Semi-continuity of set-valued monotone mappings

by

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Abstract. A slight generalization of the theorem of Kuratowski–Fort on continuity almost everywhere of a semi-continuous set-valued mapping is obtained. The result is applied to the study of monotone set-valued mappings. It is proved that every maximal monotone mapping $F: Y \to 2^X$ is upper semi-continuous. Then, due to the Kuratowski–Fort theorem, it follows that F is lower semi-continuous almost everywhere (in the sense of category). It turned out that the set-valued mapping F must be single-valued at all these points of lower semi-continuity. So, under some conditions, every maximal set-valued monotone mapping is single-valued almost everywhere. As a corollary one can obtain the theorem of Mazur that every continuous convex function, given in a Banach space Y, is Gâteaux differentiable almost everywhere, provided Y has a countable dense subset.

Introduction. In the first part of this article we will deal with the following theorem of K. Kuratowski:

THEOREM (Kuratowski [5], p. 79). *Suppose that X is a compact metric space and Y is a metric space. Let F: $Y \to 2^X$ be an upper (resp. lower) semi-continuous set-valued mapping. Then the set of points at which F is not lower (resp. upper) semi-continuous is a set of the first category (in such a case we will say that F: $Y \to 2^X$ is lower (resp. upper) semi-continuous almost everywhere).*

Later on, Fort strengthened this result:

THEOREM (Fort [2]). *Let F: $Y \to 2^X$ be an upper (resp. lower) semi-continuous mapping with compact images ($= F(y)$ being compact subsets of X for each $y \in Y$), and X be a metrizable space. Then the mapping F is lower (resp. upper) semi-continuous almost everywhere.*

It is important to point out that the Kuratowski–Fort result is symmetric with respect to both kinds of semi-continuity. If F is upper semi-continuous, then it is almost everywhere lower semi-continuous, and if F is lower semi-continuous, then it is upper semi-continuous almost everywhere.

The author has shown in [4] that for upper semi-continuous mappings the theorem of Fort can be improved. There is no need to suppose that the topological space X (let us denote its topology by τ) is metrizable. It is enough to say that there exists a metrizable topology ϱ on X which is weaker than τ, i.e. $\varrho \subseteq \tau$. Unfortunately, this result is not symmetric with respect to both kinds of semi-continuity. As Example (1.7) shows, this result does not remain true for lower semi-continuous mappings.

One of the possibilities for obtaining a "symmetric" generalization of the theorem of Kuratowski–Fort is to put conditions only on the mapping $F\colon Y \to 2^X$ or on the set $\mathfrak{A} = \{F(y)\colon y \in Y\} \subset 2^X$, instead of any demands on the whole space X. The results of this sort are gathered in § 1. In § 2 it is shown how to use the results of § 1 for the study of set-valued monotone mappings ([1]). The main result here is that every set-valued monotone mapping has to be single-valued almost everywhere (under some conditions of course). This gives us an approach to the study of the differentiable properties of a convex function given in a Banach space. It is known (Rockafellar [8]) that the subgradient ∂ of the continuous convex function $\varphi\colon E \to R$ (E being a Banach space and R being the real line) is a monotone mapping. Thus, $\partial(x)$ is a single-point set for almost all $x \in E$. On the other hand, the function $\varphi\colon E \to R$ is Gâteaux differentiable at some point $x \in E$ if and only if $\partial(x)$ is a single-point set. Thus we obtain the theorem (Mazur [6]) that $\varphi\colon E \to R$ is almost everywhere differentiable, provided E has a countable dense subset.

The author is indebted to V. A. Geiler for the useful discussion of the results of § 1; in particular, Theorem (1.9) is a product of his influence.

§ 1. Set-valued mappings. Let X and Y be topological spaces.

