On the shape of MAR and MANR-spaces

by

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Abstract. It is proved in the paper that MAR-spaces [8] are the same as spaces with trivial shape in the sense of Fox [7]. If $X \in$ MANR [8] and $\operatorname{Sh} X \geqslant \operatorname{Sh} Y$ (in the sense of Fox [7]) then $Y \in$ MANR. Moreover, a homotopy extension theorem for mutations [7] is proved.

In [3] K. Borsuk introduced the notions of a FAR-space and a FANR-space and proved ([4], [5]) that they are invariants of shape (in the classical Borsuk sense [5]). In [8], replacing *fundamental sequences* ([2]) by *mutations* ([7]) I introduced the analogous notions of the MAR-space and the MANR-space. The aim of this paper is to prove that these notions are invariants of shape in the sense of Fox ([7]) and in the sense of Borsuk ([6]).

§ 1. Shape in the sense of Fox. In this section we recall some notions introduced by R. H. Fox in [7].

Consider the category $\operatorname{ANR}(\mathfrak{M})$ of metrizable absolute neighborhood retracts with continuous mappings and the relation of homotopy between mappings denoted by $\simeq$. By the Kuratowski–Wojdysławski theorem ([1]), p. 78) any metrizable space X may be considered as a closed subset of an $\operatorname{ANR}(\mathfrak{M})$-space P. By the first theorem of Hanner ([1], p. 96) every open subset of an $\operatorname{ANR}(\mathfrak{M})$-space is an $\operatorname{ANR}(\mathfrak{M})$-space. Hence, the family of all open neighborhoods of X in P with the inclusions is an inverse system in the category $\operatorname{ANR}(\mathfrak{M})$. This system is called the *complete neighborhood system of X in P* ([7], p. 54) and denoted by $\boldsymbol{U}(X, P)$.

Consider two arbitrary complete neighborhood systems $\boldsymbol{U}(X, P)$ and $\boldsymbol{V}(Y, Q)$. A *mutation* ([7], p. 49) $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ is defined as a collection of maps $f\colon U \to V$, where $U \in \operatorname{Ob} \boldsymbol{U}(X, P)$, $V \in \operatorname{Ob} \boldsymbol{V}(Y, Q)$, such that

(1.1) If $f \in \boldsymbol{f}$, $u \in \operatorname{Mor} \boldsymbol{U}(X, P)$, $v \in \operatorname{Mor} \boldsymbol{V}(Y, Q)$ and the composition vfu is defined, then $vfu \in \boldsymbol{f}$.

(1.2) Every object of $\boldsymbol{V}(Y, Q)$ is the range of a map belonging to $\boldsymbol{f}$.

(1.3) If $f_1, f_2 \in \boldsymbol{f}$ and $f_1, f_2\colon U \to V$, then there exists a $u \in \operatorname{Mor} \boldsymbol{U}(X, P)$, $u\colon U' \to U$, such that $f_1 u \simeq f_2 u$.

If $f\colon U \to V$, then we also write $U = \operatorname{domain} f$ and $V = \operatorname{range} f$.

The collection $\boldsymbol{u} = \operatorname{Mor} \boldsymbol{U}(X, P)$ is a mutation from $\boldsymbol{U}(X, P)$ to itself ([7], p. 50).

Consider two mutations $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ and $\boldsymbol{g}\colon \boldsymbol{V}(Y, Q) \to \boldsymbol{W}(Z, R)$. The *composition* $\boldsymbol{gf}\colon \boldsymbol{U}(X, P) \to \boldsymbol{W}(Z, R)$ of the mutations $\boldsymbol{f}$ and $\boldsymbol{g}$ is the mutation being the collection of all compositions gf such that $f \in \boldsymbol{f}$, $g \in \boldsymbol{g}$, and gf is defined ([7], p. 50).

Two mutations $\boldsymbol{f}, \boldsymbol{g}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ are *homotopic* (notation: $\boldsymbol{f} \simeq \boldsymbol{g}$, [7], p. 50) if

(1.4) For every $f \in \boldsymbol{f}$ and $g \in \boldsymbol{g}$ such that $f, g\colon U \to V$ there exists a $u \in \operatorname{Mor} \boldsymbol{U}(X, P)$ such that $U = \operatorname{range} u$ and $fu \simeq gu$.

