Semi-continuity of set-valued monotone mappings

by

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Abstract. A slight generalization of the theorem of Kuratowski-Fort on continuity almost everywhere of a semi-continuous monotone set-valued mapping is obtained. The result is applied to the study of monotone set-valued mappings. It is proved that every maximal monotone mapping $F : Y \to 2^X$ is upper semi-continuous. Then, due to the Kuratowski-Fort theorem, it follows that $F$ is lower semi-continuous almost everywhere (in the sense of category). It turned out that the set-valued mapping $F$ must be single-valued at all those points of lower semi-continuity. So, under some additional conditions, every maximal set-valued monotone mapping is single-valued almost everywhere. As a corollary one can obtain the theorem of Mazur that every continuous convex function, given in a Banach space $X$, is Fréchet differentiable almost everywhere, provided $X$ has a countable dense subset.

Introduction. In the first part of this article we will deal with the following theorem of K. Kuratowski:

Theorem (Kuratowski [5], p. 79). Suppose that $X$ is a compact metric space and $Y$ is a metric space. Let $F : Y \to 2^X$ be an upper (resp. lower) semi-continuous set-valued mapping. Then the set of points at which $F$ is not lower (resp. upper) semi-continuous is a set of the first category (in such a case we will say that $F : Y \to 2^X$ is lower (resp. upper) semi-continuous almost everywhere).

Later on, Fort strengthened this result:

Theorem (Fort [2]). Let $F : Y \to 2^X$ be an upper (resp. lower) semi-continuous mapping with compact images (i.e. $F(y)$ is a compact subset of $X$ for each $y \in Y$), and $X$ be a metrizable space. Then the mapping $F$ is lower (resp. upper) semi-continuous almost everywhere.

It is important to point out that the Kuratowski-Fort result is symmetric with respect to both kinds of semi-continuity. If $F$ is upper semi-continuous, then it is almost everywhere lower semi-continuous, and if $F$ is lower semi-continuous, then it is upper semi-continuous almost everywhere.

The author has shown in [4] that for upper semi-continuous mappings the theorem of Fort can be improved. There is no need to suppose that the topological space $X$ (let us denote its topology by $\tau$) is metrizable. It is enough to say that there exists a metrizable topology $\tau'$ on $X$ which
is weaker than \( \tau \), i.e., \( \varrho \subseteq \tau \). Unfortunately, this result is not symmetric with respect to both kinds of semi-continuity. As Example (1.7) shows, this result does not remain true for lower semi-continuous mappings.

One of the possibilities for obtaining a "symmetric" generalization of the theorem of Kuratowski–Fort is to put conditions only on the mapping \( F : Y \to 2^X \) or on the set \( W \subseteq \{ F(y) : y \in Y \} \subset 2^X \), instead of any demands on the whole space \( X \). The results of this sort are gathered in \( \S 1 \) and in \( \S 2 \) it is shown how to use the results of \( \S 1 \) for the study of set-valued monotone mappings (1). The main result here is that every set-valued monotone mapping has to be single-valued almost everywhere (under some conditions of course). This gives us an approach to the study of the differentiable properties of a convex function given in a Banach space. It is known (Rockafellar [8]) that the subgradient \( \partial \) of the continuous convex function \( f : E \to R \) (where \( E \) is a Banach space and \( R \) being the real line) is a monotone mapping. Thus, \( \partial(x) \) is a single-point set for almost all \( x \in E \). On the other hand, the function \( E \to R \) is Gâteaux differentiable at some point \( x \in E \) if and only if \( \partial(x) \) is a single-point set. Thus we obtain the theorem (Masur [6]) that \( F : E \to 2^E \) is almost everywhere differentiable, provided \( E \) has a countable dense subset.

The author is indebted to V. A. Gaider for the useful discussion of the results of \( \S 1 \); in particular, Theorem (1.9) is a product of his influence.

\( \S 1 \). Set-valued mappings. Let \( X \) and \( Y \) be topological spaces.

\( (1.1) \) Definition. The (set-valued) mapping \( F \) assigning to each \( y \in Y \) a subset \( F(y) \subseteq X \), is said to be upper semi-continuous (resp. lower semi-continuous) or, for brevity, u.s.c. (resp. l.s.c.) at the point \( y_0 \in Y \), if, for every open set \( V \supseteq F(y_0) \) (resp. \( V \subseteq F(y_0) \)), there exists an open neighbourhood \( V \supseteq y_0 \), such that \( V \setminus F(y_0) \) (resp. \( V \cap F(y_0) \)) whenever \( V \supseteq y \).