(1.1) Definition. The (set-valued) mapping F, assigning to each $y \in Y$ a subset $F(y) \subset X$, is said to be *upper semi-continuous* (resp. *lower semi-continuous*) or, for brevity, u.s.c. (resp. l.s.c.) at the point $y_0 \in Y$ if, for every open set $0 \supset F(y_0)$ (resp. $0 \cap F(y_0) \neq \emptyset$), there exists an open neighbourhood $V \ni y_0$, such that $0 \supset F(y)$ (resp. $0 \cap F(y) \neq \emptyset$) whenever $y \in V$.

Mappings like F will be denoted by $F\colon Y \to 2^X$, or $F\colon Y \to 2^{(X,\tau)}$ where τ is the topology of X.

(1.2) Definition. We will say that the mapping $F\colon Y \to 2^{(X,\tau)}$ is *countably* u.s.c. (resp. *countably* l.s.c.) on Y if there exists a pseudometric topology ϱ on X such that

a) ϱ is weaker than τ (i.e. τ contains ϱ),

b) F is ϱ-u.s.c. (resp ϱ-l.s.c.) at some $y \in Y$ if and only if it is τ-u.s.c. (resp. τ-l.s.c.) at the same point y.

(1.3) Theorem. *Let $F\colon Y \to 2^{(X,\tau)}$ be an* u.s.c. (*resp.* l.s.c.) *mapping of the topological space Y into the topological space X, and let the following two conditions hold:*

a) *$F(y)$ is a compact subset of X whenever $y \in Y$,*

([1]) All definitions are given in their appropriate place in § 2.

b) *F is countably* l.s.c. (*resp. countably* u.s.c.).
Then F is l.s.c. (*resp.* u.s.c.) *almost everywhere.*

The proof of this theorem is a simple combination of the theorem of Fort cited above and Definition (1.2).

(1.4) Corollary. *If $F\colon Y \to 2^{(X,\tau)}$ is an* u.s.c. *mapping with compact images* ($= F(y)$ *being a compact subset of X for $y \in Y$*) *and there exists a metrizable topology ϱ on X such that $\varrho \subset \tau$, then $F\colon Y \to 2^{(X,\tau)}$ is τ-*l.s.c. *almost everywhere.*

Proof. If we know that F is countably l.s.c., we can apply Theorem (1.3). To show that this is the case we will use the following general proposition.

(1.5) Proposition. *Let $\tau_1 \geqslant \tau_2$ be two Hausdorff topologies on X, and $F\colon Y \to 2^X$ be a τ_1-*u.s.c. *mapping with τ_1-compact images. Then F is τ_1-*l.s.c. *at some point $y \in Y$ if and only if F is τ_2-*l.s.c. *at the same point.*

Proof. It is sufficient to show that if F is not τ_1-l.s.c. at the point $y_0 \in Y$, then it is not τ_2-l.s.c. at the same point y_0. Suppose that F is not τ_1-l.s.c. at y_0. Then there exist a point $x_0 \in F(y_0)$ and a τ_1-open set $0 \ni x_0$ such that, for each neighbourhood $U \ni y_0$, the equality $F(y_U) \cap 0 = \emptyset$ holds for some $y_U \in U$. Choose such a $y_U \in U$ for every neighbourhood $U \ni y_0$ and consider the set $B_V = \bigcup_{U \subset V} F(y_U)$, where V is an arbitrary (but fixed) neighbourhood of y_0. Evidently, the sequence $\{B_V\}_{V \ni y_0}$ is a filter basis. Now we will need the following lemma.

