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Groups in the category of f-manifolds

by

Richard S. Millman (Carbondale, Ill.)

Abstract. A structure on a n-dimensional differentiable manifold given by a tensor field of type (1,1) and constant rank r which satisfies $f^3+f=0$ is called an *f-structure*. An *f-map* is a map between f-manifolds whose differential commutes with the f-structure. An *f-Lie group* is a group in the category of f-manifolds and f-maps.

THEOREM A. *Every f-Lie group is the quotient of the product of a complex Lie group and a Lie group with trivial f-structure. An f-Lie group is an f-contact Lie group if the kernel f (as a sub-bundle of the tangent bundle) is parallelizable by commuting vector fields.*

THEOREM B. *A compact f-contact Lie group is isomorphic (as a Lie group) to a torus.*

1. A structure on an n-dimensional differentiable manifold given by a tensor field f of type $(1,1)$ and constant rank r which satisfies f^3+ $+f=0$ is called an *f-structure*. This notion has been studied by Yano and Ishihara (among others) [4]. An f-structure is *integrable* if about each point there is a coordinate system in which f has the constant components

$$f=\begin{bmatrix}0 & -I_p & 0\\ I_p & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \tag{1}$$

where I_p is the $(p\times p)$ identity matrix ($p=\frac{1}{2}r$). In [1] it is shown that the integrability of f is equivalent to the vanishing of the Nijenhuis tensor of f,

$$N(X,Y)=[fX,fY]-f[fX,Y]-f[X,fY]+f^2[X,Y]$$

where X and Y are vector fields on M. We shall write $\chi(M)$ for the set of all vector fields on M, $T_m(M)$ for the tangent space of M at $m\in M$ and $T(M)$ for the tangent bundle of M. For $m\in M$, let

$$(\ker f)_m=\{X\in T_mM|\ f_m(X)=0\}$$

and

$$(\operatorname{im} f)_m=\{X\in T_mM|\ X=f_mY \text{ for some } Y\in T_mM\}.$$

If $(\ker f)_m = 0$ for all $m \in M$ then f is an almost complex structure. If $(\operatorname{im} f)_m = 0$ for all $m \in M$ then f is the trivial f-structure, $f \equiv 0$.

Suppose that M_i is an f-manifold with f-structure f_i $(i = 1, 2)$ and $\varphi\colon M_1 \to M_2$, then φ is an *f-map* if $f_2\varphi_*(X) = \varphi_* f_1(X)$ for all $X \in T_m M_1$, $m \in M_1$. Let G be a Lie group with f-structure. If both $L_g\colon G \to G$ (left multiplication by $g \in G$) and $R_g\colon G \to G$ (right multiplication) are f-maps and f is integrable then G is an *f-Lie group*. This is clearly the appropriate notion of group in the category of f-manifolds. f-Lie groups have been used in, for example, generalizing Weil's approach to the classical Cousin problem of several complex variables [2]. We will prove:

THEOREM A. *Every f-Lie group is the quotient of the product of a complex Lie group and a Lie group with trivial f-structure by a discrete subgroup.*

We will also give an example of an f-Lie group which is not the product of a complex Lie group and a Lie group with trivial f-structure. We say that the f-Lie group, G, is a *f-contact Lie group* if there are $\xi_1, \ldots, \xi_{n-r} \in (\ker f)_e$ which are linearly independent and $[\xi_i, \xi_j] = 0$ for all $1 \leqslant i, j \leqslant n-r$. We also prove:

THEOREM B. *A compact f-contact Lie group is isomorphic (as a Lie group) to a torus.*

2. Let $\hat{G}$ be the Lie algebra of G and $g \in G$, $X \in \hat{G}$. As usual we define $\operatorname{ad} g\colon G \to G$ by $\operatorname{ad} g(x) = g x g^{-1}$ and $\operatorname{Ad} X\colon \hat{G} \to \hat{G}$ by $\operatorname{Ad} X(Y) = [X, Y]$. An f-structure is *bi-invariant* if both left and right multiplication are f-maps.

PROPOSITION 1. *If f is a bi-invariant f-structure on a Lie group, then $f[X, Y] = [f(X), Y]$ for all $X, Y \in \hat{G}$.*

Proof. Since $f(L_g)_* = (L_g)_* f$ and $f(R_g)_* = (R_g)_* f$ we have $f(\operatorname{ad} g)_* = (\operatorname{ad} g)_* f$ for all $g \in G$. If $g = \exp tX$ where $t \in R$ then $f(\operatorname{ad}\exp tX(Y)) = \operatorname{ad}\exp tX f(Y)$ hence by a standard result in Lie groups:

$$f\big(e^{\operatorname{Ad} tX}(Y)\big) = e^{\operatorname{Ad} tX} f(Y)$$

or

$$f\Big(Y + t[X, Y] + \frac{t^2}{2!}[X, [X, Y]] + \ldots\Big)$$
$$= f(Y) + t[X, f(Y)] + \frac{t^2}{2!}[X, [X, f(Y)]] + \ldots$$

hence

$$(1) \quad f[X, Y] + \frac{t}{2!} f([X, [X, Y]]) + \ldots = [X, f(Y)] + \frac{t}{2!}[X, [X, f(Y)]] + \ldots$$

Letting $t \to 0$ in (1) gives us the desired result. ■

The proof of the following corollary is immediate since from Proposition 1 the Nijenhuis torsion of a bi-invariant f-structure must vanish at e.