Mappings like \( F \) will be denoted by \( F : Y \to 2^X \), or \( F : Y \to 2^{(X,n)} \) where \( \tau \) is the topology of \( X \).

\( (1.2) \) Definition. We will say that the mapping \( F : Y \to 2^{(X,n)} \) is countably u.s.c. (resp. countably l.s.c.) on \( Y \) if there exists a pseudometric topology \( \tau \) on \( X \) such that

(a) \( \varrho \) is weaker than \( \tau \) (i.e. \( \tau \) contains \( \varrho \));

(b) \( F \) is \( \tau \)-u.s.c. (resp. \( \tau \)-l.s.c.) at some \( y \in Y \) if and only if \( y \) is \( \tau \)-u.s.c. (resp. \( \tau \)-l.s.c.) at the same point \( y \).

\( (1.3) \) Theorem. Let \( F : Y \to 2^{(X,n)} \) be an u.s.c. (resp. l.s.c.) mapping of the topological space \( Y \) into the topological space \( X \), and let the following two conditions hold:

a) \( F(y) \) is a compact subset of \( X \) whenever \( y \in Y \),

b) \( F \) is countably u.s.c. (resp. countably u.s.c.).

Then \( F \) is l.s.c. (resp. u.s.c.) almost everywhere.

The proof of this theorem is a simple combination of the theorem of Fort cited above and Definition (1.2).

\( (1.4) \) Corollary. If \( F : Y \to 2^{(X,n)} \) is an u.s.c. mapping with compact images \( \{ F(y) \} \) being a compact subset of \( X \) for \( y \in Y \) and there exists a metrizable topology \( \varrho \) on \( X \) such that \( \varrho \subseteq \tau \), then \( F : Y \to 2^{(X,n)} \) is \( \varrho \)-l.s.c. almost everywhere.

Proof. If we know that \( F \) is countably l.s.c., we can apply Theorem (1.3). To show that this is the case we will use the following general proposition.

\( (1.5) \) Proposition. Let \( \tau_1 \supseteq \tau_2 \) be two Hausdorff topologies on \( X \), and \( F : Y \to 2^X \) be a \( \tau_1 \)-u.s.c. mapping with \( \tau_1 \)-compact images. Then \( F \) is \( \tau_2 \)-l.s.c. at some point \( y \in Y \) if and only if \( F \) is \( \tau_2 \)-l.s.c. at the same point.

Proof. It is sufficient to show that if \( F \) is not \( \tau_1 \)-l.s.c. at the point \( y \in Y \) then it is not \( \tau_2 \)-l.s.c. at the same point \( y \). Suppose that \( F \) is not \( \tau_1 \)-l.s.c. at \( y \). Then there exists a point \( y_0 \in F(y) \) and a \( \tau_1 \)-open set \( \widetilde{O} \subset Y \) such that, for each neighbourhood \( U \supseteq y_0 \), the equality \( F(y) \cap 0 = \emptyset \) holds for some \( y_0 \in U \). Choose such a \( y_0 \in U \) for every neighbourhood \( U \supseteq y_0 \) and consider the set \( B = \bigcup_{\tau_1} F(y) \), where \( F \) is an arbitrary (but fixed) neighbourhood of \( y_0 \). Evidently, the sequence \( (B_r)_{r \in \mathbb{N}} \) is a filter basis. Now we will need the following lemma.

\( (1.6) \) Lemma. Let \( \xi \) be the filter generated by \( \{ B_r \}_{r \in \mathbb{N}} \), and \( \xi \) be an ultralimit \( \tilde{\xi} \). Then \( \tilde{\xi} \tau_1 \)-converges to some point of \( F(y) \).

Proof. Suppose the contrary. Then each \( \xi \in F(y) \) is contained in a \( \tau_1 \)-open set \( O \subseteq \tilde{\xi} \). Since \( F(y) \) is compact, we can find a finite sequence \( y_1, y_2, \ldots, y_k \) such that \( \bigcup_{i=1}^k O_{y_i} \cap F(y) \). As the mapping \( F \) is \( \tau_1 \)-u.s.c.,

\( F(y) = \bigcup_{i=1}^k O_{y_i} \cap F(y) \) whenever \( y \in \tilde{\xi} \). Since \( \xi \) is an ultralimit, it has to contain at least one \( O_{y_i} \), \( i = 1, 2, \ldots, k \), but this is impossible because \( O_{y_i} \not\subseteq \tilde{\xi} \).

We get a contradiction and the lemma is proved.