(1.6) Lemma. *Let ξ be the filter generated by $\{B_V\}_{V \ni y_0}$, and $\hat{\xi}$ be an ultrafilter $\hat{\xi} \supset \xi$. Then $\hat{\xi}$ τ_1-converges to some point of $F(y_0)$.*

Proof. Suppose the contrary. Then each $x \in F(y_0)$ is contained in a τ_1-open set $O_x \notin \hat{\xi}$. Since $F(y_0)$ is compact, we can find a finite sequence $x_1, x_2, \ldots, x_k$ such that $\bigcup_{i=1}^{k} O_{x_i} \supset F(y_0)$. As the mapping F is τ_1-u.s.c., $F(V) = \bigcup_{y \in V} F(y) \subset \bigcup_{i=1}^{k} O_{x_i}$ for some neighbourhood $V \in y_0$. Then $B_V \subset \bigcup_{i=1}^{k} O_{x_i}$ and $\bigcup_{i=1}^{k} O_{x_i} \in \xi \subset \hat{\xi}$. Since $\hat{\xi}$ is an ultrafilter, it has to contain at least one O_{x_i}, $i = 1, 2, \ldots, k$, but this is impossible because $O_{x_i} \notin \hat{\xi}$.

We get a contradiction and the lemma is proved.

Let us consider now the set $C = \bigcap_{B \in \xi} \bar{B}^{\tau_1}$ ($\bar{B}^{\tau_1}$ being the τ_1-closure of the set B); it follows from the lemma that C is non-empty and $C \subset F(y_0)$ because for each $x \in C$ there is an ultrafilter $\hat{\xi} \supset \xi$ that converges to x. The set C is τ_1-compact as a τ_1-closed subset of $F(y_0)$ and it does not contain the point x_0 ($\bar{B}_V^{\tau_1} \cap 0 = \emptyset$ whenever $V \ni y_0$). Let $W_1 \ni x_0$ and $W_2 \supset C$ be two τ_2-open disjoint subsets of X. Such a pair of sets exists because the topology τ_2 is Hausdorff, $x_0 \notin C$ and C is τ_2-compact ($\tau_1 \geqslant \tau_2$).

Let us now remark that the set W_2 must contain some B_V (otherwise there would be an ultrafilter $\hat{\xi} \supset \xi \cup (X \setminus W_2)$ which would converge to some point $x \in C \cap (X \setminus W_2) = \emptyset$). If $B_V \subset W_2$, then $B_V \cap W_1 \subset W_1 \cap \cap W_2 = \emptyset$. This shows that $F: Y \to 2^X$ is not τ_2-l.s.c. at y_0. Proposition (1.5) is proved, and thus also Corollary (1.4).

Unfortunately, Corollary (1.4) is not "symmetric" with respect to both kinds of semi-continuity. The following example gives us a l.s.c. mapping $F: Y \to 2^X$ which is nowhere u.s.c., although the demands of Corollary (1.4) are satisfied.

(1.7) EXAMPLE. Let X be the usual two-dimensional plane with a coordinate system Oxy, and ϱ be the usual metric topology on X. By τ we will denote another topology on X which is defined by the following rule: the subset $U \subset X$ is τ-open if and only if, for every line g which is parallel to Ox or Oy, the set $g \cap U$ is open with respect to the usual topology on g. Obviously $\tau \geqslant \varrho$. Let Y denote the real line $(-\infty, +\infty)$, and $F: Y \to 2^X$ be the mapping given by the formula $F(y) = \{(y, a) \in X: 0 \leqslant a \leqslant 1\}$. It is not difficult to see that F is τ-l.s.c. at every point $y \in Y$, and $F(y)$ is τ-compact for each $y \in Y$. Despite these facts, F is nowhere u.s.c. To show this we will consider the set

$$U_{y_0} = X \setminus \{(a, |y_0 - a|): -\infty < a < +\infty;\ a \neq y_0\}.$$

It is τ-open and among all the sets $F(y)$, $y \in Y$, it contains $F(y_0)$ only. Thus F is not u.s.c. at the point y_0.

Our second example will show that the metrizability condition in Corollary (1.4) cannot be omitted. More exactly, an u.s.c. mapping F, which is nowhere l.s.c., will be given.

