On the rim-types of hereditarily locally connected continua

by

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Abstract. To every rational curve it is possible to assign a countable ordinal number called the rim-type of the curve. In this paper it is shown that for each ordinal number $\alpha$ that is at most countable there exists a hereditarily locally connected continuum of rim-type $\alpha$.

It is known (see Kuratowski [1], p. 290) that the rim-type of a rational curve is an ordinal number that is strictly smaller than the first uncountable ordinal $\Omega$. A continuum is regular if and only if it is of rim-type 1. Hereditarily locally connected continua are rational (see [3], p. 94) and regular continua are hereditarily locally connected. A. Lelek asked in a letter if the rim-type of every hereditarily locally connected continuum is at most 2. It is our purpose to prove that for each ordinal $\alpha$ such that $1 \leq \alpha < \Omega$ there exists a planar hereditarily locally connected continuum of rim-type $\alpha$.

Our notation follows Whyburn [3]. A continuum is a compact, connected metric space. A continuum is said to be hereditarily locally connected if each of its subcontinua is locally connected.

If $A$ is a subset of a space $X$ we let $A'$ denote the derived set of $A$. We let $A^{(0)} = A$. If $\alpha$ is the successor of the ordinal $\nu$ we let $A^{(\nu)} = (A^{(\nu)})'$. If $\alpha$ is a limit ordinal we let $A^{(\alpha)} = \bigcap \{A^{(\nu)} \mid \nu < \alpha\}$.

Let $X$ be a rational continuum. Let $\alpha$ be the smallest ordinal number such that for each $x \in X$ and for each neighbourhood $U$ of $x$ there is a neighbourhood $V$ of $x$ such that $V \subseteq U$ and $(\partial V)^{\alpha} = \emptyset$ where $\partial V$ denotes the boundary of $V$. The rim-type of $X$ is defined to be $\alpha$.

We shall need the following lemma:

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LEMMA. If $Y_1, Y_2, \ldots$ is a sequence of pairwise disjoint hereditarily locally connected continua in $E^3$ whose diameters converge to 0 and if $A$ is an arc which meets each $Y_i$ then $X = A \cup Y_1 \cup Y_2 \cup \ldots$ is a hereditarily locally connected continuum.

Proof. $X$ is connected since $A, Y_1, Y_2, \ldots$ are connected sets and each $Y_i$ meets $A$. $X$ is compact since $A, Y_1, Y_2, \ldots$ are compact sets and the sequence $Y_i$ is eventually in every neighborhood of $A$. To prove that $X$ is hereditarily locally connected it suffices to prove (by [2], p. 89) that $X$ contains no continuum of convergence.

Let $\varepsilon > 0$ be given. Suppose $B_1, B_2, \ldots$ is a sequence of pairwise disjoint continua of $X$ each of which has diameter at least $\varepsilon$. We may suppose that the sequence $B_1, B_2, \ldots$ converges to a continuum $K$ in the space of closed subsets of $X$. One of the following two situations occurs. Either for each $\delta > 0$ the sequence $B_1, B_2, \ldots$ is eventually in the $\delta$-neighborhood of $A$ or there is a natural number $n$ and $\varepsilon_1 > 0$ such that infinitely many of the sets $B_i$ meet $Y_n$ in a connected set of diameter at least $\varepsilon_1$. In the second case we may suppose without loss of generality that each $B_i$ meets $Y_n$ in a connected set of diameter at least $\varepsilon_1$. In this case it follows from the fact that $Y_n$ is hereditarily locally connected that almost every $B_i$ meets $K$. If the second case fails to hold then it is easy to see that $X$ is a line segment in $A$ and almost every $B_i$ meets $K$. This completes the proof that $X$ does not contain a continuum of convergence.

THEOREM. If $\alpha$ is an ordinal number such that $1 \leq \alpha < \Omega$ then there exists a planar hereditarily locally connected continuum of rim-type $\alpha$.

Proof. The proof is by transfinite induction. Let $X_0 = [0, 1] \times \{0\}$. Then $X_0$ is of rim-type 1. Let $\alpha$ be an ordinal number such that $1 < \alpha < \Omega$. Suppose that for each ordinal number $\beta$ such that $1 < \beta < \alpha$ $X_0$ is a planar hereditarily locally connected continuum of rim-type $\beta$. Suppose also that for each $\beta$ such that $1 < \beta < \alpha$ and $\beta$ is the successor of an ordinal $\gamma$ the following hold:

(i) $[0, 1] \times \{0\} \subset X_0 \subset [0, 1] \times [-1, 1],$
(ii) $A_\alpha = X_0 \cap ([0, 1] \times [-1, 1])$ and $B_\alpha = X_\alpha \cap ([0, 1] \times [-1, 1])$ are countable sets,
(iii) for each $A \subset X_\alpha$ such that $A^{(\omega)} = \emptyset$ there exists an arc in $X_\alpha \setminus A$ with one endpoint in $A_\alpha$ and the other in $B_\alpha$.

We consider three cases.

