Projections of knots

by

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Abstract. In this paper we define and study projections of PL n-manifolds in $\mathbb{R}^{n+2}$ and derive an algorithm for calculating the fundamental group of the complement of the $n$-manifold similar to the Wirtinger presentation for knotted circles in $\mathbb{R}^3$.

In this paper, we define and study projections of PL $n$-manifolds in $\mathbb{R}^{n+2}$, and derive an algorithm for calculating the knot group (i.e., the fundamental group of the complement of the manifold in $\mathbb{R}^{n+2}$). This algorithm generalizes the classical presentation of knot groups as in [2]. In Yajima [3] knot groups of orientable 2-manifolds in $\mathbb{R}^4$ are studied; our results extend this further in two ways: to higher dimensions, where general position is more of a problem, and to non-orientable manifolds.

We will be concerned with closed (i.e., compact without boundary) $n$-dimensional PL submanifolds of $\mathbb{R}^{n+2}$. If $M^n$ is such a submanifold and $i_M : M \to \mathbb{R}^{n+2}$ is the inclusion map, then the knot type of $(\mathbb{R}^{n+2}, M)$ is the PL ambient isotopy class of $i_M$. Frequently, to avoid excessive notation, we will use the letter $M$ to also denote the image of $i_M$.

In order to describe the general position of a map we will use the following definitions as in Zeeman [4]. If $f : X \to Y$ is a map and $r$ an integer, we will define $S_r(f) = \{ x \in X : f^{-1}(y) \text{ contains at least } r \text{ points} \}$. We will define $S(f)$ to be the closure of $S_r(f)$, and let $Br(f) = \{ x \in X : \text{no neighborhood of } x \text{ is embedded by } f \}$. Then, we will have $X = S(f) \supset \cdots \supset S_1(f) \supset \cdots$, and $S(f) = \bigcup S_r(f) \cup Br(f)$. If $f$ is a PL map of polyhedra, then these subsets of $X$ will be subcomplexes of some subdivision of $X$. Also, a map is called non-degenerate if it embeds each simplex.

We will consider $\mathbb{R}^{n+2}$ as the set of all $(n+2)$-tuples of real numbers, and $\mathbb{R}^{n+2} = \{(x_1, \ldots, x_{n+2}) \in \mathbb{R}^{n+2} \text{ with } x_{n+2} = 0 \}$. Then $H : \mathbb{R}^{n+2} \to \mathbb{R}^{n+1}$ defined by $H(x_1, \ldots, x_{n+2}, x_{n+3}) = (x_1, \ldots, x_{n+2}, 0)$, will be called the projection of $\mathbb{R}^{n+2}$ onto $\mathbb{R}^{n+1}$. The map $H$, called the height function, is
defined by $h(x_1, \ldots, x_{n+3}) = x_{n+2}$. Generally, if $X \subseteq \mathbb{R}^{n+2}$, we will let $\mathbb{X}^\ast$ denote $H(X)$.

**Proposition 1.** Given $(\mathbb{R}^{n+2}, M)$ there is an isotopy $\varphi_t$ of $\mathbb{R}^{n+2}$ such that $H(M) = \mathbb{X}^\ast$.

**Proof.** Let $x_1, \ldots, x_n$ denote the vertices of $M$. Let $\Omega(x_1, \ldots, x_n)$ be the union of all $k$-dimensional subspaces, $K_k$ of $\mathbb{R}^{n+2}$ with $k < n + 2$, such that $K$ is parallel to some $k$-dimensional affine subspace of $\mathbb{R}^{n+2}$ spanned by a subset of $\{x_1, \ldots, x_n\}$. If $S^{n+1}$ is the unit sphere of $\mathbb{R}^{n+2}$, then $S^{n+1} - \Omega(x_1, \ldots, x_n)$ will be dense in $S^{n+1}$ and, in particular, non-empty. Choose $v \in S^{n+1} - \Omega(x_1, \ldots, x_n)$ and consider $v$ as a vector, let $H$ be the $(n+1)$-subspace of $\mathbb{R}^{n+2}$ orthogonal to $v$. Then the orthogonal projection of $\mathbb{R}^{n+2}$ onto $H$ will embed each simplex of $M$. If $\varphi_t$ is an isotopy which takes $H$ onto $\mathbb{R}^{n+2}$ by means of a rigid rotation, then $\varphi_t$ is the isotopy we wish.

For the definition of $t$-shifts, see Armstrong and Zeeman [1]; the construction of a $t$-shift will be contained in the proof of the following proposition, we will not need Brouwer triangulations, since our embeddings are Euclidean space. This proposition will be used in applying established general position arguments to our particular situation.

**Proposition 2.** (Shift lifting lemma). In suitable triangulations, if $\sigma$ is a $t$-simplex of $M$ with a local $t$-shift of $H + i\mathbb{X}$ with respect to $v$, then there is a local $t$-shift $\tilde{\sigma}$ of $\tilde{\mathbb{X}}$ with respect to $v$ such that $H + \tilde{\tilde{\sigma}} = \tilde{\sigma}$ and such that $\tilde{\tilde{\sigma}}$ is isotopic to $i\mathbb{X}$, and $\tilde{\tilde{\sigma}}$ is the restriction of an isotopy of $\mathbb{R}^{n+2}$.