(1.8) EXAMPLE. Let Y denote the unit segment $[0, 1]$ with its usual topology, and X_1 be the same set with the discrete topology. Put $X = \beta X_1$ — the Čech-Stone compactification of X_1. The identity mapping $f: X_1 \to Y$ is continuous and it can be extended by continuity on $\beta X_1 = X$. The extended mapping (we will denote it by the same letter f) is closed, i.e. the images of closed sets are closed. This implies that the set-valued mapping $F = f^{-1}: Y \to 2^X$ is u.s.c. Let now $y_0 \in Y$. The point y_0 is an open subset of $X = \beta X_1$, which intersects $F(y_0) = f^{-1}(y_0)$ (because $y_0 \in f^{-1}(y_0)$), but, among the sets of the form $F(y) = f^{-1}(y)$, $F(y_0)$ is the only set that contains y_0. Therefore F is not l.s.c. at y_0.

Let now X be uniform space with the uniformity

$$\mathfrak{U} = \{U:\ U \subset X \times X\}$$

(Kelley [3]). By $U[A]$, where A is a subset of X and $U \in \mathfrak{U}$, we will denote, as usual, the set $\{x \in X: (a, x) \in U \text{ for some } a \in A\}$. On the set 2^X we will consider the so-called uniformity of Hausdorff $\widetilde{\mathfrak{U}}$. It has a basis made up of sets of the form $\widetilde{U} = \{(A, B) \in 2^X \times 2^X:\ U[A] \supset B \text{ and } U[B] \supset A\}$, where $U \in \mathfrak{U}$.

(1.9) THEOREM. *Let $F: Y \to 2^X$ be an u.s.c. (resp. l.s.c.) mapping, with compact images, of the topological space Y into the uniform space $(X, \mathfrak{U})$. Suppose that the uniformity induced in $\mathfrak{A} = \{F(y): y \in Y\} \subset 2^X$ by $\widetilde{\mathfrak{U}}$ is metrizable. Then F is almost everywhere* l.s.c. (*resp.* u.s.c.).

Proof. We shall use Theorem (1.3) once more; so we have to prove first that F is countably l.s.c. (resp. countably u.s.c.). Let $\{U_i\}_{i=1}^{\infty} \subset \mathfrak{U}$ be such sets that $\{\widetilde{U}_i\}_{i=1}^{\infty}$ generate the metrizable uniformity of $\mathfrak{A}$. We can assume that $U_{i+1} \circ U_{i+1} \circ U_{i+1} \subset U_i$, $i = 1, 2, 3, \ldots$ and that all U_i are symmetric. Then there exists (Kelley [3],) a pseudo-metric $d(x_1, x_2)$ on X such that

$$U_{n+1} \subset \{(x_1, x_2) \in X \times X:\ d(x_1, x_2) < 1/2^n\} \subset U_n.$$

Evidently, $d(x_1, x_2)$ generates on $\mathfrak{A}$ the same uniformity as $\widetilde{\mathfrak{U}}$ does, i.e. for each $U \in \mathfrak{U}$ there exists an $\varepsilon > 0$ such that the inequalities $O_\varepsilon(F(y_0)) = \{x \in X:\ d(x, F(y_0)) < \varepsilon\} \supset F(y)$ and $O_\varepsilon(F(y)) \supset F(y_0)$ imply the inequalities $U[F(y_0)] \supset F(y)$ and $U[F(y)] \supset F(y_0)$.

The following lemma will complete the proof.

(1.10) LEMMA. *Let F, Y and X be as those in* (1.9), *and F be* l.s.c. (*resp.* u.s.c.) *at some point $y_0 \in Y$ with respect to the topology of $d(x_1, x_2)$. Then F is* l.s.c. (*resp.* u.s.c.) *at y_0 concerning the uniform topology.* (*The meaning of this lemma is that F is countably semi-continuous in the sense of Definition* (1.2)).