Two metrizable space X and Y are said to be of the same *shape* ([7], p. 55) in the sense of Fox (notation: $\operatorname{Sh} X = \operatorname{Sh} Y$) if there exist two mutations $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ and $\boldsymbol{g}\colon \boldsymbol{V}(Y, Q) \to \boldsymbol{U}(X, P)$ such that

(1.5) $$\boldsymbol{fg} \simeq \boldsymbol{v} = \operatorname{Mor} \boldsymbol{V}(Y, Q) \quad \text{and} \quad \boldsymbol{gf} \simeq \boldsymbol{u} = \operatorname{Mor} \boldsymbol{U}(X, P).$$

By Theorem (3.2) of [7] the choice of $\operatorname{ANR}(\mathfrak{M})$-spaces P, Q and the manner of imbedding of X and Y in P and Q, respectively, is immaterial.

If mutations $\boldsymbol{f}$ and $\boldsymbol{g}$ satisfy the first of the conditions (1.5), then we say that the shape (in the sense of Fox) of X *dominates* the shape of Y (notation: $\operatorname{Sh} X \geqslant \operatorname{Sh} Y$).

§ 2. A homotopy extension theorem for mutations. Let X be a closed subset of a metrizable space X' considered as a closed subset of an $\operatorname{ANR}(\mathfrak{M})$-space P and let Y be a closed subset of an $\operatorname{ANR}(\mathfrak{M})$-space Q. We say that a mutation $\boldsymbol{f}'\colon \boldsymbol{U}(X', P) \to \boldsymbol{V}(Y, Q)$ is an *extension* ([8]) of a mutation $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ if

(2.1) for every $f \in \boldsymbol{f}$ there exists an $f' \in \boldsymbol{f}'$ such that $\operatorname{range} f' = \operatorname{range} f$ and $f'(x) = f(x)$ for every $x \in X$.

Then $\boldsymbol{f}$ is called a *restriction* ([8]) of $\boldsymbol{f}'$.

In [8] (Theorem (2.3)) we have proved that

(2.2) *If mutations* $\boldsymbol{f}, \boldsymbol{g}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ *are both restrictions of a mutation* $\boldsymbol{f}'\colon \boldsymbol{U}'(X', P) \to \boldsymbol{V}(Y, Q)$, *then* $\boldsymbol{f} \simeq \boldsymbol{g}$.

Let us prove the following

(2.3) THEOREM. *If* $\boldsymbol{f} \simeq \boldsymbol{g}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ *and* $\boldsymbol{f}'\colon \boldsymbol{U}'(X', P) \to \boldsymbol{V}(Y, Q)$ *is an extension of* $\boldsymbol{f}$, *then there exists an extension* $\boldsymbol{g}'\colon \boldsymbol{U}'(X', P) \to \boldsymbol{V}(Y, Q)$ *of* $\boldsymbol{g}$ *such that* $\boldsymbol{f}' \simeq \boldsymbol{g}'$.

Proof. It is easy to see that the collection $\boldsymbol{i}$ of all inclusions $i\colon U \to U'$, where $U \in \operatorname{Ob} \boldsymbol{U}(X, P)$, $U' \in \operatorname{Ob} \boldsymbol{U}'(X', P)$, $U \subset U'$, is a mutation $\boldsymbol{i}\colon \boldsymbol{U}(X, P) \to \boldsymbol{U}'(X', P)$. Consider the composition $\boldsymbol{f}'\boldsymbol{i}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$. It is evident that $\boldsymbol{f}'\boldsymbol{i}$ is a restriction of $\boldsymbol{f}'$. Hence by (2.2) it follows that $\boldsymbol{f}'\boldsymbol{i} \simeq \boldsymbol{f}$. Therefore $\boldsymbol{f}'\boldsymbol{i} \simeq \boldsymbol{g}$.