Let us consider now the set \( C = \bigcap_{r \in \mathbb{N}} B_r \) (being the \( \tau_1 \)-closure of the set \( B \)); it follows from the lemma that \( C \) is non-empty and \( C \subseteq F(y_0) \) because for each \( \xi \in F(y) \) there is a \( \tau_1 \)-open set \( O_{y_i} \) that converges to \( y \). The set \( C \) is \( \tau_1 \)-compact as a \( \tau_1 \)-closed subset of \( F(y_0) \) and it does not contain the point \( y_0 \). Let \( W_1 \supseteq W_2 \supseteq \ldots \supseteq W \subseteq \tau_1 \)-open disjoint subset of \( F(y_0) \). Such a pair of sets exists because the topology \( \tau_1 \) is Hausdorff. This is a contradiction and the theorem is proved.
Let us now remark that the set $W_1$ must contain some $B_1$ (otherwise, there would be an ultrafilter $\mathcal{F} \supseteq \mathcal{G} \cup (X \setminus W_1)$ which would converge to some point $x \in G \cup (X \setminus W_1) = \emptyset$). If $B_1 \subseteq W_1$, then $B_1 \cap W_1 \subseteq W_1 \cap W_1 = \emptyset$. This shows that $F:Y \to 2^X$ is not $\tau_1$-l.s.c. at $y_1$. Proposition (1.5) is proved, and thus also Corollary (1.4).

Unfortunately, Corollary (1.4) is not “symmetric” with respect to both kinds of semi-continuity. The following example gives us a l.s.c. mapping $F: Y \to 2^X$ which is nowhere u.s.c., although the demands of Corollary (1.4) are satisfied.

(1.7) Example. Let $X$ be the usual two-dimensional plane with a coordinate system $Oxy$, and $g$ be the usual metric topology on $X$. By $\tau$ we will denote another topology on $X$ which is defined by the following rule: the subset $U \subseteq X$ is $\tau$-open if and only if, for every line $g$ which is parallel to $Ox$ or $Oy$, the set $g \cap U$ is open with respect to the usual topology on $g$. Obviously $\tau \supseteq g$. Let $Y$ denote the real line $(-\infty, +\infty)$, and $F:Y \to 2^X$ be the mapping given by the formula $F(y) = \{(y, a) \in X: 0 < a < 1\}$. It is not difficult to see that $F$ is $\tau$-l.s.c. at every point $y \in Y$, and $F(y)$ is $\tau$-compact for each $y \in Y$. Despite these facts, $F$ is nowhere u.s.c. To show this we will consider the set

$U_{\alpha} = \{x \in a, y \neq a\} = (-\infty, a) \cup (a, +\infty)$.

It is $\tau$-open and among all the sets $F(y), y \in Y$, it contains $F(y_0)$ only. Thus $F$ is not u.s.c. at the point $y_0$.

Our second example will show that the metrizability condition in Corollary (1.4) cannot be omitted. Exactly, an u.s.c. mapping $F$, which is nowhere l.s.c., will be given.

(1.8) Example. Let $Y$ denote the unit segment $[0, 1]$ with its usual topology, and $X_1$ be the same set with the discrete topology. Put $X = \beta X_1$ — the Stone-Čech compactification of $X_1$. The identity mapping $f: X_1 \to Y$ is continuous and it can be extended by continuity on $\beta X_1 = X$. The extended mapping (we will denote it by the same letter $f$) is closed, i.e. the images of closed sets are closed. This implies that the set-valued mapping $F = f^*: Y \to 2^X$ is u.s.c. Let now $y \in Y$. The point $y_0$ is an open subset of $X = \beta X_1$, which intersects $F(y_0) = f^{-1}(y_0)$ (because $y_0 \in f^{-1}(y_0)$) among the sets of the form $F(y) = f^{-1}(y)$, $F(y_0)$ is the only set that contains $y_0$. Therefore $F$ is not l.s.c. at $y_0$.

Let now $X$ be uniform space with the uniformity

$U = \{U \subseteq X \times X\}$

(Kelley [3]). By $U(A)$, where $A$ is a subset of $X$ and $U \subseteq U$, we will denote, as usual, the set $\{a \in X: (a, b) \in U \text{ for some } a \in A\}$. On the set $2^X$ we will consider the so-called uniformity of Hausdorff $U$. It has a basis made up of sets of the form $U = \{(A, B) \subseteq 2^X \cup 2^X: U(A) \subseteq B \text{ and } U(B) \subseteq A\}$, where $U \subseteq U$.