Proof. There is no need to recall that F is uniformly l.s.c. (resp. uniformly u.s.c.), if and only if, for each $U \in \mathfrak{U}$, an open $V \ni y_0$ exists such that $U[F(y)] \supset F(y_0)$ (resp. $U[F(y_0)] \supset F(y)$) as soon as $y \in V$ (here we essentially use the compactness of images $F(y)$, $y \in Y$).

Suppose now that F is uniformly u.s.c. at $y_0 \in Y$ and l.s.c. with respect to the pseudometric topology. Let $U \in \mathfrak{U}$ and $\varepsilon > 0$ be such that $U[F(y_1)] \supset F(y_2)$ and $U[F(y_2)] \supset F(y_1)$ as soon as $O_\varepsilon(F(y_1)) \supset F(y_2)$ and $O_\varepsilon(F(y_2)) \supset F(y_1)$. Then an open $V \ni y_0$ exists such that $O_\varepsilon(F(y_0)) \supset F(y)$ and $O_\varepsilon(F(y)) \supset F(y_0)$ for each $y \in V$. This means that $U[F(y)] \supset F(y_0)$ whenever $y \in V$. Hence F is uniformly l.s.c.

Replacing "u.s.c." by "l.s.c." and "l.s.c." by "u.s.c." in the last part of the proof we deduce the "(resp. u.s.c.)" part of the lemma.

§ 2. Monotone mappings. We shall now obtain some results on the semi-continuity of monotone mappings.

Let E be a Hausdorff locally convex space, and E' be its conjugate (= the set of all continuous linear functionals on E). By $\langle x, y \rangle$ we shall, as usual, denote the value of the functional $y \in E'$ at the point $x \in E$.

(2.1) Definition. The set-valued mapping $T\colon E \to 2^{E'}$ is said to be *monotone* if $\langle x_1 - x_2, y_1 - y_2 \rangle \geqslant 0$ for $y_i \in T(x_i)$, $i = 1, 2$. A set $A \subset E \times E'$ is said to be *monotone* if, for each pair of its elements $(x_i, y_i) \in A$, $i = 1, 2$, $\langle x_1 - x_2, y_1 - y_2 \rangle \geqslant 0$. A is a maximal monotone set if it is not a proper part of a monotone set.

The graph $G = \{(x, y) \in E \times E'\colon y \in T(x)\}$ of the monotone mapping $T\colon E \to 2^{E'}$ is a monotone set. If this graph happens to be a maximal monotone set, then T is (by definition) a maximal monotone mapping.

By means of the Zorn lemma, it follows that every monotone set can be included into a maximal set, i.e. for every monotone mapping $T\colon E \to 2^{E'}$, there exists a maximal monotone mapping $\tilde{T}\colon E \to 2^{E'}$ such that $T(x) \subset \tilde{T}(x)$ whenever $x \in E$.

In what follows, we will consider E to be a Banach space and $T(x) \neq \emptyset$ for each $x \in E$.

The next theorem is of great importance for our considerations.

(2.2) Theorem (Rockafellar [9]). *Every maximal monotone mapping $T\colon E \to 2^{E'}$ is locally bounded, i.e. for each $x_0 \in E$, there exists an open $V \ni x_0$ such that $T(V) = \bigcup_{x \in V} T(x)$ is a bounded subset of E'.*

(2.3) Proposition. *The graph $G = \{(x, y) \in E \times E\colon y \in T(x)\}$ of every maximal monotone mapping $T\colon E \to 2^{E'}$ is a closed subset of $E \times (E', \sigma(E', E))$ where $\sigma(E, E')$ is the weakest topology on E' with respect to which all elements of E, regarded as linear functionals on E', are continuous.*