**Proof.** Let $K_0, (L_0, K_0^*)$ be triangulations of $M, (\mathbb{R}^{n+2}, M_0)$, and $(\mathbb{R}^{n+2}, K_0^*)$ such that $A_{l+1}$ is simplicial, $H: \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+2}$ is simplicial, and $H + i\mathbb{X}$ is non-degenerate. Let $\sigma$ be a $t$-simplex of $\tilde{\mathbb{X}}$, $\tilde{\sigma} = i_{l+1}(\sigma)$, and $\tilde{\sigma}^* = \sigma^*(\tilde{\mathbb{X}})$. Let $K_0^*, (L_0^*, K_0'^*)$, and $(L_0^*, K_0'^*)$ denote the second barycentric subdivisions of $K_0, (L_0, K_0^*)$, and $(L_0, K_0^*)$, respectively.

Let $\tilde{\mathbb{X}}$ be the regular neighborhood in $\mathbb{L}^n$ of $\tilde{\sigma}$ modulo its boundary (i.e., $\tilde{\mathbb{X}}$ consists of all closed simplices of $\mathbb{L}^n$ which meet the interior of $\tilde{\sigma}$); $\tilde{\mathbb{X}}$ will be an $(n+2)$-ball. Let $B_0 = H(B)$; then $B_0$ will be an $(n+1)$-ball, and in fact will be the regular neighborhood of $\sigma$ in $\mathbb{L}^n$ modulo its boundary.

Let $B_{l+1}(\tilde{\mathbb{X}} \cap \mathbb{X})$. Let $\tilde{v}, \tilde{v}'$, and $\tilde{\sigma}^*$ denote the barycentres of $\sigma, \tilde{\sigma}$, and $\tilde{\sigma}^*$, respectively. Then $\tilde{B}_0 = [\tilde{v} \times \mathbb{Z}], [\tilde{v} \times \mathbb{Z}], [\tilde{v} \times \mathbb{Z}], [\tilde{v} \times \mathbb{Z}]$, and $\tilde{\sigma}^* = [\tilde{v} \times \mathbb{Z}]$. Let $f: B \rightarrow B_0$ denote the restriction of $H + i\mathbb{X}$; then $f$ is the join of two maps: the restrictions of $H + i\mathbb{X}$ to $v$ and to $\mathbb{X}$.

Our $t$-shift will now be defined. First we need to find a point $u^* \in B_0$ near $v^*$ such that:

(i) $u^*$ is contained in the open star of $v^*$ in $B_0$;
(ii) $u^*$ is joinable to $\tilde{\mathbb{X}}$;
(iii) $u^*$ is in general position with respect to the vertices of $B_0$.

We next define a homeomorphism $f$ of $B_0$ to be the join of the map which sends $v^*$ to $u^*$ and the map which is the identity on $\tilde{\mathbb{X}}$. Then $f$ is a homeomorphism of $B_0$ which is fixed on $\tilde{\mathbb{X}}$. If we then define $g_0$ by $g_0 = f^* \circ f$, then $g_0$ is an ambient isotopy to $f_\ast$ keeping $\tilde{\mathbb{X}}$ fixed. The map $g$ defined by

$$g = \begin{cases} H + i\mathbb{X} & \text{on } \mathbb{X}, \\ f_\ast & \text{on } B \end{cases}$$

will be called a $t$-shift of $H + i\mathbb{X}$ with respect to $v$. (Note that $g$ is homotopic to $H + i\mathbb{X}$.)

We may finally lift this shift by choosing a point $\tilde{u}$ in $\pi^{-1}(u)$ such that:

(i) $\tilde{u}$ is contained in the open star of $\tilde{v}$ in $\mathbb{L}$;
(ii) $\tilde{u}$ is joinable to $\tilde{\mathbb{X}}$;
(iii) $\tilde{u}$ is in general position with respect to the vertices of $\mathbb{L}$.

We then proceed as before, defining a homeomorphism $f_\ast$ of $\mathbb{X}$ to be the join of the map which sends $\tilde{v}$ to $\tilde{u}$ and the identity on $\tilde{\mathbb{X}}$; then defining $g_\ast = f_\ast \circ f_\ast$, where $f_\ast$ denotes the restriction of $i_{l+1}$ to $B$. We may now define $g$ by

$$g = \begin{cases} i_{l+1} & \text{on } \mathbb{X}, \\ \tilde{f}_\ast & \text{on } B \end{cases}$$

We will then have $\tilde{g}$ isotropic to $i_{l+1}$, and $H + \tilde{g} = g$.

**Proposition 3.** (Given $(\mathbb{R}^{n+2}, M)$ we may find an isotopy $\varphi_t$ of $\mathbb{R}^{n+2}$ such that $H + \varphi_t \ast i_{l+1}$ is in general position.

**Proof.** By Zeeman [4] one may inductively define a sequence of arbitrarily small $t$-shifts, $\varphi^{(i)}$, $i = 1, \ldots, p$ such that $\varphi^{(i)}$ is obtained by a $t$-shift of $H + i\mathbb{X}$; $\varphi^{(0)}$ is obtained by a $t$-shift of $g^{(i-1)}$ for $i = 2, \ldots, p$, and such that $\varphi^{(0)}$ is in general position.

We may then define $\varphi_t$ as the composition of the compositions $\varphi^{(i)}$, $i = 1, \ldots, p$.