(1.9) Theorem. Let $F: Y \to 2^X$ be an u.s.c. (resp. l.s.c.) mapping, with compact images, of the topological space $Y$ into the uniform space $(X, U)$. Suppose that the uniformity induced in $\Xi = \{F(y): y \in Y\} \subseteq 2^X$ by $U$ is metrizable. Then $F$ is almost everywhere l.s.c. (resp. u.s.c.).

Proof. We shall use Theorem (1.3) once more; so we have to prove first that $F$ is countably l.s.c. (resp. countably u.s.c.). Let $\{U_i\}_{i=1}^\infty$ be such sets that $U_i \cap \overline{U}_{i+1}$ generate the metrizable uniformity of $\Xi$. We can assume that $U_1 \subseteq U_4 \subseteq \cdots \subseteq U_i \subseteq U_1, i = 1, 2, 3, \ldots$ and that all $U_i$ are symmetric. Then there exists (Kelley [3]) a pseudo-metric $d(x_1, x_2)$ on $X$ such that

$U_{x_1} \subseteq \{(x_1, x_2) \in X \times X: d(x_1, x_2) < 1/n \} \subseteq U_{x_2}$.

Evidently, $d(x_1, x_2)$ generates on $\Xi$ the same uniformity as $U$ does, i.e. for each $U \subseteq U$ there exists an $\varepsilon > 0$ such that the inequalities $O(U)[d(y)] = \{x \in X: d(x, F(y)) < \varepsilon \} \supseteq \varphi(U[F(y)])$ and $O(U)[F(y)] \supseteq \varphi(U[F(y)])$ imply the inequalities $\varphi(U)[F(y)] \supseteq \varphi(U[F(y)])$ and $\varphi(U)[F(y)] \supseteq \varphi(U[F(y)])$.

The following lemma will complete the proof.

(1.10) Lemma. Let $F$ be l.s.c. (resp. u.s.c.) at some point $y \in Y$ with respect to the topology of $d(x_1, x_2)$. Then $F$ is l.s.c. (resp. u.s.c.) at $y_1$ concerning the uniform topology. (The meaning of this lemma is that $F$ is countably semi-continuous in the sense of Definition (1.3)).

Proof. There is no need to recall that $F$ is uniformly l.s.c. (resp. uniformly u.s.c.), if and only if, for each $U \subseteq U$, an open $V \subseteq Y$ exists such that $U[F(y)] \supseteq \varphi(U[F(y)])$ and $U[F(y)] \supseteq \varphi(U[F(y)])$ as soon as $g \subseteq V$ (here we essentially use the compactness of images $F(y), y \in Y$).

Suppose now that $F$ is uniformly u.s.c. at $y_1 \in Y$ and l.s.c. with respect to the pseudometric topology. Let $U \subseteq U$ be such that $U[F(y)] \supseteq \varphi(U[F(y)])$ and $U[F(y)] \supseteq \varphi(U[F(y)])$ as soon as $O(U[F(y)]) \supseteq \varphi(U[F(y)])$ and $O(U)[F(y)] \supseteq \varphi(U[F(y)])$ for each $y \in Y$. This means that $U[F(y)] \supseteq \varphi(U[F(y)])$ whenever $g \subseteq V$. Hence $F$ is uniformly l.s.c.

Replacing “u.s.c.” by “l.s.c.” and “l.s.c.” by “u.s.c.” in the last part of the proof we deduce the “(resp. u.s.c.)” part of the lemma.

§ 2. Monotone mappings. We shall now obtain some results on the semi-continuity of monotone mappings.

Let $E$ be a Hausdorff locally convex space, and $E'$ be its conjugate (= the set of all continuous linear functionals on $E$). By $(x, y)$ we shall, as usual, denote the value of the functional $y \in E'$ at the point $x \in E$. 
(2.4) **Corollary** (Browder [11]). Let the suppositions of Proposition (2.3) be satisfied. Then \( T(\alpha) \) is a convex and \( e(E', E) \)-compact subset of \( E' \).

**Proof.** The \( e(E', E) \)-compactness of \( T(\alpha) \) is a simple consequence of Theorem (2.2) and Proposition (2.3).

If \( y, y \in T(\alpha), y \in T(\beta) \) and \( 0 < \alpha < \beta \), then

\[
\langle x - x, y - (\alpha) y \rangle = \langle x - x, y - y \rangle + \langle 1 - \alpha \rangle \langle x - x, y - y \rangle > 0.
\]

By virtue of the maximality of \( T \), it follows that \( ay + (1 - \alpha)y \in T(\alpha) \).