Take an arbitrary $g \in \boldsymbol{g}$, $g\colon U \to V$. By (1.2) there exists an $f' \in \boldsymbol{f}'$ with range $f' = V$, $f'\colon U' \to V$. Since $\boldsymbol{f}'\boldsymbol{i} \simeq \boldsymbol{g}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$, then by (1.1) and (1.4) it follows that there exists a $U_0 \in \operatorname{Ob} \boldsymbol{U}(X, P)$ such that $\overline{U}_0 \subset U' \cap U$ and $f'|\overline{U}_0 \simeq g|\overline{U}_0\colon \overline{U}_0 \to V$. By the first theorem of Hanner ([1], p. 96) and the Borsuk homotopy extension theorem ([1], p. 94) it follows that there exists an extension $g'\colon U' \to V$ of $g|\overline{U}_0$ such that $g' \simeq f'$. It is easy to see that the set of all extensions g' of maps $g \in \boldsymbol{g}$ obtained this way is a mutation $\boldsymbol{g}'\colon \boldsymbol{U}'(X', P) \to \boldsymbol{V}(Y, Q)$ homotopic to $\boldsymbol{f}'$ and being an extension of $\boldsymbol{g}$. Thus, the proof is concluded.

Remark. Theorem (2.3) is related to Patkowska's theorem on the extension of a homotopy for fundamental sequences ([10], p. 87).

§ 3. Shape of MAR-spaces. Let X be a closed subset of a metrizable space X' considered as a closed subset of an $\operatorname{ANR}(\mathfrak{M})$-space P. A mutation $\boldsymbol{r}\colon \boldsymbol{U}'(X', P) \to \boldsymbol{U}(X, P)$ is called a *mutational retraction* ([8]) if $r(x) = x$ for every $r \in \boldsymbol{r}$ and for every $x \in X$. A closed subset X of a metrizable space X' is called a *mutational retract* ([8]) of X' if there exists a mutational retraction $\boldsymbol{r}\colon \boldsymbol{U}'(X', P) \to \boldsymbol{U}(X, P)$. A metrizable space X is called a *mutational absolute retract* (shortly: MAR, [8]) if, for every metrizable space X' containing X as a closed subset, the set X is a mutational retract of X'. If $\operatorname{Sh} X = \operatorname{Sh}(a)$, where (a) is a space consisting of only one point a, then we say that the shape $\operatorname{Sh} X$ is *trivial*.

In [8] (Theorem (4.9)) we have proved that

(3.1) MAR-*spaces are the same as mutational retracts of* $\operatorname{AR}(\mathfrak{M})$-*spaces*.

Let us prove that

(3.2) *If* $\operatorname{Sh} X \geqslant \operatorname{Sh} Y$ *and* $\operatorname{Sh} X$ *is trivial, then* $\operatorname{Sh} Y$ *is trivial.*

Proof. By hypothesis there exist mutations $\boldsymbol{f}\colon \boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ and $\boldsymbol{g}\colon \boldsymbol{V}(Y, Q) \to \boldsymbol{U}(X, P)$ such that

(3.3) $$\boldsymbol{fg} \simeq \boldsymbol{v} = \operatorname{Mor} \boldsymbol{V}(Y, Q).$$

It remains to show that

(3.4) $$\boldsymbol{gf} \simeq \boldsymbol{u} = \operatorname{Mor} \boldsymbol{U}(X, P).$$

Since $\operatorname{Sh} X$ is trivial and the relation of domination between shapes is transitive, we can assume that X contains only one point a, $X = (a)$. Since the choice of an $\operatorname{ANR}(\mathfrak{M})$-space P containing X as a closed subset is immaterial ([7], Theorem (3.2), p. 55), we can also assume that $P = (a)$.

Therefore $U(X, P) = U((a), (a))$ is a rudimentary system consisting of only one object (a) and only one morphism $\mathrm{id}_{(a)}$, which is the identity on (a). Hence the mutation gf: $U((a), (a)) \to U((a), (a))$ consists of only one constituent, namely $\mathrm{id}_{(a)}$. Therefore $gf = (\mathrm{id}_{(a)}) = u = \mathrm{Mor}\, U(X, P)$. Hence we obtain (3.4) and the proof is concluded.

Remark. The statement (3.2) is related to the statement (7.2) of [4], due to K. Borsuk and concerning the shape of compacta.

Let us prove the following

(3.5) THEOREM. *A metrizable space* Y *is a* MAR*-space if and only if the shape* $\mathrm{Sh}\, Y$ *is trivial.*

Proof. Suppose that $Y \in \mathrm{MAR}$. Then by (3.1) there exists an $\mathrm{AR}(\mathfrak{M})$-space X such that $\mathrm{Sh}\, X \geqslant \mathrm{Sh}\, Y$. By Corollary (3.3) of [7] $\mathrm{Sh}\, X$ is trivial. Hence by (3.2) $\mathrm{Sh}\, Y$ is trivial.

