[8] S. Godlewski, *Mutational retracts and extensions of mutations*, Fund. Math. 84 (1974), pp. 47–65.
[9] — and S. Nowak, *On two notions of shape*, Bull. Acad. Polon. Sci. 20 (1972), 20 (1972), pp. 387–393.
[10] H. Patkowska, *A homotopy extension theorem for fundamental sequences*, Fund. Math. 64 (1969), pp. 87–89.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
INSTYTUT MATEMATYCZNY PAN

Accepté par la Rédaction le 21. 8. 1973

Two theorems of functional analysis effectively equivalent to choice axioms

by

D. A. Edwards (Oxford)

Abstract. The first main result, when combined with a recent theorem of Gardiner, shows that the Boolean prime ideal theorem is equivalent, as an axiom of set theory, to the existence, for a certain class of vector lattices, of an extreme positive linear functional. The second main result provides an equivalence in the same spirit for the axiom of choice.

1. Introduction. It is well known that certain theorems of functional analysis are effectively equivalent to certain choice axioms. For example, Luxemburg [14] has shown that Alaoglu's theorem is equivalent to the Boolean prime ideal theorem PI (*viz* that every Boolean algebra has a prime ideal). Bell and Fremlin [3] have shown that the axiom of choice, AC, is equivalent to the statement that the unit ball of the Banach dual of a normed vector space always has an extreme point. The object of the present paper is to present two new results (Theorems A and B of § 2) in this spirit.

Theorem A provides, when taken together with the subsequent work of Gardiner [8], a new functional analytic equivalent of PI. Luxemburg [13] proved that PI implies the Hahn–Banach theorem, and Pincus [15] has shown that the converse is false. But Bell and Jellett [4] have shown that the Hahn–Banach and Krein–Milman theorems together imply PI. Theorem A somewhat resembles the Bell–Jellett theorem, but by restricting the statement suitably to vector lattices we have been able to achieve a strict equivalence to PI; the proof of our Theorem A is also more direct than Bell and Jellett's argument.

Halpern [9] has shown that PI is strictly weaker than AC. Theorem B provides a functional analytic statement equivalent to AC. It is not perhaps so simple as Bell and Fremlin's result mentioned above, but it has some independent interest: it is related both to Theorem A and to a theorem of Klimovsky, Bell and Fremlin [11, 2] which formulates AC in terms of maximal ideals in lattices of sets.

The axioms of set theory assumed in the present work may be taken to be those of Zermelo–Fraenkel theory (see [5, 17]) together with which ever of the two axioms stated in § 2 is supposed to be in force.

A summary of an earlier version of this paper was published in [7].

I am most grateful to Dr. Paul Bacsich for his valuable suggestions for this paper.

2. Axioms and main results. In the axioms and theorems which we are about to state Z will denote a real vector lattice with a distinguished order unit (denoted by 1), and $P(Z)$ will denote the (obviously convex) set of all positive linear functionals u on Z such that $u(1) = 1$. (For the terminology see, for example, [10]).

Axiom HBKML. *The convex set $P(Z)$ has at least one extreme point.*

In classical (i.e. with Zorn's lemma) functional analysis this is a theorem, a straightforward consequence of the Hahn–Banach and Krein–Milman theorems.

For each subset E of Z we define the *quasi-order* $\lesssim$ *on* $P(Z)$ *associated with* E as follows: if $u_1, u_2 \in P(Z)$ then $u_1 \lesssim u_2$ whenever $u_1(z) \leqslant u_2(z)$ for all $z \in E$. Our second axiom is

Axiom BL. *Let E be a sublattice of Z. Then there exists an extreme point of the convex set $P(Z)$ which is minimal in $P(Z)$ for the quasi-order on $P(Z)$ associated with E.*

For the special case in which $E = Z$ the quasi-order here is trivial, and we see therefore that BL implies HBKML. In classical functional analysis BL is a theorem, a special case of a result stated by Lumer [12] (but see also [1, 6]).

The main results of this paper may now be stated.

Theorem A. *Axiom* HBKML *effectively implies* PI.

This theorem is rather close in spirit to one of Bell and Jellett [4], but it has a much more direct proof. Gardiner in [8], a sequel to the present paper, has proved that PI effectively implies HBKML, so that, in fact, PI *is effectively equivalent to* HBKML.

Theorem B. *Axiom* BL *is effectively equivalent to* AC.

