* has codimension 3 in R^{n+1}),

4) B^* intersects $\mathring{D}^*$ transversely in a finite number of points $q_1^*, \ldots, q_s^*$, and B^* intersects each $\mathring{\Sigma}_i^*$ transversely.

(In the case $n = 1$, we will need to use the following in place of (4): if σ^* is a 2-simplex of B^* then $\sigma^* \cap D^* \subseteq \operatorname{Int}\sigma^*$, and $\partial\sigma^*$ meets each arc $\mathring{\Sigma}_i^*$ transversely.) Such a map, g, will be called a nulhomotopy in general position. If $X \subseteq D^2$, let $X' = g(X)$. If x and y are points of ∂D^2 then $[x', y']$ will denote the directed subpath of $g(\partial D^2)$ from x' to y'.

Let $J = (\Pi \circ g)^{-1}(M^*)$, $K = (\Pi \circ g)^{-1}(\mathring{D}^*)$. K is a collection of points in $\operatorname{Int} D^2$, and J is the union of proper arcs in D^2 and simple closed curves in $\operatorname{Int} D^2$ whose intersections are transverse and constitute K (an arc $a = D^2$ is proper if $a \cap \partial D^2$ corresponds to the endpoints of a).

Proposition 11. *If $g\colon D^2 \to R^{n+2}-M$ is a nulhomotopy in general position with $K = \emptyset$, and z_1 is any point in $\partial D^2 - J$, $z_1 \neq z_0$, then $w(\varrho[z_0', z_1']) = [w(\varrho[z_1', z_0'])]^{-1}$.*

Proof. We will use induction on the number of components of J. If J has one component then $J = (\Pi g)^{-1}(\mathring{\Sigma}_k^*)$ for some k. If $J \cap \partial D^2 = \emptyset$ (i.e., if J is a simple closed curve in the interior of D^2), then for any z_1, $w(\varrho[z_0', z_1']) = w(\varrho[z_1', z_0']) = 0$. If $J \cap \partial D^2 \neq \emptyset$, then $J \cap \partial D^2$ consists of two points p and r with say $p < r$. Let q be a point of ∂D^2 such that $p < q < r$. Now by Proposition 9, $\varrho([z_0', q']) \in \sigma_k^{\varepsilon((p')^*)}$. We will show that 1) if $\varepsilon((p')^*) = 0$, then $\varepsilon((r')^*) = 0$; 2) otherwise $\varepsilon((p')^*) = -\varepsilon((r')^*)$. Let $p'' = \Pi^{-1}((p')^*) \cap M$, $r'' = \Pi^{-1}((r')^*) \cap M$. If $\varepsilon((p')^*) = 0$, then $h(p') > h(p'')$; by connectivity of J', we may argue that therefore all points of J' lie above the corresponding points in M, in particular, $h(r') > h(r'')$, and thus $\varepsilon((r')^*) = 0$. To consider case (2); suppose that $\varepsilon((p')^*) = 1$. Then $h(p') < h(p'')$ and $(z_0')^*$ lies in the same component, X, of $R^{n+1}-M^*$ as a_k^*. As above, we see that all points of J' lie below corresponding points of M and so, $h(r') < h(r'')$, and since $(z_0')^* \in X$, $\varepsilon((r')^*) = -1$. The case $\varepsilon((p')^*) = -1$ is similar. Now by checking the three cases $z_0 < z_1 < p$, $p < z_1 < r$, $r < z_1 < z_0$, one may verify our proposition. For example, in the first case, $w(\varrho[z_0', z_1']) = 1$, $w(\varrho[z_1', z_0']) = \sigma_k^{\varepsilon((p')^*)}\sigma_k^{-\varepsilon((p')^*)}$.

Next, suppose that J has $n+1$ components, $n > 1$. It is not difficult to find a proper arc, β, in D^2 with one end point z_0, the other, say z_2, such that $D^2-\beta$ contains some components of J, β meets each component of J at most once, see Fig. 2a). Then the closure of each component is a nulhomotopy in general position satisfying our inductive hypothesis. There are two cases: $z_0 < z_1 < z_2$, and $z_0 < z_2 < z_1$. We will examine the first case, the second is similar. Let $w_1 = w(\varrho[z_0', z_1'])$, $w_2 = w(\varrho[z_1', z_2'] * \beta^{-1})$, $w_3 = w(\varrho(\beta))$, $w_4 = w(\varrho[z_2', z_0'])$, $w_5 = w(\varrho[z_1', z_0'])$. We wish to show that $w_5 = w_1^{-1}$. By our inductive hypothesis we have $w_2 = w_1^{-1}$ and $w_4 = w_3^{-1}$. Thus we have $w_5 = w_2w_3w_4 = w_1^{-1}w_3w_3^{-1} = w_1^{-1}$.

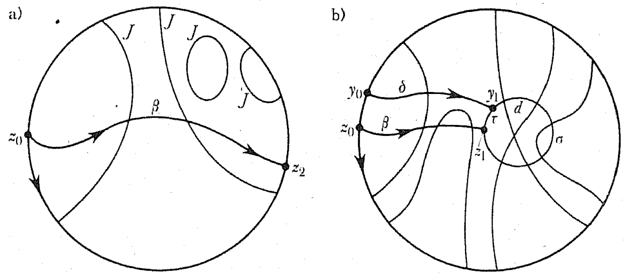

Fig. 2

From the above proposition, we see that if $K = \emptyset$, then the word $a = w(f(a))$ is equivalent to $1 \in F$ and therefore $a \in R$ in general by induction on the number of points in K.