Now, suppose that $\mathrm{Sh}\, Y$ is trivial. Then there exist mutations f: $U((a), (a)) \to V(Y, P)$ and g: $V(Y, P) \to U((a), (a))$ such that

$$fg \simeq v = \mathrm{Mor}\, V(Y, P)\,. \tag{3.6}$$

By the Kuratowski–Wojdysławski theorem ([1], p. 78) we can assume that $P \in \mathrm{AR}(\mathfrak{M})$. It is easy to see that the mutation g': $V'(P, P) \to U((a), (a))$ consisting of the constant map g': $P \to (a)$ is an extension of the mutation g: $V(Y, P) \to U((a), (a))$. Hence the mutation fg': $V'(P, P) \to V(Y, P)$ is an extension of the mutation fg: $V(Y, P) \to V(Y, P)$. From (3.6) by the homotopy extension theorem for mutations (2.3) it follows that the mutation v: $V(Y, P) \to V(Y, P)$ has an extension v': $V'(P, P) \to V(Y, P)$. Consider the collection r of all maps $r \in v'$ such that $r(y) = y$ for every $y \in Y$. We have proved in [8] (see the proof of Theorem (4.14)) that r: $V'(P, P) \to V(Y, P)$ is a mutational retraction. Therefore Y is a mutational retract of P and $P \in \mathrm{AR}(\mathfrak{M})$. Hence by (3.1) we obtain $Y \in \mathrm{MAR}$. Thus, the proof is finished.

Remark. Theorem (3.5) is related to Theorem (7.1) of [4], due to K. Borsuk and concerning the shape of compacta.

From (3.2) and Theorem (3.5) we obtain the following

(3.7) COROLLARY. *If* $X \in \mathrm{MAR}$ *and* $\mathrm{Sh}\, X \geqslant \mathrm{Sh}\, Y$, *then* $Y \in \mathrm{MAR}$.

Remark. Corollary (3.7) is related to Corollary (7.3) of [4], due to K. Borsuk and concerning the shape of compacta.

From (3.7) we obtain the following

(3.8) COROLLARY. *If* $X \in \mathrm{MAR}$ *and* $\mathrm{Sh}\, X = \mathrm{Sh}\, Y$, *then* $Y \in \mathrm{MAR}$, *i.e.* MAR *is an invariant of shape in the sense of Fox.*

Consider an arbitrary metrizable space X and denote by $\mathrm{Sh}(X)$ the *shape* of X in the sense of Borsuk [6]. This notion is well known and it will be recalled here.

In [9] (Theorem (3.6)) S. Nowak and the author have proved that

(3.9) *If* $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$ *then* $\mathrm{Sh}\, X \geqslant \mathrm{Sh}\, Y$.

From (3.7) and (3.9) we obtain the following

(3.10) COROLLARY. *If* $X \in \mathrm{MAR}$ *and* $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$, *then* $Y \in \mathrm{MAR}$.

(3.11) COROLLARY. *If* $X \in \mathrm{MAR}$ *and* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$, *then* $Y \in \mathrm{MAR}$, *i.e.* MAR *is an invariant of shape in the sense of Borsuk.*

§ 4. Shape of MANR-spaces. A closed subset X of a metrizable space X' is called a *mutational neighborhood retract* ([8]) of X' if there exists a closed neighborhood W of X in X' such that X is a mutational retract of W. A metrizable space X is said to be a *mutational absolute neighborhood retract* (shortly: MANR, [8]) if for every metrizable space X' containing X as a closed subset, the set X is a mutational neighborhood retract of X'.

In [8] (Theorem (4.11)) we have proved that

(4.1) MANR*-spaces are the same as mutational retracts of* $\mathrm{ANR}(\mathfrak{M})$*-spaces.*

In [8] (Lemma (2.4)) we have proved the following

(4.2) LEMMA. *If* X *is a closed subset of a metrizable space* X', Z *is an* $\mathrm{ANR}(\mathfrak{M})$*-space and* $\hat{f}, \tilde{f}$: $X' \to Z$ *are both extensions of a map* f: $X \to Z$, *then there exists a neighborhood* U *of* X *in* X' *such that* $\hat{f}|U \simeq \tilde{f}|U$.