Preliminary versions of Theorems A and B were announced in [7]. Since then, Bell and Fremlin [3] have shown that AC is effectively equivalent to the statement that the unit ball of the dual of a normed vector space has an extreme point. Their result does not seem to imply Theorem B (nor vice versa). Some of the interest of Theorem B lies in the fact that, as we shall see, it is related to a reformulation of AC in terms of existence of maximal ideals in distributive lattices.

3. Proof of Theorem A. Axiom HBKML will be in force throughout this section.

Proposition 1. *Let Y be a proper vector lattice ideal of Z and let*

$$Q = \{u \in P(Z):\ u \perp Y\}.$$

Then Q is a convex set which has at least one extreme point.

To prove this, consider the quotient vector lattice $\frac{Z}{Y}$ together with the natural map $\pi\colon Z \to \frac{Z}{Y}$. The positive cone in $\frac{Z}{Y}$ is $\pi(Z_+)$, where Z_+ is that in Z. Since $1 \notin Y$, $\pi(1)$ is an order unit for $\frac{Z}{Y}$. Accordingly it is clear that the natural map $\varphi \to \varphi \circ \pi$, from $P\left(\frac{Z}{Y}\right)$ into Q, is an affine bijection. By HBKML the convex set $P\left(\frac{Z}{Y}\right)$ has an extreme point φ_0. Let $u_0 = \varphi_0 \circ \pi$, and suppose that $u_0 = \frac{1}{2}(u_1 + u_2)$, where $u_1, u_2 \in Q$. Then there exist φ_i in $P\left(\frac{Z}{Y}\right)$ such that $u_i = \varphi_i \circ \pi$ $(i = 1, 2)$ and we deduce that $\varphi_0 = \frac{1}{2}(\varphi_1 + \varphi_2)$. Hence $\varphi_0 = \varphi_1 = \varphi_2$, and so u_0 is an extreme point of Q as desired.

Theorem 2. *Let Ω be a non-empty set, $\mathfrak{B}$ a Boolean subalgebra of the power set $\mathfrak{P}(\Omega)$, and $\mathfrak{J}$ a proper ideal of $\mathfrak{B}$. Then $\mathfrak{J}$ is contained in a maximal ideal $\mathcal{M}$ of $\mathfrak{B}$.*

For the proof consider the vector space L over $\boldsymbol{R}$ of all real functions on Ω that are finite linear combinations of characteristic functions χ_a of elements a of $\mathfrak{B}$. For each function $f\colon \Omega \to \boldsymbol{R}$ let $S(f)$ denote the set $\{\omega\colon f(\omega) \neq 0\}$ and let

$$V = \{f \in L:\ S(f) \in \mathfrak{J}\}.$$

It is easy to see that L is a vector lattice with respect to the standard lattice operations on real functions and that V is an ideal of L that contains $\{\chi_a\colon a \in \mathfrak{J}\}$. Since $\mathfrak{J}$ is a proper ideal of $\mathfrak{B}$, V cannot contain the order unit $1 \equiv \chi_\Omega$ of L.

Now consider

$$K = \{u \in P(L):\ u \perp V\}.$$

By Proposition 1, K is a convex set which possesses an extreme point u_0. Now L is obviously a function algebra, and I claim that u_0 is a multiplicative linear functional on L. To see this we have merely to adapt a standard argument. Suppose $g \in L$ with $0 \leqslant g \leqslant 1$, and let $v(f) = u_0(fg)$ for all $f \in L$. Obviously $0 \leqslant v(f) \leqslant u_0(f)$ for all $f \in L_+$. Moreover, for all $f \in V$ we have $|f| \in V$ and hence

$$|v(f)| \leqslant v(|f|) \leqslant u_0(|f|) = 0 .$$

Thus v is multiple of some element of K, and, since u_0 is extreme in K, that element must be u_0. It follows that $v = \lambda_g u_0$ for some real $\lambda_g \in [0, 1]$. To compute λ_g note that

$$\lambda_g = \lambda_g u_0(1) = v(1) = u_0(g) .$$

Thus $u_0(fg) = u_0(f) u_0(g)$ for all f in L. It follows that u_0 is multiplicative on L.

Now let $\alpha(a) = u_0(\chi_a)$ for all $a \in \mathfrak{B}$. Then α is an epimorphism of $\mathfrak{B}$ onto the two-element Boolean algebra *2*. Since u_0 annihilates V it is clear that $\mathfrak{J} \subseteq \operatorname{Ker}\alpha$, so that we may take $\mathcal{M} = \operatorname{Ker}\alpha$ as the desired maximal ideal.