We will say that $\mathbb{X}$ is self-transverse if for all $g$, if $x \in S^2(\mathbb{X} + i\mathbb{X})$ and $y \in S^2(\mathbb{X} + i\mathbb{X})$ (i.e., $x$ is a point of order exactly $g$) then there is a neighborhood, $U$, of $x \in \mathbb{X}$ such that $H^{-1}(U) \cap M$ is a disjoint collection of $g$ open $n$-disks, $B_i$, and a homeomorphism $f: U \rightarrow \mathbb{R}^{n+2}$ such that $f(x) = 0$, and each $f(\mathbb{X}(B_i))$ is an $n$-plane containing $0$, and the $g$ $n$-planes, $\{f(\mathbb{X}(B_i))\}$ are in general position in $\mathbb{R}^{n+2}$ (i.e., the intersection of any $r$ of them is $(n+1) - r$ dimensional).
Lemma 4. Given \((\mathbb{R}^{n+1}, M)\), there exists an isotopy, \(\varphi_t\), of \(\mathbb{R}^{n+1}\) such that \(\varphi_1(M)\) is self transverse.

**Proof.** For each \(i = 1, \ldots, n+1\), let \(D_i = \{x \in M \mid \text{there exists exactly } i \text{ points, } p_j, \text{ such that } \varphi_t(p_j) = \varphi_t(x) \text{ and } h(p_j) < h(x)\}\), \(D_i\) denotes the closure of \(D_i\) in \(M\); let \(D = D_1 \cup \cdots \cup D_{n+1}\).

Let \((L, \mathcal{K}, (L^*, \mathcal{K}^*))\) be triangulations of \(M\), \((\mathbb{R}^{n+1}, M)\), \((\mathbb{R}^{n+1}, M)\), respectively, such that \(\mathcal{K}\) is simplicial, \(\mathcal{K}\) is simplicial, non-degenerate and in general position. Let \(\mathcal{K}'\), \((L', \mathcal{K}'), (L'^*, \mathcal{K}'^*)\) denote the second barycentric subdivisions of these triangulations. Let \(N_i\) be the derived neighborhood of \(D_i \mod D_i\) in \(M\). Then \(\int N_i\) is a locally flat embedding; it is an embedding since there are no self intersections, locally flat since all non-locally flat points of \(M^*\) are images of branch points of \(\varphi_t\).

Consider the simplexes of \(N_i\), mod \(\partial N_i\), \(\partial_1, \ldots, \partial_p\), ordered in decreasing dimension. As in Theorem 4 of Armstrong and Zeeman [1] we may define \(t\)-shifts for these simplexes and associated maps \(g^t\), \(t = 1, \ldots, p\) such that \(g^0\) will be transversal simplexes of the \(\bigcup \partial j\); \(g^0\) will agree with \(\varphi_t\) of \(N_i\). Let \(N_i\) be the derived neighborhood of \(D_i \mod D_i\) in \(M\). Then the proof of Lemma 6 of [1] shows that \(g^0(\int N_i)\) will be transverse to \(g^0(\int N_i)\). By Proposition 2, we can find a map \(g^0\) isotopic to \(\varphi_t\), by an isotopy fixed off of \(N_i\) such that \(\varphi_t \circ g^0 = g^0 \circ \varphi_t\). \(g^0\) is the restriction of an isotopy of \(\mathbb{R}^{n+1}\). To complete the proof, we proceed similarly with \(N_{i+1}, \ldots, N_{n+1}\) inductively finding isotopies \(g^1\) of \(N_i\) in \(\mathbb{R}^{n+1}\) such that \(g^1\) will be transversal, and therefore transverse to each \(N_i\) since \(g_i < g_i\).

If \(M \subseteq \mathbb{R}^{n+1}\) is such that \(\varphi_t\) is in general position, and if \(M^*\) is self transverse, then we will say that \(M\) is in general position with respect to projection. In this case we need to consider the following subsets. Let \(D = \bigcup D_i = \partial_0(\varphi_t)\); \(\mathcal{D}\) denote the closure of \(D\) in \(M\); \(\mathcal{D}\) will be called the set of pure double points. Also let \(Z = \bigcup (\partial_0(\varphi_t)\); \(\Sigma\) denotes the closure of \(Z\) in \(M\). Let \(\Sigma_i = \{k = 1, \ldots, p\}\) be the components of \(M - D_i\); \(\Sigma_i\) denotes the closure of \(\Sigma_i\) in \(M\). Then \(\varphi_t\) is an embedding for each \(\Sigma_i\); however, \(\varphi_t\) may not be an embedding since it may fail to be an embedding on \(\partial \Sigma_i\). Each \(\Sigma_i\) is a relative \(n\)-manifold, see Spanier [5]. We also note that \(\Sigma_i^c\) will be locally flat in \(\mathbb{R}^{n+1}\) since the non-locally flat points must correspond to branch points of the projection and thus be contained in \(D^*\).

Lemma 5. (Separation lemma). Each \(\Sigma_i^c\) is an open orientable \(n\)-manifold, and is two-sided in \(\mathbb{R}^{n+1}\) (i.e., \(\Sigma_i^c\) lies on the boundary of exactly two components of \(\mathbb{R}^{n+1} - M^*\)).

**Proof.** We have already established that \(\Sigma_i^c\) is an open locally flat \(n\)-manifold in \(\mathbb{R}^{n+1}\). We will need the following proposition:

**Proposition 6.** The map \(H^k(M^*) 	o H^k(M)\) induced by the restriction of \(\varphi_t\) to \(M^*\) is an onto map for all coefficients, in particular \(H^k(M^*; Z) = 0\) since \(H^k(M^*; Z) = 0\). Also, the map \(H^k_{\Sigma^c}(\Sigma; Z) 	o H^k_{\Sigma^c}(\Sigma; Z)\) induced by the restriction of \(\varphi_t\) to \(\Sigma^c\) is an isomorphism.