Case 1. $\alpha$ is a limit ordinal. Let $a_1, a_2, \ldots$ be a strictly increasing sequence of ordinals which converges to $\alpha$. For each $i$ let $Z_i$ be a hereditarily locally connected plane continuum of rim-type $a_i$ and diameter less than 1/1 such that for each $i \neq j$ $Z_i \cap Z_j = (\{0\}, \{0\})$. Let $Z = Z_1 \cup Z_2 \cup \ldots$. Then $Z$ is easily seen to be hereditarily locally connected continuum of rim-type $\alpha$.
Let \( X_n = X_{n-1} \cup X_1 \cup \ldots \). By the lemma \( X_n \) is a hereditarily locally connected continuum.

We check that the rim-type of \( X_n \) is no greater than \( n \). If \( x \in X_n \setminus X_n \) then \( x \) has a set homeomorphic to \( X_n \) for a neighbourhood. Since the rim-type of \( X_n \) is \( n \) there exist arbitrarily small neighbourhoods \( V \) of \( x \) in \( X_n \) such that \( \partial V \not\subseteq \emptyset \). If \( x \in X_n \) then it follows from the construction that there exist arbitrarily small neighbourhoods \( V \) of \( x \) such that \( \partial V \cap X_n \) contains at most two points and for each \( j \in \{0, 1, \ldots, n\} \) \( \partial V_j \cap X_n \) is open in \( \partial V \) for each \( j \in \{0, 1, \ldots, n\} \) \( \partial V \cap X_n \) is finite \( \partial V \not\subseteq \emptyset \).

Next we show that the rim-type of \( X_n \) is at least \( n \). Let \( A \) be a set in \( X_n \) which separates \( (x, 0) \) and \( (y, 0) \) in \( X_n \), where \( x, y \in [0, 1] \). Since \( X_n \) is completely normal we may assume without loss of generality that \( A \) is a closed set. Just suppose that \( A^{(n)} = \emptyset \).

There exist at most finitely many \( j \in \{1, 2, \ldots\} \) and \( i \in \{1, 2, \ldots, m_j\} \) such that \( f_{j, i}(X_n) \subset A^{(n)} \) for each ordinal \( m < n \) since the sets \( f_{j, i}(X_n) \) are pairwise disjoint and their diameters converge to zero. We may suppose without loss of generality, therefore, that for each \( j = 1, 2, \ldots \) and for each \( i = 1, 2, \ldots, m_j \) there is an ordinal \( m < n \) such that \( f_{j, i}(X_n) \cap A^{(n)} = \emptyset \). Thus by condition (iii) on the continuum \( X_n \) for each \( j, i \) there exists an arc \( K_{ji} \) in \( f_{j, i}(X_n) \) with one endpoint \( A \cap K_{ji} \) in \( f_{j, i}(A_n) \) and the other in \( f_{j, i}(B_n) \).

Let \( Y = X_n \cup \{K_{ji} \mid j \in \{1, 2, \ldots\} \text{ and } i \in \{1, 2, \ldots, m_j\}\} \). It is not difficult to see that \( Y \) is a continuum and that the set of local cutpoints of \( Y \) is contained in \( \cup K_{ji} \).

Now, \( Y \cap A \subset Y \setminus \cup K_{ji} \) and \( Y \cap A \) separates \( (x, 0) \) and \( (y, 0) \) in \( Y \). By a result in [3], p. 62 \( Y \cap A \) contains a perfect set since \( Y \cap A \) does not contain a local cutpoint of \( Y \). This contradicts the assumption that \( A^{(n)} = \emptyset \). Thus, if \( A \) is any set in \( X_n \) that separates \( (x, 0) \) and \( (y, 0) \), then \( A^{(n)} \not\subseteq \emptyset \). We have proved that the rim-type of \( X_n \) is \( n \).

Clearly, \( X_n \cap \{(1) \times [-1, 1]\} \) is countable for \( i = 0, 1 \).

Finally, let \( A \) be a set in \( X_n \) such that \( A^{(n)} = \emptyset \). We must prove that there is an arc in \( X_n \setminus A \) stretching from \( (0) \times [-1, 1] \) to \( (1) \times [-1, 1] \). By the above \( X_n \setminus A \) is contained in one arc component of \( X_n \setminus A \). Since the sequence of continua \( f_{j, i}(X_n) \), \( j = 1, 2, \ldots \), converges to a point and \( A^{(n)} = \emptyset \) it follows that there is a natural number \( n \) and an ordinal \( m < n \) such that \( A \cap f_{j, i}(X_n) = \emptyset \). By (iii) there exists an arc from \( f_{j, i}(A_n) \) to \( f_{j, i}(B_n) \) in \( f_{j, i}(X_n) \setminus A \). Similarly there is a natural number \( k \) and an arc from \( f_{j, i}(A_n) \) to \( f_{j, i}(B_n) \) in \( f_{j, i}(X_n) \setminus A \). Thus, there is an arc in \( X_n \setminus A \) stretching from \( (0) \times [-1, 1] \) to \( (1) \times [-1, 1] \).

Case 3. \( n \) is the successor of the limit ordinal \( \alpha \). Let \( \alpha, \alpha_2, \ldots \) be a strictly increasing sequence of ordinal numbers which converges to \( n \) such that each \( \alpha \) is not a limit ordinal. Take everything to be as in Case 2 except...