Proof. Let $(x_\alpha, y_\alpha) \in G$ be a convergent net in $E \times (E', \sigma(E', E))$ and $\lim(x_\alpha, y_\alpha) = (x_0, y_0)$. This means that $x_\alpha \to x_0$ in E and $y_\alpha \to y_0$ in $(E', \sigma(E', E))$. Then $x_\alpha - x \to x_0 - x$ in E and $y_\alpha - y \to y_0 - y$ in $(E', \sigma(E', E))$, where $(x, y) \in G$. Let us prove that $\langle x_\alpha - x, y_\alpha - y \rangle \to \langle x_0 - x, y_0 - y \rangle$. Indeed,

$$|\langle x_\alpha - x, y_\alpha - y \rangle - \langle x_0 - x, y_0 - y \rangle| = |\langle (x_\alpha - x) - (x_0 - x), y_\alpha - y \rangle + \langle x_0 - x, (y_\alpha - y) - (y_0 - y) \rangle| \leqslant |\langle x_\alpha - x_0, y_\alpha - y \rangle| + |\langle x_0 - x, y_\alpha - y_0 \rangle| .$$

The second term on the right-hand side of the last inequality tends to 0 by the definition of $\sigma(E', E)$. The set $\{y_\alpha\colon \alpha \geqslant \alpha_0\}$ is bounded when α_0 is large enough for T is locally bounded. Then the set $\{y_\alpha - y, \alpha \geqslant \alpha_0\}$ is also bounded, and

$$|\langle x_\alpha - x_0, y_\alpha - y \rangle| \leqslant C \|x_\alpha - x_0\| \to 0 .$$

Thus

$$0 \leqslant \lim \langle x_\alpha - x, y_\alpha - y \rangle = \langle x_0 - x, y_0 - y \rangle .$$

Due to the maximality of T, it follows that $(x_0, y_0) \in G$. The proof is completed.

(2.4) Corollary (Browder [1]). *Let the suppositions of Proposition (2.3) be satisfied. Then $T(x)$ is a convex and $\sigma(E', E)$-compact subset of E'.*

Proof. The $\sigma(E', E)$-compactness of $T(x)$ is a simple consequence of Theorem (2.2) and Proposition (2.3).

If $y_1, y_2 \in T(x_0)$, $y \in T(x)$ and $0 < \alpha < 1$, then

$$\langle x - x_0, y - (\alpha y_1 + (1 - \alpha) y_2) \rangle = \alpha \langle x - x_0, y - y_1 \rangle + (1 - \alpha) \langle x - x_0, y - y_2 \rangle \geqslant 0 .$$

By virtue of the maximality of T, it follows that $\alpha y_1 + (1 - \alpha) y_2 \in T(x_0)$.

(2.5) Theorem. *Every maximal monotone mapping $T\colon E \to 2^{(E', \sigma(E', E))}$ is upper semi-continuous* (u.s.c.).

Proof. According to the local boundedness of T, there exists an open $V \ni x_0$ such that the set $T(V)$ is bounded. This means that $T(V)$ is a relatively $\sigma(E', E)$-compact subset of E'. Admitting that T is not u.s.c. at x_0 we can find a $\sigma(E', E)$-open set $U \supset T(x_0)$ and a net $\{x_\alpha\}_\alpha \subset V$, $x_\alpha \to x_0$ such that $T(x_\alpha) \cap (E' \setminus U) \neq \emptyset$. Let $y_\alpha \in T(x_\alpha) \cap (E' \setminus U) \subset T(V)$. Without loss of generality we can consider that the net $\{y_\alpha\}_\alpha$ $\sigma(E', E)$-converges to some $y_0 \in E'$. Since the graph of T is closed (Proposition (2.3)), $(x_0, y_0) \in G$, i.e. $y_0 \in T(x_0)$. On the other hand, the set $E' \setminus U$ is $\sigma(E', E)$-closed and has to contain y_0. Thus $y_0 \in (E' \setminus U) \cap T(x_0) = \emptyset$. We reach a contradiction and the proposition is proved.