**Proof.** Let \(C^k(X)\) denote the \(j\)-dimensional simplicial cochains on \(X\), \(\partial^k\) denote the \(j\)-dimensional coboundary operator. Let \(H^k: C^k(M^*) 	o C^k(M)\) be the cochain map induced by \(\varphi_t\). Since \(\varphi_t\) is a 1-1 mapping of the \(n\)-simplexes of \(M^*\) to the \(n\)-simplexes of \(M^*\), \(H^k\) is an isomorphism and its restriction gives an isomorphism of the cycle groups \(Z^k(M^*) 	o Z^k(M)\). Now \(H^k: H^k(M^*) 	o H^k(M)\) will be onto if \(H^k(M^*) 	o H^k(M)\), where \(H^k\) refers to the coboundary groups; but this follows since \(H^k(M^*) \to H^k(M)\).

Similarly, since \(\Sigma^c\) can be considered as \(\Sigma^c\) with identifications on \(\partial \Sigma^c\), then the cochain map induced by \(\varphi_t\) from \(C^k(\Sigma^c) \to C^k(\Sigma^c)\) is an isomorphism for all coefficients and for all \(j\), and thus \(H^k(\Sigma^c) \to H^k(\Sigma^c)\) is isomorphic to \(H^k(\Sigma^c) \to H^k(M)\).

We will now show that \(\Sigma^c\) is two-sided. Let \(A = \{X \subseteq M^* \mid \Sigma^c\} \subseteq \{X \subseteq M^* \mid \Sigma^c\}\). We will show \(\Sigma^c\) is a subcollection of \(\{X \subseteq M^* \mid \Sigma^c\}\). A is partially ordered by inclusion, let \(\leq\) denote strict inclusion. For each \(j\), let \(\mathcal{A}_j = \{X \subseteq M^* \mid \Sigma^c\} \subseteq \{X \subseteq M^* \mid \Sigma^c\}\) such that \(\mathcal{A}_j \subseteq \mathcal{A}_j\) and such that for any \(\Sigma^c \subseteq X\), the map \(H^k(\Sigma^c) \to H^k(X - \Sigma^c)\) induced by inclusion, is not an isomorphism.

Next we show \(\mathcal{A}_j \neq \emptyset\) by showing that \(\mathcal{A}_j \neq \emptyset\). Let \(N = M - \Sigma^c\), then \(\Sigma^c = M^* - \Sigma^c\). Consider the diagram below, which is part of the homomorphism of the cohomology sequence of the pair \((M^c, N)\) induced by the restriction of \(\varphi_t\) to \(M\). (coefficients are to be taken in \(Z_2\)).

\[
H^k(M^*, N) \xrightarrow{\delta^*} H^k(M^*) \xrightarrow{\alpha^*} H^k(N) \xrightarrow{\beta^*} 0
\]

Now \(H^k(M^*), H^k(M^*; N)\) and \(H^k(M^c, N)\) are isomorphic to \(Z_2\) by the Lefschetz duality theorem (Spanier [5]) since each is compact relative manifold orientable over \(Z_2\); the map \(\delta^*\) is an isomorphism by exactness of the bottom row since \(H^k(N) = 0\) (\(N\) collapses to an \((n-1)\)-dimensional subcomplex); \(\beta^*\): \(H^k(M^c, N) \to H^k(M, N)\) is an isomorphism by Proposition 6 since, by excision, this map is equivalent to the map \(H^k(\Sigma^c, \Sigma^c) \to H^k(\Sigma^c, \Sigma^c)\) thus by commutativity of the left hand square, \(\delta^*\) is not a zero map; therefore, by the exactness of the top row, the map \(\delta^*\), although onto, is not 1-1.

Let \(W_k\) be a minimal element of \(\mathcal{A}_j\) (i.e., \(W_k \in \mathcal{A}_j\) and if \(X \in \mathcal{A}_j\) with \(X \subset W_k\), then \(X \notin \mathcal{A}_j\)). Such a minimal element will be called a \(\Sigma^c\)-cycle. We will show that \(H^k(W_k; Z)\) is isomorphic to \(Z_2\). First we show:

\[
H^k(\Sigma^c) \to H^k(M) \to H^k(N) \to 0
\]
PROPOSITION 7. If $X \in A$, $Y \subseteq X$ with $X-Y = \Sigma^*_p$ and $H^0(Y; Z) = 0$, then $H^0(X; Z) = 0$, so $H^0(X; Z)$ is isomorphic to either 0 or $Z_n$.

Proof. By excision, $H^0(X, Y; Z) \approx H^0(\Sigma^*_p, \partial \Sigma^*_p; Z) \approx Z_n$, and thus around dimension $n$, the cohomology sequence of the pair $(X, Y)$ is $Z_n \Rightarrow H^0(X; Z) \Rightarrow 0$. Thus $H^0(X) = Z_n$ is the image under $J$ of $Z_n$; therefore $H^0(X)$ is 0 or $Z_n$. 