Suppose that K consists of one point, say d. Let B be a small regular neighborhood of d. Let z_1 be a point of $\partial B - J$ and let β be a proper arc in $D^2 - \operatorname{Int} B$ from z_0 to z_1 which meets J transversely; let δ be an arc parallel to β, with the same properties from, $y_0 \in \partial D^2$ to $y_1 \in \partial B$ such that $J \cap \partial D^2 \subseteq (z_0, y_0)$. Orient ∂B counterclockwise and let σ be the directed path from y_1 to z_1, τ the directed path from z_1 to y_1, see Fig. 2b). Let $w_1 = w[\varrho([z_0', y_0'])]$, $w_2 = w[\varrho(\delta)]$, $w_3 = w[\varrho(\sigma)]$, $w_4 = w[\varrho(\beta^{-1})]$, $w_5 = w[\varrho(\tau)]$, and $w_6 = w[\varrho([y_0', z_0'])]$. Clearly, it can be arranged that $w_5 = 1$ and $w_6 = 1$. By Proposition 11, considering the disk bounded by the path $[y_0, z_0] * \beta * \tau * \delta^{-1}$, we see that $w_2 = w_4^{-1}$. Considering the disk bounded by the path $[z_0, y_0] * \delta * \sigma * \beta^{-1}$ we see that $w_1 = w_2w_3w_4 = w_4^{-1}w_3w_4$. By the remark following the statement of our theorem, with S^* corresponding to $[(\partial B)']^*$, we see that $w_3w_5 = w_3 \in R$. Now $w(f(a)) = w_1w_6 = w_1 \cdot 1 = w_1$ and w_1 is a conjugate by w_4 of an element of R, therefore $w(f(a)) \in R$.

Now suppose that K consists of $n+1$ points. Let β be a proper arc in ∂D^2 from z_1 to, say, z_2 such that each component of $D^2-\beta$ contains some points of K. Then

$$\begin{aligned} w(f(a)) &= w[\varrho([z_0', z_2']) * \beta^{-1} * \beta * \varrho([z_2', z_0'])] \\ &= w[\varrho([z_0', z_2']) * \beta^{-1}] w[\beta * \varrho([z_2', z_0'])] \end{aligned}$$

is the product of words in R. This completes the proof of the theorem. ∎

As an example of this theorem, consider the crosscap; this is the projection, in general position, of an embedding of the projective plane in R^4. To find its knot group we note that there is only one region, Σ_1, and only one arc of double points. Thus we have one generator, σ_1, and two relations; the first is $\sigma_1 = \sigma_1^{-1}$, the second is $\sigma_1\sigma_1^{-1}\sigma_1\sigma_1^{-1} = 1$; thus the knot group is isomorphic to Z_2. Similarly one may consider the embedding of the Klein bottle in R^4 whose projection in R^3 is a surface with a single circle of self-intersection and find the knot group of this embedding to be isomorphic to Z_2.

References

[1] M. A. Armstrong and E. C. Zeeman, *Transversality for piecewise linear manifolds*, Topology 6 (1967), pp. 433–466.

[2] R. H. Crowell and R. H. Fox, *Introduction to Knot Theory*, 1963.

[3] T. Yajima, *On the fundamental groups of knotted 2-manifolds in the 4-space*, J. Math. Osaka City Univ. 13 (1963), pp. 63–71.

[4] E. C. Zeeman, *Seminar on Combinatorial Topology* (mimeographed notes), I. H. E. S., Paris 1963.

[5] E. Spanier, *Algebraic Topology*, 1966.

Accepté par la Rédaction le 15. 10. 1973

Two notes on abstract model theory
II. Languages for which the set of valid sentences is semi-invariantly implicitly definable

by

Solomon Feferman * (Stanford, Cal.)

Abstract. It is shown that if L is an abstract model-theoretic language, the syntax of L is represented in a structure $\mathfrak{A} = (A, \ldots)$, the Löwenheim-Skolem property holds down to card (A) and $\models_L$ is uniformly i.i.d. in L then the set of L-valid consequences of a set S of sentences is s.i.i.d. in L whenever S itself is so definable. This generalizes a theorem of Kunen for admissible fragments of $L_{\infty,\omega}$. The final part of the paper relates this to a program of study of good properties of model-theoretic languages.

Introduction. The aim of this note is quite different from that of the preceding [F2], though it makes use of the same general preliminaries. In content, it is a sequel to [F1], § 3 where some connections were studied, for arbitrary languages L, between implicit definability of the satisfaction relation $\models_L$ and logical properties of L. The basic relevant notions of [F1] are recalled below, in particular that of the *syntax* of L being *represented in a structure* $\mathfrak{A} = (A, \ldots)$ and (relative to any such representation) that of $\models_L$ being *uniformly invariantly implicitly definable* (uiid) *in* L. We add here a related notion of a subset S of A being *semi-invariantly implicitly definable* (siid) *in* L (1). This includes Kunen's definition in [Ku] of siid for admissible structures $\mathfrak{A} = (A; \varepsilon, R_1, \ldots, R_k)$ as a special case, and in the same line extends model-theoretic generalizations of recursion theory. Kunen showed that being siid is equivalent to other proposed

* Guggenheim Fellow 1972–73. The preparation of this paper was begun at the U. E. R. de Mathématiques, Université Paris VII and completed at Stanford University under NSF Grant GP 34091X during that period.

(1) To be more precise, several variants of the notions siid are introduced and compared in § 2. In accordance with one of these, uiid is rewritten $\#$-uiid$_{\mathfrak{A}}$.