The above use of the subspace V is similar to a device in one of Luxemburg's proofs (see p. 131 of [14]; Luxemburg's argument requires correction however, because he wrongly asserts that his V is closed).

To complete the proof of Theorem A we must be able to handle abstract Boolean algebras. For this we need

THEOREM 3. *Let B be a Boolean algebra. Then there exists a Boolean algebra of sets $\mathfrak{B}$ together with an epimorphism β: $\mathfrak{B} \to B$.*

This is attributed by Luxemburg [14] to Tarski. Luxemburg gives no reference and only the barest hint of a proof. As I have been unable to trace an effective proof I supply one here. This depends on the following extension theorem (for an effective proof of which see pp. 36–37 of [16]).

THEOREM 4. *Let A, B be Boolean algebras, let S be a non-empty set of generators for A and let α: $S \to B$ be a map. Then α extends to a (necessarily unique) homomorphism $\bar{\alpha}$: $A \to B$ if and only if whenever*

$$\varepsilon_1 a_1 \wedge \varepsilon_2 a_2 \wedge \ldots \wedge \varepsilon_n a_n = 0 ,$$

with $n \geqslant 1$, and $a_r \in S$, $\varepsilon_r = \pm 1$ for $r = 1, 2, \ldots, n$, it follows that

$$\varepsilon_1 \alpha(a_1) \wedge \varepsilon_2 \alpha(a_2) \wedge \ldots \wedge \varepsilon_n \alpha(a_n) = 0 .$$

(The convention here is that $\varepsilon a = a$ if $\varepsilon = 1$, and $\varepsilon a = a'$, the complement of a, if $\varepsilon = -1$.)

To prove Theorem 3 consider, for each $b \in B$, the principal ideal $[b]$ of B generated by b, and let A be the Boolean subalgebra of the power set $\mathfrak{I}(B)$ generated by $S = \{[b]: b \in B\}$. Let α: $S \to B$ be defined by $\alpha([b]) = b$ for all $b \in B$. We show that α satisfies the condition of Theorem 4.

Suppose that $b_1, b_2, \ldots, b_m$, $c_1, c_2, \ldots, c_n$ belong to B with

(1) $$[b_1] \cap [b_2] \cap \ldots \cap [b_m] \cap [c_1]' \cap [c_2]' \cap \ldots \cap [c_n]' = \emptyset .$$

We want to deduce that

$$b_1 \wedge b_2 \wedge \ldots \wedge b_m \wedge c_1' \wedge c_2' \wedge \ldots \wedge c_n' = 0 .$$

Since $\bigcap_{r=1}^{m} [b_r] = [\bigwedge_{r=1}^{m} b_r]$ it is enough, for this, to check that the condition

(2) $$[b] \cap [c_1]' \cap [c_2]' \cap \ldots \cap [c_n]' = \emptyset$$

implies that

(3) $$b \wedge c_1' \wedge c_2' \wedge \ldots \wedge c_n' = 0 .$$

Now (2) says that $[b] \subseteq \bigcup_{r=1}^{n} [c_r]$, so that $b \in [c_r]$, and hence $b \wedge c_r' = 0$, for some r; *a fortiori* (3) is true.

Next, we must consider two degenerate forms of the condition (1), *viz*

$$[b_1] \cap [b_2] \cap \ldots \cap [b_m] = \emptyset$$

and

$$[c_1]' \cap [c_2]' \cap \ldots \cap [c_n]' = \emptyset .$$

A moment's reflection will show that the first of these cannot occur. As for the second, it can be rewritten as

$$[1] \cap [c_1]' \cap \ldots \cap [c_n]' = \emptyset$$

so that by the earlier part of the proof $1 \wedge \bigwedge_{r=1}^{n} c_r' = 0$, and hence

$$c_1' \wedge \ldots \wedge c_n' = 0 .$$

We have thus shown that the map α: $S \to B$ satisfies the condition of Theorem 4. Consequently it can be extended to a homomorphism $\bar{\alpha}$: $A \to B$. Since α is clearly onto B, so is $\bar{\alpha}$, and the proof of Theorem 3 is complete.

It is now easy to complete the proof of Theorem A. Let B be a Boolean algebra, let $\mathfrak{B}$ and β be as in Theorem 3, and let $\mathfrak{J} = \operatorname{Ker}\beta$. Then, by Theorem 2, $\mathfrak{J}$ is contained in a maximal ideal $\mathcal{M}$ of $\mathfrak{B}$. Evidently $\beta(\mathcal{M})$ is a maximal ideal of B, as desired.