Now if $W_b = \Sigma^*_b$, then it follows from the above proposition with $X = \Sigma^*_a$, and $Y = \partial \Sigma^*_a$ that $H^0(W_b; Z) \approx Z_n$, otherwise $H^0(W_b; Z) = 0$, and $H^0(W_b, \partial \Sigma^*_a; Z)$ would be both zero and thus isomorphic contradicting the assumption that $W_b \in A$. Next suppose that $W_b \neq \Sigma^*_b$. Then we must have $H^0(\Sigma^*_a; Z) = 0$. We may suppose that $W_b = \bigcup_{n=0}^{\infty} \Sigma^*_n$ where $\Sigma^*_n = \Sigma^*_b$. Let $V_n = \bigcup_{n=0}^{\infty} \Sigma^*_n$. By the minimality of $W_b$, for all $q < p$, we have, with $Z_n$ coefficients, $H^0(V_n; Z)$ isomorphic to $H^0(V_n, \Sigma^*_n)$ and since $V_n = \Sigma^*_n$, it follows that for all $q < p$, $H^0(V_n; Z) = 0$, in particular, $H^0(V_{p-1}; Z) = 0$. Applying Proposition 7 with $X = W_b$ and $Y = \partial V_{p-1}$, we conclude that $H^0(W_b; Z) = 0$; but it cannot be 0 since then $W_b$ would not be in $A$, therefore the assumption that $W_b = \Sigma^*_b$ is false.

By Alexander duality, $H^0(W_b)$ is isomorphic to $H^0(\Sigma^*_a; W_b)$ for any coefficients, where $\Sigma^*_a$ denotes reduced zero-th homology. The number of components of $E^{n+1} - W_b$ is one more than the rank of $H^0(\Sigma^*_a; W_b)$, that is, the number of components is two.

In order to complete the proof of Lemma 5 we need:

PROPOSITION 8. Let $K$ be a component of $E^{n+1} - M$. If $\Sigma^*_a \cap K \neq \emptyset$, then $\Sigma^*_a \subset K$.

Proof. Using the fact that each $\Sigma^*_a$ is locally flat and connected, one can show that the set of points of $\Sigma^*_a \cap K$ is both open and closed in $\Sigma^*_a$; thus $\Sigma^*_a \subset K$ and it follows that $\Sigma^*_a \subset K$.

Now let the components of $E^{n+1} - W_b$ be $C$ and $D$. If $B$ is a component of $E^{n+1} - M$, then either $\Sigma^*_a \subset C$ or $\Sigma^*_a \subset D$. Let $C'$ be the component of $E^{n+1} - M$ contained in $C$ such that $\Sigma^*_a \subset C'$, such exist since some component contained in $C$ must meet $\Sigma^*_a$ and thus by the above proposition its closure must contain $\Sigma^*_a$. Similarly let $D'$ be the component of $E^{n+1} - M$ contained in $D$ such that $\Sigma^*_a \subset D'$. Now $\Sigma^*_a$ is contained in the boundary of these two components and only these components of $E^{n+1} - M$.

It remains to be shown that $\Sigma^*_a$ is orientable. If $\Sigma^*_a$ were non-orientable then $\Sigma^*_a$ would be non-orientable and $H^0(\Sigma^*_a, \partial \Sigma^*_a; Z) = 0$. Let $N = W_b - \Sigma^*_a$; since $H^0(W_b; Z) \approx Z_n$. It follows from the definition of $W_b$ that $H^0(N; Z) = 0$, thus, by Alexander duality, $N$ does not separate $E^{n+1}$, and $H^0(N, Z) = 0$. By excision, $H^0(W_b, N) \approx H^0(\Sigma^*_a, \partial \Sigma^*_a) \approx 0$. Thus in dimension $n$, the cohomology exact sequence of $(W_b, N)$ with $Z$ coefficients becomes $0 \rightarrow H^0(W_b, Z) \rightarrow 0$, and $H^0(W_b; Z) = 0$, thus $W_b$ does not separate $E^{n+1}$. But we have already established that it does separate; thus $\Sigma^*_a$ could not have been non-orientable. This concludes the proof of Lemma 5.

Let $H$ be an $(n+1)$-plane in $E^{n+2}$ parallel to $E^{n+1}$. We may assume that $M$ is between $H$ and $E^{n+1}$. If $X \subseteq E^{n+1}$, then $X$ will be $H^{-1}(U(X)) \cap H$.

Now for our basepoint in the calculation of the knot group, we will take a point $x_0 \in H$ such that $x_0 \in M$. If $x \in E^{n+2}$, then $x_0$ will be the line segment between $x$ and $x_0$. If $X \subseteq E^{n+1}$, then $X = \bigcup_{x \in X} x_0 = \bigcup_{x \in X} X$.

A directed path is a path with a particular ordering of the points; we will denote this ordering by $<$. If $\{a_1, \ldots, a_k\}$ are points of $E^{n+1}$, by the path $(a_1, \ldots, a_k)$ we mean the directed path from $a_1$ to $a_k$ consisting of straight line segments joining $a_i$ to $a_{i+1}$ at times, to conserve notation, this symbol will also refer to the unordered path. If $L$ is a directed polygonal path, $-L$ will denote the path $L$ directed in the opposite direction. Suppose $L$ is a polygonal path in $E^{n+1} - M$ with endpoints $a$ and $b$, directed from $a$ to $b$, and such that $a^*\neq b^*$ are not points of $M$, then $\varphi(L)$ will denote the loop in $E^{n+1} - M$ given by the polygonal path $(a_0, a_1, a_2, a_3, \ldots, b)$, where the operation $\ast$ is path composition, and note that $\varphi(-L) = \varphi(-L)$.