(2.6) Proposition. *If the set-valued monotone mapping is lower semi-continuous* (l.s.c.) *at some point $x_0 \in E$, then the set $T(x_0)$ has only one element.*

Proof. Suppose the contrary: there are $y_0, \bar{y}_0 \in T(x_0)$ and $y_0 \neq \bar{y}_0$. Then there exists an $e \in E$ such that $\varepsilon = \langle e, \bar{y}_0 - y_0 \rangle > 0$. The sequence $x_n = x_0 + (1/n) e$ converges to x_0 and, since T is l.s.c. at x_0, $T(x_n) \cap \{y \in E'\colon |\langle e, y_0 - y \rangle| < \frac{1}{2}\varepsilon\} \neq \emptyset$ when n is sufficiently large. For some $y_m \in T(x_m) \cap \{y \in E'\colon |\langle e, y_0 - y \rangle| < \frac{1}{2}\varepsilon\}$ we have

$$\begin{aligned} 0 \leqslant \langle x_m - x_0, y_m - \bar{y}_0 \rangle &= (1/m) \langle e, y_m - \bar{y}_0 \rangle \\ &= (1/m)(\langle e, y_m - y_0 \rangle + \langle e, y_0 - \bar{y}_0 \rangle) = (1/m)(\langle e, y_m - y_0 \rangle - \varepsilon) \\ &< (1/m)(\tfrac{1}{2}\varepsilon - \varepsilon) < 0 . \end{aligned}$$

The proposition is proved.

(2.7) Theorem. *Let E be a separable Banach space, and $T\colon E \to 2^{E'}$ be a monotone set-valued mapping. Then T is almost everywhere single-valued, i.e. the set $\{x \in E\colon T(x)$ has more than one element$\}$ is of the first category in E.*

Proof. Without loss of generality we can consider T to be a maximal monotone mapping. In this case $T\colon E\to 2^{E'}$ is u.s.c. with respect to the topology $\sigma(E', E)$, and the sets $T(x)$, $x\in X$, are $\sigma(E', E)$-compact (Theorem (2.5) and Corollary (2.4)). On the other hand, there is a metrizable topology ϱ on $E\varrho \leqslant \sigma(E', E)$ because E has a countable subset which is everywhere dense in E. Applying Corollary (1.4) we obtain that the mapping $T\colon E\to 2^{(E',\sigma(E',E))}$ is l.s.c. almost everywhere. As Proposition (2.6) shows, at all these points of lower semi-continuity, the set $T(x)$ has only one element. The proof is completed.

Let us now discuss the connection between the monotone mappings and convex functions given on E.

Suppose $\varphi\colon E\to R$ (where R denotes the usual real line) is a convex function. It is known that, for every $x_0\in E$, there is at least one $y_0\in E'$ such that the inequality $\varphi(x)-\varphi(x_0)\geqslant \langle x-x_0, y_0\rangle$ holds for each $x\in E$. For fixed $x_0\in E$ put $\partial(x_0)=\{y\in E'\colon \varphi(x)-\varphi(x_0)\geqslant\langle x-x_0, y\rangle$ whenever $x\in E\}$. It is known (Rockafellar [8]) that $\partial\colon E\to 2^{E'}$ is a maximal monotone mapping. Having this and Theorem (2.5) in mind, we obtain the following result of Moreau.

(2.8) COROLLARY (Moreau [7]). *The mapping* $\partial\colon E\to 2^{(E',\sigma(E',E))}$ *is upper semi-continuous.*

It is not difficult to see that the continuous convex function $\varphi\colon E\to R$ is differentiable in the sense of Gateaux at the point $x_0\in E$, if and only if $\partial(x_0)$ is a single-point set. Thus, in this case, Theorem (2.7) can be rewritten in the following way:

(2.9) COROLLARY (Mazur [6]). *Let* $\varphi\colon E\to R$ *be a continuous convex function on the separable Banach space. Then* φ *is almost everywhere differentiable in the sense of Gâteaux, i.e. the set of points, at which* φ *is not differentiable in the sense of Gâteaux, is of the first category in* E.

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