4. Proof of Theorem B. A preliminary step, which does not depend on the axioms of § 2, is the proof of

PROPOSITION 5. *Let Ω be a non-empty set, $\mathcal{A}$ a sublattice of $\mathfrak{I}(\Omega)$ containing Ω and at least one other element, and let $\mathfrak{J}$ be a (lattice) ideal of $\mathcal{A}$. Let $\mathfrak{B}$ be a Boolean subalgebra of $\mathfrak{I}(\Omega)$ generated by $\mathcal{A}$. Then there exists a (Boolean algebra) ideal $\mathcal{J}$ of $\mathfrak{B}$ such that $\mathcal{J} \cap \mathcal{A} = \mathfrak{J}$.*

I claim that

$$\mathcal{J} = \{x \cap b: x \in \mathfrak{J}, b \in \mathfrak{B}\}$$

has the right properties.

First we show that $\mathfrak{J}$ is an ideal of $\mathfrak{B}$. If $y \in \mathfrak{J}$ and $c \in \mathfrak{B}$ then, obviously, $y \cap c \in \mathfrak{J}$. Next, if $x, y \in \mathfrak{I}$ and $b, c \in \mathfrak{B}$, then $(x \cap b) \cup (y \cap c) \in \mathfrak{J}$. To see this, let $b_1 = x \cap b$, $c_1 = y \cap c$, $z = x \cup y$. Then $z \in \mathfrak{I}$ and $b_1 = z \cap b_1$, $c_1 = z \cap c_1$, so that $b_1 \cup c_1 = z \cap (b_1 \cup c_1)$, which shows that $b_1 \cup c_1 \in \mathfrak{J}$, as desired. Thus $\mathfrak{J}$ is an ideal of the Boolean algebra $\mathfrak{B}$.

Obviously $\mathfrak{I} \subseteq \mathfrak{J}$. Now let $a \in \mathcal{A} \cap \mathfrak{J}$. Then $a = x \cap b$ for some $x \in \mathfrak{I}$, $b \in \mathfrak{B}$. Then

$$a = x \cap b \cap a \subseteq x \cap a \subseteq a ,$$

so that $a = x \cap a$, which shows that $a \in \mathfrak{I}$. Consequently $\mathcal{A} \cap \mathfrak{J} \subseteq \mathfrak{I}$ and hence $\mathcal{A} \cap \mathfrak{J} = \mathfrak{I}$.

THEOREM 6. *Suppose axiom* BL *is in force. Then, under the conditions of Proposition* 5, *the lattice* $\mathcal{A}$ *has a maximal ideal.*

Let $\mathfrak{B}$ be as in Proposition 5 and let L, $P(L)$ be constructed from $\mathfrak{B}$ exactly as in the proof of Theorem 2. Evidently $E = \{\chi_a\colon a \in \mathcal{A}\}$ is a sublattice of L, and we shall take $P(L)$ to have the quasi-order associated with this set (see § 2). By axiom BL there is an extreme point u_0 of $P(L)$ that is minimal in $P(L)$ for this quasi-order. Since u_0 is extreme in $P(L)$ it is multiplicative on L (as in the proof of Theorem 2). If, therefore, we write $\alpha(a) = u_0(\chi_a)$ for all $a \in \mathcal{A}$ then we see that α is either identically 1 on $\mathcal{A}$ or else is a lattice epimorphism of $\mathcal{A}$ onto $\mathbf{2} = \{0, 1\}$. Let $\mathfrak{K} = \{a \in \mathcal{A}\colon \alpha(a) = 0\}$. I claim that $\mathfrak{K}$ is a maximal ideal of $\mathcal{A}$.

For suppose not. Then $\mathfrak{K}$ is either empty or a non-maximal ideal. In either case we can then choose a proper ideal $\mathfrak{I}$ of $\mathcal{A}$ such that $\mathfrak{K} \underset{\neq}{\subset} \mathfrak{I}$. (If $\mathfrak{K}$ is empty it is enough, for instance, to take for $\mathfrak{I}$ the principal ideal generated by some element of $\mathcal{A}$ other than the largest element Ω). By Proposition 5 we can find an ideal $\mathfrak{J}$ of the Boolean algebra $\mathfrak{B}$ such that $\mathfrak{I} = \mathfrak{J} \cap \mathcal{A}$. Now let