COROLLARY. *A bi-invariant f-structure on a Lie group is integrable.*

We now prove Theorem A. Let $L_k = (\ker f)_e$ and $L_i = (\operatorname{im} f)_e$. It is clear from Proposition 1 that both L_k and L_i are Lie subalgebras of $\hat{G}$. Now if $X = f(Z) \in L_i \cap L_k$ then $f^2(Z) = 0$ hence since $f(Z) + f^3(Z) = 0$, $X = f(Z) = 0$ and so $L_i \cap L_k = (0)$. By dimensions $\hat{G}$ is therefore the direct sum (as a vector space) of L_i and L_k. Furthermore if $X = f(Z) \in L_i$ and $Y \in L_k$ then again applying Proposition 1,

$$[X, Y] = f[Z, Y] = [Z, f(Y)] = 0 .$$

Thus $\hat{G} = L_i \oplus L_k$ as Lie algebras and by standard results of Lie theory we have Theorem A. ■

3. Before proving Theorem B we need to recall some results of [3]. The *kernel of f*, $\ker f$, is $\bigcup_m (\ker f)_m$ and the *image of f*, $\operatorname{im} f$, is $\bigcup_m (\operatorname{im} f)_m$. An f-manifold is *k-framed* if there are $\xi_1, \ldots, \xi_{n-r} \in \chi(M)$ such that $\{\xi_1(m), \ldots, \xi_{n-r}(m)\}$ forms a basis for $(\ker f)_m$ for all $m \in M$. We write $n_0 = n - r$. If M_1 and M_2 are k-framed f-manifolds then we define an almost complex structure J on $M_1 \times M_2$. We shall denote the k-framing on M_i by $\{\xi_1^i, \ldots, \xi_{n_0}^i\}$ and the f-structure on M_i by f_i. If in addition $[\xi_k^i, \xi_l^i] = 0$ for all $1 \leqslant k, l \leqslant n_0$ then M_i is called an *f-contact manifold*. The concept of f-contact manifold generalizes the basic features of almost contact structure to f-manifold of higher nullity (i.e. lower rank). In [3, Lemma 2] we have associated to the framing $\{\xi_1^i, \ldots, \xi_{n_0}^i\}$ differential forms η_j^i for $i = 1, 2$, $j = 1, \ldots, n_0$. We define the almost complex structure J on $M_1 \times M_2$ as follows: if $X_1 \in T_p M_1$, $X_2 \in T_q M_2$ where $p \in M_1$, $q \in M_2$ then

$$J_{p,q}(X_1, X_2) = \Big(f_1(X_1) - \sum \eta_i^2(X_2)\,\xi_i^1(p),\, f_2(X_2) + \sum \eta_i^1(X_1)\,\xi_i^2(q)\Big) .$$

We also proved the following theorem in [3].

THEOREM. *Let M_1 and M_2 be two k-framed f-manifolds of the same rank. If f_1 and f_2 are integrable then the almost complex structure J is integrable then the almost complex structure J is integrable if and only if both M_1 and M_2 are f-contact manifolds.*

To prove Theorem B we note that if G is an f-contact Lie group then $G \times G$ is a complex Lie group. (This is essentially showing that the η_j are bi-invariant which follows immediately from the bi-invariance of f). Hence if G is compact then $G \times G$ is a compact complex Lie group, hence abelian and the result follows. ■

Theorem B is proven in the special case that f defines a structure of an almost contact manifold in [2].

If we let $G = C \times R$ where C is the complex line (considered as a complex manifold) and R is a Lie group with trivial f-structure and $D = \{(n+in, n) \mid n$ is an integer$\}$ then G/D is an f-Lie group which is not the product of a complex Lie group and an f-Lie group with trivial f-structure. (G/D is of course diffeomorphic to $C \times S^1$ but the f-structure on G/D is not the product f-structure of $C \times S^1$). This is the example mentioned in the introduction.

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Reducing hyperarithmetic sequences

by

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Abstract. Every a'-sequence is isomorphic to an a^*-sequence. This implies: Every a'-theory T with an a-language has an a^*-model. If T has an infinite normal-model then T has an normal a^*-model.

§ 1. Introduction. If you analyse a mathematical construction to evaluate its complexity e.g. in terms of the hyperarithmetic hierarchy, it is not difficult to get a'-bounds ($a \in O$, O Kleene's system of ordinal notations, $a' = 2^a$), for you can employ recursive processes to describe the construction. If you try to get a^*-bounds (a predicate is a^*-bounded if it is a Boolean combination of $\Sigma_1^0(a)$-predicates) you must analyse some tricky constructions often related to wait and see methods.

In this paper we prove a theorem on hyperarithmetic sequences by which in some cases we can avoid this analysis and get an a^*-bound by means of a'-bound. In § 5 examples regarding models and structures will be discussed.

A model is called *normal* if its universe is the set of natural numbers and the first predicate is the identity. In [3] Hensel and Putnam have shown that every axiomatized consistent theory based on a finite number of predicates which has an infinite model with "=" interpreted as identity, has a normal model in $B^*(1)$, i.e. all predicates are 1^*-bounded. Among its consequences the theorem has an analogue to the Hensel-Putnam result for arbitrary hyperarithmetic theories with a recursive language. We can drop the assumption that the theory must be based on a finite number of predicates, and different to Putnam [5] and Hensel-Putnam [3] the result yields a method which solves Mostowski's problem [4, p. 39] simultaneously for theories with and without identity.

§ 2. The hyperarithmetic hierarchy. Let O be Kleene's system of ordinal notations with the ordering $<_0$, $a' = 2^a$ the successor of a in O, A' the recursive jump of A; we write $A \leqslant B$ if A is recursive in B. $H_1 := \emptyset$, $H_{a'} := H'_a$ for a in O, $H_{3\cdot 5^a} := \{\langle x, y\rangle : y <_0 3\cdot 5^a \;\&\; x \in H_y\}$, where $3\cdot 5^a$ is a notation of a limit ordinal.