Let us prove the following

(4.3) LEMMA. *Suppose* f, g: $U(X, P) \to V(Y, Q)$ *are mutations. If for every* $V \in \mathrm{Ob}\, V(Y, Q)$ *there exist* $f \in f$ *and* $g \in g$ *such that* $\mathrm{range} f = \mathrm{range} g = V$ *and* $f|X = g|X$ *then* $f \simeq g$.

Proof. Take arbitrary two maps $f_0 \in f$ and $g_0 \in g$ with a common domain and a common range, f_0, g_0: $U_0 \to V_0$. By hypothesis there exist maps $f_1 \in f$ and $g_1 \in g$ such that range $f_1 =$ range $g_1 = V_0$ and $f_1|X = g_1|X$. By (1.1) we can assume that f_1 and g_1 have a common domain U_1 contained in U_0, f_1, g_1: $U_1 \to V_0$, $U_1 \subset U_0$. It follows by the first theorem of Hanner and by Lemma (4.2) that there exists a $U_2 \in \mathrm{Ob}\, U(X, P)$ such that $U_2 \subset U_1$ and $f_1|U_2 \simeq g_1|U_2$. Let us put $f_2 = f_1|U_2$ and $g_2 = g_1|U_2$. Then $f_2 \in f$, $g_2 \in g$, f_2, g_2: $U_2 \to V_0$, $f_2 \simeq g_2$. Let $f_3 = f_0|U_2$, $g_3 = g_0|U_2$. Then $f_3 \in f$, $g_3 \in g$, f_3, g_3: $U_2 \to V_0$. Since $f_2, f_3 \in f$ and f_2, f_3: $U_2 \to V_0$, by (1.3) there exists a $U_3 \in \mathrm{Ob}\, U(X, P)$ such that $U_3 \subset U_2$ and $f_2|U_3 \simeq f_3|U_3$. Analogously, there exists a $U_4 \in \mathrm{Ob}\, U(X, P)$ such that $U_4 \subset U_2$ and $g_2|U_4 \simeq g_3|U_4$. Putting $U_5 = U_3 \cap U_4$, we obtain $U_5 \in \mathrm{Ob}\, U(X, P)$ and $f_0|U_5 = f_3|U_5 \simeq f_2|U_5 \simeq g_2|U_5 \simeq g_3|U_5 = g_0|U_5$. Thus $f \simeq g$ and the proof is finished.

(4.4) COROLLARY. *Suppose* f, g: $U(X, P) \to V(Y, Q)$ *are mutations.*

If for every $V \in \mathrm{Ob}\,\boldsymbol{V}(Y, Q)$ *there exists a map* f *with range* $f = V$ *such that* $f \in \boldsymbol{f}$ *and* $f \in \boldsymbol{g}$, *then* $\boldsymbol{f} \simeq \boldsymbol{g}$.

Let us prove the following

(4.5) THEOREM. *If* $X \in \mathrm{MANR}$ *and* $\mathrm{Sh}\,X \geqslant \mathrm{Sh}\,Y$, *then* $Y \in \mathrm{MANR}$.

Proof. By (4.1) X is a mutational retract of an $\mathrm{ANR}(\mathfrak{M})$-space P. Therefore there exists a mutational retraction $\boldsymbol{r}$: $\boldsymbol{W}(P, P) \to \boldsymbol{U}(X, P)$. By hypothesis there exist mutations $\boldsymbol{f}$: $\boldsymbol{U}(X, P) \to \boldsymbol{V}(Y, Q)$ and $\boldsymbol{g}$: $\boldsymbol{V}(Y, Q) \to \boldsymbol{U}(X, P)$ such that

$$\boldsymbol{fg} \simeq \boldsymbol{v} = \mathrm{Mor}\,\boldsymbol{V}(Y, Q).$$

It is easy to see that the family $\boldsymbol{i}$ consisting of all inclusions i: $U \to P$, where $U \in \mathrm{Ob}\,\boldsymbol{U}(X, P)$ is a mutation $\boldsymbol{i}$: $\boldsymbol{U}(X, P) \to \boldsymbol{W}(P, P)$. It is proved in [8] (statement (3.12)) that

$$\boldsymbol{ri} \simeq \boldsymbol{u} = \mathrm{Mor}\,\boldsymbol{U}(X, P).$$

Hence we obtain

(4.6) $$\boldsymbol{frig} \simeq \boldsymbol{fg} \simeq \boldsymbol{v}\colon\ \boldsymbol{V}(Y, Q) \to \boldsymbol{V}(Y, Q).$$