$$W = \{f \in L\colon S(f) \in \mathfrak{J}\}$$

and

$$F = \{u \in P(L)\colon u \perp W\} .$$

Since axiom BL implies HBKML the proof of Theorem 2 shows that F has an extreme point, u_1 say, and that, if we define β by $\beta(b) = u_1(\chi_b)$ for all $b \in \mathfrak{B}$, we obtain an epimorphism $\beta\colon \mathfrak{B} \to \mathbf{2}$ whose kernel covers $\mathfrak{J}$. If γ is the restriction of β to $\mathcal{A}$ then γ is a lattice epimorphism of $\mathcal{A}$ onto $\mathbf{2}$ whose kernel covers $\mathfrak{I}$. Consequently $\operatorname{Ker}\gamma \underset{\neq}{\supset} \operatorname{Ker}\alpha$, and hence $\alpha(a) \geqslant \gamma(a)$ for all $a \in \mathcal{A}$, with strict inequality somewhere in $\mathcal{A}$. This means that $u_0(\chi_a) \geqslant u_1(\chi_a)$ for all $a \in \mathcal{A}$, with strict inequality somewhere. But that contradicts the minimality of u_0 in $P(L)$ for the given quasi-order, and we are forced to conclude that $\mathfrak{K}$ is, after all, a maximal ideal of $\mathcal{A}$.

To complete the proof of Theorem B it is now enough to recall a theorem of Klimovsky, Bell and Fremlin [11, 2]. To state it we require

AXIOM KBF. *Each lattice of sets having a greatest element and at least one other element has a maximal ideal.*

Klimovsky, Bell and Fremlin showed that Axiom KBF is effectively equivalent to AC. Since we have shown in Theorem 6 that axiom BL implies KBF, Theorem B now follows.

References

[1] H. Bauer, *Minimalstellen von Funktionen und Extremalpunkte*, Arch. Math. 9 (1958), pp. 389–393.

[2] J. L. Bell and D. H. Fremlin, *The maximal ideal theorem for lattices of sets*, Bull. Lond. Math. Soc. 4 (1972), pp. 1–2.

[3] — — *A geometric form of the axiom of choice*, Fund. Math. 77 (1972), pp. 167–170.

[4] — and F. Jellett, *On the relationship between the Boolean prime ideal theorem and two principles in functional analysis*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 19 (1971), pp. 191–194.

[5] P. Bernays and A. A. Fraenkel, *Axiomatic set theory*, Amsterdam 1958.

[6] D. A. Edwards, *An extension of Choquet boundary theory to certain partially ordered compact convex sets*, Studia Math. 36 (1970), pp. 177–193.

[7] — *Functional analysis and the axiom of choice*, 5 pp., in Papers from the "Open House in Functional Analysts", March 20–28, 1972, Various Publications No. 23, Mathematical Institute, Aarhus 1972.

[8] G. Gardiner, *The equivalence of the Boolean Prime ideal theorem and a theorem of functional analysis* Fund. Math. 84 (1974), pp. 81–86.

[9] J. D. Halpern, *The independence of the axiom of choice from the Boolean prime ideal theorem*, Fund. Math. 55 (1964), pp. 57–66.

[10] G. Jameson, *Ordered linear spaces*, Berlin 1970.

[11] G. Klimovsky, *Zorn's theorem and the existence of maximal filters and ideals in distributive lattices*, Rev. Un. Mat. Argentina 18 (1958), pp. 160–164.

[12] G. Lumer, *Points extrémaux associés; frontières de Shilov et Choquet; principe du minimum*, C. R. Acad. Sci. Paris 256 (1963), pp. 858–861.

[13] W. A. J. Luxemburg, *Two applications of the method of construction by ultrapowers to analysis*, Bull. Amer. Math. Soc. 68 (1962), pp. 416–419.

[14] — *Reduced powers of the real number system and equivalents of the Hahn-Banach extension theorem*, in W. A. J. Luxemburg (ed), Applications of model theory to algebra, analysis and probability, New York 1969.

[15] D. Pincus, *Independence of the prime ideal theorem from the Hahn-Banach theorem*, Bull. Amer. Math. Soc. 78 (1972), pp. 766–770.

[16] R. Sikorski, *Boolean Algebras*, Berlin–Heidelberg–New York 1969.

[17] P. Suppes, *Axiomatic Set Theory*, Princeton N. J. 1960.

MATHEMATICAL INSTITUTE, OXFORD

Accepté par la Rédaction le 1. 10. 1973