For each region $\Sigma^*_a$ choose a polygonal path $L_i$ in $E^{n+1}$ with endpoint $a_i$ and $b_i$ such that $L_i \cup M$ consists of a single point of $\Sigma^*_a$ between $a_i$ and $b_i$, that intersection being transverse. Let $\varphi_i$ be the homotopy class of the loop $\varphi(L_i)$, then $\varphi(-L_i) = \varphi(-L_i)$. (In the case that $M$ is orientable, we might wish to choose the direction of $L_i$ to be consistent with the orientation of $M$ as in the classical case [2], Chapter VI, or as in Yajima [3].)

PROPOSITION 9. Let $L$ be a path with endpoints $a$ and $b$ in $E^{n+1} - M$ directed from $a$ to $b$, such that $L \setminus M$ consists of a single point $p \in \Sigma^*_a$, and so does $\varphi(L)$. Let $L' = \varphi(L) - b$ and such that $L'$ is transverse to $M$ at $p$. Then $\varphi(L)$ represents $e(p)$ where $e(p)$ is defined as follows. Let $p = \varphi(L) \cap M$, then $e(p)$ is $0$ if $h(p) > h(q)$; $e(p) = 1$ if $h(p) = h(q)$ and if $a^* \neq b^*$ is in the same component, $X$, of $E^{n+1} - M$ as $a^*$; $e(p) = -1$ otherwise (i.e., $h(p) < h(q)$ and $b^* \neq X$).

Proof. If $e(p) = 0$, then $L' \setminus M = \emptyset$ and $\varphi(L)$ is nullhomotopic in $E^{n+1} - M$, the nullhomotopy carried by $L' \cup \cup_{q \in \partial L}(L_q)$, where $\cup_{q \in \partial L}(L_q)$ denotes the cone on $Y$ from $x$, that is, this is set is the image of a map of $I \times X$ into $E^{n+1} - M$ which gives the nullhomotopy.

Let $Y$ be the component of $E^{n+1} - M$ which contains $b$. If $e(p) = 1$, let $Q$ be a path in $X$ from $a^*$ to $a^*_0$, $R$ a path in $Y$ from $b^*_0$ to $b^*$. Let $b$ be the dihedral, perhaps singular, in $E^{n+1}$ whose boundary is the path $Q \ast L^* \ast X$. 

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Consider the projection of \( D^* \) onto \( X \). We have \( p_i^* = g^{-1}(08 \cap P_i) \) for each \( i \), and similarly for \( p_i^- \).

**Theorem 10.** \( \Pi_*([\mathbf{R}^{n+2} - M]) \) has the following presentation. There is one generator, \( \sigma_i \), for each component, \( \Sigma^2 \), of \( M^* - D^* \). For each component, \( \gamma \) of \( D^* \), there are two relations as follows: choose an \( \alpha_i \in \gamma \):

1. \( \sigma_i = \sigma_i \) if \( \Sigma_i \cap \Sigma_j \) is the outer surface at \( x_i \),
2. \( \Pi_*(\sigma_{i_1}^n \sigma_{i_2}^m) = \Pi_*(\sigma_{i_2}^m \sigma_{i_1}^n) = 1 \),

where \( \epsilon(\sigma_i) \) is defined as in Proposition 9 via the path \( (r_{i-1}, r_i) \) for \( i \neq 1 \), \( \epsilon(p_i^+) \) determined via \( (r_i^+, r_i^-) \).

**Proof.** We first remark that we have shown that relations of type I do not depend on the choice of \( \alpha_i \); similarly one can show that relation II does not depend on the choice of \( \alpha_i \). We also remark that we may describe relation II as obtained as follows: let \( S^* \) be a small oriented simple closed curve about \( \gamma \) which meets \( \Sigma_i, \Sigma_j, \Sigma_k, \Sigma_l \) in one point, as transversely, and let \( S^* - M^* \), then consider \( S^* \) as a directed path from \( x_i \) back to \( x_i \), relation II is obtained by setting \( \omega(\sigma_i) = 1 \) where \( \omega(\sigma^*(S^*)) \) is defined below. We can see that this relation does not depend on the choice of \( S^* \) or the orientation of \( S^* \).

Let \( F \) denote the free group on the symbols \( \{a_i\} \), let \( B \) be the smallest normal subgroup of \( F \) containing the words \( a_i a_i^{-1} \) and \( a_i^m b_i^m a_i^{-m} b_i^{-m} \), we shall show that \( \Pi_*([\mathbf{R}^{n+2} - M]) \) is isomorphic to the quotient group \( F/R \) by defining a homomorphism \( j: F \rightarrow \Pi_*([\mathbf{R}^{n+2} - M]) \) which is onto, such that the kernel of \( f \) is \( B \). The map which sends each generator, \( a_i \) of \( F \) to the element representing the loop \( \sigma_i \) in \( \Pi_*([\mathbf{R}^{n+2} - M]) \) extends to a unique homomorphism of \( F \) to \( \Pi_*([\mathbf{R}^{n+2} - M]) \), this will be our map \( j \). If \( a \in F \) and we represent \( a \) by the word \( w = a_1^m a_2^m \cdots a_r^m \), then \( f(a) \) is represented by the loop \( a_1^m a_2^m \cdots a_r^m \), conversely we note that if \( a = a_1^m a_2^m \cdots a_r^m \) represents an element of \( \Pi_*([\mathbf{R}^{n+2} - M]) \) and \( f(a) = a \) then \( a \) is represented by the word \( a_1^m a_2^m \cdots a_r^m \). We will let this word be denoted by \( \omega(a) \). Next we show that \( f \) is onto. If \( a \in \Pi_*([\mathbf{R}^{n+2} - M]) \) we may represent it by a directed polygonal path, \( L \), in \( \mathbf{R}^{n+2} - M \) with the following properties:

1. \( L \) contains no points of \( D^* \) (this can be done since \( D^* \) has co-dimension 2 in \( \mathbf{R}^{n+2} \)),
2. \( L \) intersects \( M^* - D^* \) transversely in a finite number of points \( p_1^*, \ldots, p_n^* \), where \( p_1^* < p_2^* < \ldots < p_n^* \). Let \( c_i \) be a point of \( L \) such that \( c_i = \sigma_i^m b_i^m a_i^{-m} b_i^{-m} \). We define \( \epsilon(p_i^*) \) via the path \( (c_{i-1}, c_i) \) as in Proposition 9, and then by that proposition we will have \( L \) homotopic to the loop \( \omega(c_i, c_{i+1}) \).
and thus represented by the word $\sigma_1^{p_1}\sigma_2^{p_2}\ldots\sigma_{n-1}^{p_n}$, where $p_i \equiv \Sigma_i$, we will denote this word by $w([F])$.

We next will show that the kernel of $f$ is $\mathbb{R}$. Suppose that $\alpha \in F$ with $f(\alpha) = 1$; then the loop $f(\alpha)$ is nullhomotopic in $\mathbb{R}^{n+1}_x$. Let $g : D^2 \to \mathbb{R}^{n+1}_x$ give this homotopy as follows: let $x_0$ be the point of $\partial \mathbb{R}$ corresponding to $(1,0)$, then if we consider $\partial \mathbb{R}$ to be the directed path from $x_0$ to $x_0$ counterclockwise about $\partial \mathbb{R}$, then $g(\partial \mathbb{R})$ represents $f(\alpha)$. Let $B = g(D^2)$.

By choosing $g$ appropriately, we may assume that

1) $g$ is piecewise-linear,

2) $B \cap \mathbb{Z} = \emptyset$ (since $\mathbb{Z}$ has codimension 3 in $\mathbb{R}^{n+1}$),

3) $B \cap \mathbb{Z} = \emptyset$ (since $\mathbb{Z}$ has codimension 3 in $\mathbb{R}^{n+1}$),

4) $B$ intersects $\Sigma_i$ transversely in a finite number of points $q_1^i, \ldots, q_i^i$, and $B\cap \Sigma_i$ intersects each $\Sigma_i$ transversely.

(In the case $n = 1$, we will need to use the following in place of (4): if $\alpha$ is a 2-simplex of $B$ then $\alpha \cap \partial \mathbb{R} \cap \mathbb{Z}$ meets each $\Sigma_i$ transversely.) Such a map, $g$, will be called a nullhomotopy in general position. If $X \subseteq \mathbb{R}$, let $X^* = g(X)$. If $x$ and $y$ are points of $\partial \mathbb{R}$ then $[x^*, y^*]$ will denote the directed subpath of $g([x, y])$ from $x^*$ to $y^*$.

Let $J = (f + g)^{-1}(M')$, $K = (f + g)^{-1}(M'')$. $K$ is a collection of points in $\partial \mathbb{R}$, and $J$ is the union of proper arcs in $\partial \mathbb{R}$ and simple closed curves in $\partial \mathbb{R}$ whose intersections are transverse and constitute $K$ (an arc $a = \mathbb{R}$ is proper if $a \cap \partial \mathbb{R}$ corresponds to the endpoints of $a$).

Proposition 11. If $g : D^2 \to \mathbb{R}^{n+1}_x$ is a nullhomotopy in general position with $K = \emptyset$, and $x_i$ is any point in $\partial \mathbb{R}$, $x_i \neq x_0$, then $w([x_i, x_0]) = w([x_i, x_0])^{-1}$.

Proof. We will use induction on the number of components of $J$. If $J$ has one component then $J = (Ig)^{-1}(\mathbb{R}^{n+1}_x)$ for some $k$. If $J \cap \partial \mathbb{R} = \emptyset$ (i.e., if $J$ is a simple closed curve in the interior of $\mathbb{R}$), then for any $x_i$, $w([x_i, x_0]) = w([x_i, x_0])^{-1} = 0$. If $J \cap \partial \mathbb{R} = \emptyset$, then $J \cap \partial \mathbb{R}$ consists of two points $p$ and $q$ with $s = r$, let $G$ be a point of $\partial \mathbb{R}$. Then $B = \mathbb{R}^{n+1}_x$ such that $p < q$. Now by Proposition 9, $g([x_i, p]) \in \mathbb{Z}^{n+1}_x$. We will show that $\mathbb{L}$ if $s([p]) = 0$, then $e(s([p])) = 1$; $= 0$; $s([p]) = s([p])$. Let $p' = (s([p])) \cap \mathbb{M}/\mathbb{M''}$. Let $e([p]) = 0$, then $h([p]) = h([p'])$ by connectivity of $J'$, we may argue that therefore all points of $J'$ lie above the corresponding points of $\mathbb{M}$, in particular, $h([p']) = h([p'])$ and thus $e([p]) = 0$. To consider case 1, suppose that $s([p]) = 1$. Then $h([p']) = h([p'])$ and $e([p]) = 1$ is similar. Now by checking the three cases $s_i < s_i < p_1$, $p < s_i < r$, $r < s_i < x_i$, one may verify our proposition. For example, in the first case, $e([x_i, x_0]) = 1$, then $e([x_i, x_0]) = e([x_i, x_0])^{-1}$.