Since $\boldsymbol{g}$: $\boldsymbol{V}(Y, Q) \to \boldsymbol{U}(X, P)$ is a mutation and $P \in \mathrm{Ob}\,\boldsymbol{U}(X, P)$, by (1.2) there exists a map $g_1 \in \boldsymbol{g}$ with range $g_1 = P$, g_1: $V_1 \to P$, $V_1 \in \mathrm{Ob}\,\boldsymbol{V}(Y, Q)$. Take $V_0 \in \mathrm{Ob}\,\boldsymbol{V}(Y, Q)$ such that $\overline{V}_0 \subset V_1$. Consider the family $\boldsymbol{h}$ consisting of all maps belonging to the mutation $\boldsymbol{frig}$ of the form frg_1v, where $f \in \boldsymbol{f}$, $r \in \boldsymbol{r}$, $v \in \boldsymbol{v}$ with domain $v \supset \overline{V}_0$.

Let us observe that $\boldsymbol{h}$: $\boldsymbol{V}'(\overline{V}_0, Q) \to \boldsymbol{V}(Y, Q)$ is a mutation. The verification of (1.1) and (1.2) is trivial. Let us verify (1.3). Take two arbitrary maps $h_1, h_2 \in \boldsymbol{h}$ with a common domain and a common range. By the definition of $\boldsymbol{h}$ they are of the following form: $h_1 = f_1r_1g_1v$, $h_2 = f_2r_2g_1v$, where $f_k \in \boldsymbol{f}$, $r_k \in \boldsymbol{r}$ for $k = 1, 2$, $v \in \boldsymbol{v}$ with domain $v \supset \overline{V}_0$. Hence $f_1r_1, f_2r_2 \in \boldsymbol{fr}$, $\boldsymbol{fr}$: $\boldsymbol{W}(P, P) \to \boldsymbol{V}(Y, Q)$. Therefore, since $\boldsymbol{W}(P, P)$ is a rudimentary system, by (1.3) we have $f_1r_1 \simeq f_2r_2$. Hence $h_1 \simeq h_2$ and the condition (1.3) is satisfied. Thus, $\boldsymbol{h}$: $\boldsymbol{V}'(\overline{V}_0, Q) \to \boldsymbol{V}(Y, Q)$ is a mutation.

It is easy to see that the family $\boldsymbol{j}$ of all inclusions j: $V \to V'$, where $V \in \mathrm{Ob}\,\boldsymbol{V}(Y, Q)$, $V' \in \mathrm{Ob}\,\boldsymbol{V}'(\overline{V}_0, Q)$ and $V \subset V'$, is a mutation, $\boldsymbol{j}$: $\boldsymbol{V}(Y, Q) \to \boldsymbol{V}'(\overline{V}_0, Q)$.

Consider the composition

$$\boldsymbol{hj}\colon\ \boldsymbol{V}(Y, Q) \to \boldsymbol{V}(Y, Q).$$

It is evident that

(4.7) The mutation $\boldsymbol{h}$: $\boldsymbol{V}'(\overline{V}_0, Q) \to \boldsymbol{V}(Y, Q)$ is an extension of the mutation $\boldsymbol{hj}$: $\boldsymbol{V}(Y, Q) \to \boldsymbol{V}(Y, Q)$.

Let us show that

(4.8) $$\boldsymbol{hj} \simeq \boldsymbol{frig}\colon\ \boldsymbol{V}(Y, Q) \to \boldsymbol{V}(Y, Q).$$

Take an arbitrary $V \in \mathrm{Ob}\,\boldsymbol{V}(Y, Q)$. Let $hj \in \boldsymbol{hj}$ be such that range $hj = V$. By the definition of $\boldsymbol{h}$ we have $hj = frg_1v = fr\mathrm{id}_Pg_1v$, where $f \in \boldsymbol{f}$, $r \in \boldsymbol{r}$, $v \in \boldsymbol{v}$. Since $hj = fr\mathrm{id}_Pg_1v$ and $f \in \boldsymbol{f}$, $r \in \boldsymbol{r}$, $\mathrm{id}_P \in \boldsymbol{i}$, $g_1v \in \boldsymbol{g}$, we have $hj \in \boldsymbol{frig}$. Hence by Corollary (4.4) we obtain (4.8).

It follows by (4.6) and (4.8) that

(4.9) $$\boldsymbol{hj} \simeq \boldsymbol{v}\colon\ \boldsymbol{V}(Y, Q) \to \boldsymbol{V}(Y, Q).$$

By (4.7), (4.9) and the homotopy extension theorem for mutations (2.3) it follows that

(4.10) The mutation $\boldsymbol{v}$: $\boldsymbol{V}(Y, Q) \to \boldsymbol{V}(Y, Q)$ has an extension $\boldsymbol{v}'$: $\boldsymbol{V}'(\overline{V}_0, Q) \to \boldsymbol{V}(Y, Q)$.

By the first theorem of Hanner ([1], p. 96) $V_0 \in \mathrm{ANR}(\mathfrak{M})$; therefore we can consider systems $\boldsymbol{V}(V_0, V_0)$ and $\boldsymbol{V}(Y, V_0)$. Let $\boldsymbol{r}'$ be the family of all maps r': $V_0 \to V$, where $V \in \boldsymbol{V}(Y, V_0)$, such that r' is a restriction of a map belonging to $\boldsymbol{v}'$ and $r'(y) = y$ for every $y \in Y$. It is easy to verify that $\boldsymbol{r}'$: $\boldsymbol{V}(V_0, V_0) \to \boldsymbol{V}(Y, V_0)$ is a mutational retraction (compare the proof of Theorem (4.14) of [8]). Therefore Y is a mutational retract of V_0. Hence, by (4.1) Y is a MANR-space. Thus, the proof is concluded.

Remark. Theorem (4.5) is related to Theorem (2.3) of [5], due to K. Borsuk and concerning the shape of compacta.

From Theorem (4.5) we obtain the following

(4.11) COROLLARY. *If* $X \in \mathrm{MANR}$ *and* $\mathrm{Sh}\,X = \mathrm{Sh}\,Y$, *then* $Y \in \mathrm{MANR}$, *i.e.* MANR *is an invariant of shape in the sense of Fox.*

From (3.9) and Theorem (4.5) we obtain the following

(4.12) COROLLARY. *If* $X \in \mathrm{MANR}$ *and* $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$, *then* $Y \in \mathrm{MANR}$.

(4.13) COROLLARY. *If* $X \in \mathrm{MANR}$ *and* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$, *then* $Y \in \mathrm{MANR}$, *i.e.* MANR *is an invariant of shape in the sense of Borsuk.*

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Two theorems of functional analysis effectively equivalent to choice axioms

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Abstract. The first main result, when combined with a recent theorem of Gardiner, shows that the Boolean prime ideal theorem is equivalent, as an axiom of set theory, to the existence, for a certain class of vector lattices, of an extreme positive linear functional. The second main result provides an equivalence in the same spirit for the axiom of choice.

1. Introduction. It is well known that certain theorems of functional analysis are effectively equivalent to certain choice axioms. For example, Luxemburg [14] has shown that Alaoglu's theorem is equivalent to the Boolean prime ideal theorem PI (*viz* that every Boolean algebra has a prime ideal). Bell and Fremlin [3] have shown that the axiom of choice, AC, is equivalent to the statement that the unit ball of the Banach dual of a normed vector space always has an extreme point. The object of the present paper is to present two new results (Theorems A and B of § 2) in this spirit.

Theorem A provides, when taken together with the subsequent work of Gardiner [8], a new functional analytic equivalent of PI. Luxemburg [13] proved that PI implies the Hahn–Banach theorem, and Pincus [15] has shown that the converse is false. But Bell and Jellett [4] have shown that the Hahn–Banach and Krein–Milman theorems together imply PI. Theorem A somewhat resembles the Bell–Jellett theorem, but by restricting the statement suitably to vector lattices we have been able to achieve a strict equivalence to PI; the proof of our Theorem A is also more direct than Bell and Jellett's argument.

Halpern [9] has shown that PI is strictly weaker than AC. Theorem B provides a functional analytic statement equivalent to AC. It is not perhaps so simple as Bell and Fremlin's result mentioned above, but it has some independent interest: it is related both to Theorem A and to a theorem of Klimovsky, Bell and Fremlin [11, 2] which formulates AC in terms of maximal ideals in lattices of sets.