From the above proposition, we see that if $K = \emptyset$, then the word $w = w(f(\alpha))$ is equivalent to $1 \in F$ and therefore $\alpha = 1 \in B$. Suppose that $K$ consists of one point, say $d$. Let $B$ be a small regular neighborhood of $d$. Let $x_i$ be a point of $\partial B - J$ and let $\beta$ be a proper arc in $\partial \mathbb{R}$ from $x_i$ to $x_i$ which meets $J$ transversely; let $\beta$ be an arc parallel to $\beta$, with the same properties from $y_j \in \partial \mathbb{R}$ to $y_j \in \partial \mathbb{R}$ such that $J \cap \partial \mathbb{R} \subseteq \mathbb{R}$ (say, $y_j$). Orient $\partial B$ counterclockwise and let $\sigma$ be the directed path from $y_j$ to $x_i$, $\tau$ the directed path from $x_i$ to $y_j$, see Fig. 2b). Let $w_1 = w([x_i, y_j]), w_2 = w([y_j, y_j]), w_3 = w([y_j, x_i]), w_4 = w([\sigma, \tau])$, and $w_5 = w([x_i, y_j])$. Clearly, it can be arranged that $w_1 = 1$ and $w_1 = 1$. By Proposition 11, considering the disk bounded by the path $[y_j, y_j]$, $\mathbb{R} \cap \mathbb{R} = \mathbb{R}$, we see that $w_5 = w_5$. Considering the disk bounded by the path $[x_i, y_j]$ and $\mathbb{R} \cap \mathbb{R} = \mathbb{R}$, we see that $w_5 = w_5$. By the remark following the statement of our theorem, with $S$ corresponding to $([\partial B])^*$, we see that $w_5 = w_5$. Now $w(f(a)) = w_5 = w_5 = 1 = w_5 = w_5$. Therefore $w(f(a)) \in \mathbb{R}$.

Now suppose that $K$ consists of $w + 1$ points. Let $\beta$ be a proper arc
in $\mathcal{D}\beta$ from $\varepsilon_1$ to, say, $\varepsilon_2$ such that each component of $\mathcal{D}\beta - \beta$ contains some points of $K$. Then

$$\omega(f(\alpha)) = \omega)[[\varepsilon_1', \varepsilon_2']] \cdot \beta^{-1} \cdot \beta \cdot \omega[[\varepsilon_2', \varepsilon_1']]$$

is the product of words in $R$. This completes the proof of the theorem.

As an example of this theorem, consider the crosscap; this is the projection, in general position, of an embedding of the projective plane in $\mathbb{R}^4$. To find its knot group we note that there is only one region, $\Sigma_1$, and only one curve of double points. Thus we have one generator, $\sigma_1$, and two relations: the first is $\sigma_1 = \sigma_2^{-1}$, the second is $\sigma_1 \sigma_2^{-1} \sigma_3 \sigma_2^{-1} = 1$; thus the knot group is isomorphic to $Z_2$. Similarly one may consider the embedding of the Klein bottle in $\mathbb{R}^4$ whose projection in $\mathbb{R}^2$ is a surface with a single circle of self-intersection and find the knot group of this embedding to be isomorphic to $Z_2$.

References


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Two notes on abstract model theory

II. Languages for which the set of valid sentences is semi-invariantly implicitly definable

by

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Abstract. It is shown that if $L$ is an abstract model-theoretic language, the syntax of $L$ is represented in a structure $\mathcal{M} = (\mathcal{A}, \ldots)$, the Löwenheim-Skolem property holds down to card ($\mathcal{A}$) and $\mathcal{M}$ is uniformly i.i.d. in $L$ then the set of $L$-valid consequences of a set $S$ of sentences is i.i.d. in $L$ whenever $S$ itself is so definable. This generalizes a theorem of Kunen for admissible fragments of $L_{\omega_1\omega}$. The final part of the paper relates this to a program of study of good properties of model-theoretic languages.

Introduction. The aim of this note is quite different from that of the preceding [F2], though it makes use of the same general preliminaries. In content, it is a sequel to [F1], § 3 where some connections were studied, for arbitrary languages $L$, between implicit definability of the satisfaction relation $\models_2$ and logical properties of $L$. The basic relevant notions of [F1] are recalled below, in particular that of the syntax of $L$ being represented in a structure $\mathcal{M} = (\mathcal{A}, \ldots)$ and (relative to any such representation) that of $\models_2$ being uniformly invariently implicitly definable (i.i.d) in $L$. We add here a related notion of a subset $S$ of $\mathcal{A}$ being semi-invariantly implicitly definable (siid) in $L$ (1). This includes Kunen's definition in [Ku] of siid for admissible structures $\mathcal{M} = (\mathcal{A}; c, R_1, \ldots, R_k)$ as a special case, and in the same line extends model-theoretic generalizations of recursion theory. Kunen showed that being siid is equivalent to other proposed

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(1) To be more precise, several variants of the notions siid are introduced and compared in § 2. In accordance with one of these, siid is rewritten $\equiv \cdot \text{siid}_*$.