(2.5) **Theorem.** Every maximal monotone mapping \( T : E \to 2^{E^*} \) is upper semi-continuous (u.s.c.).

**Proof.** According to the local boundedness of \( T \), there exists an open \( V \ni x_0 \) such that the set \( T(V) \) is bounded. This means that \( T(V) \) is a relatively \( e(E', E) \)-compact subset of \( E' \). Admitting that \( T \) is not u.s.c. at \( x_0 \), we can find a \( e(E', E) \)-open set \( U \supset T(x_0) \) and a net \( \{x_n \} \subset V \setminus U \), \( x_n \to x_0 \), and \( T(x) \cap (E' \setminus U) \neq \emptyset \). Let \( y_n \in T(x_n) \cap (E' \setminus U) \subset T(V) \). Without loss of generality we can consider that the net \( \{y_n \} \), \( e(E', E) \)-converges to some \( y \in E' \). Since the graph of \( T \) is closed (Proposition (2.3)), \( (x_n, y) \in G \), i.e., \( y \in T(x_n) \). On the other hand, the set \( E' \setminus U \) is \( e(E', E) \)-closed and has to contain \( y_n \). Thus \( y_n \in (E' \setminus U) \cap T(x_0) = \emptyset \). We reach a contradiction and the proposition is proved.

(2.6) **Proposition.** If the set-valued monotone mapping is lower semi-continuous (l.s.c.) at some point \( x \in E \), then the set \( T(x) \) has only one element.

**Proof.** Suppose the contrary: there are \( y, y \in T(x) \) and \( y \neq y \). Then there exists an \( e \in E \) such that \( e \in (y, y) > 0 \). The sequence \( a_n = e + (1/n)e \) converges to \( e \) and, since \( T \) is l.s.c. at \( x_0 \), \( T(a_n) \cap (y, y) > 0 \) when \( n \) is sufficiently large. For some \( y_n \in T(a_n) \cap (y, y) > 0 \) we have

\[
0 < \langle x_n - x, y - y \rangle = \langle x_n - x, y - y \rangle + \langle x_n - x, y - y \rangle + \langle x_n - x, y - y \rangle + \langle x_n - x, y - y \rangle.
\]

The second term on the right-hand side of the last inequality tends to 0 by the convexity of \( T \). When the sequence \( a_n \) is large enough for \( T \) to be bounded. Then the set \( (y, y) \) is also bounded, and

\[
\langle x_n - x, y - y \rangle \leq C\|x_n - x\| + C\|y_n - y\| > 0.
\]

Thus

\[
0 < \lim \langle x_n - x, y - y \rangle = \langle a - x, y - y \rangle.
\]

Due to the maximality of \( T \), it follows that \( (x, y) \in G \). The proof is completed.
Proof. Without loss of generality we can consider $T$ to be a maximal monotonous mapping. In this case $T: E \to g^{\mathbb{R}}$ is u.s.c. with respect to the topology $\sigma(E', E)$, and the sets $T(x)$, $x \in X$, are $\sigma(E', E)$-compact (Theorem (2.3) and Corollary (2.4)). On the other hand, there is a metrizable topology $\tau$ on $E$ such that $E \in \sigma(E', E)$ because $E$ has a countable subset which is everywhere dense in $E$. Applying Corollary (1.4) we obtain that the mapping $T: E \to g^{\mathbb{R}}_{\sigma(E', E)}$ is u.s.c. almost everywhere. As Proposition (2.6) shows, at all these points of lower semi-continuity, the set $T(x)$ has only one element. The proof is completed.

Let us now discuss the connection between the monotone mappings and convex functions given on $E$.

Suppose $\phi: E \to R$ (where $R$ denotes the usual real line) is a convex function. It is known that, for every $x_r \in E$, there is at least one $y_r \in E'$ such that the inequality $\phi(x) - \phi(x_r) > \langle x - x_r, y_r \rangle$ holds for each $x \in E$. For fixed $x_r \in E$ put $\delta(x_r) = \{ y \in E' : \phi(x) - \phi(x_r) > \langle x - x_r, y \rangle \}$ whenever $x \in E$. It is known (Rockafellar [5]) that $\delta: E \to g^{2}$ is a maximal monotone mapping. Having this and Theorem (2.3) in mind, we obtain the following result of Moreau.

(2.8) COROLLARY (Moreau [7]). The mapping $\delta: E \to g^{2}$ is upper semi-continuous.

It is not difficult to see that the continuous convex function $\phi: E \to R$ is differentiable in the sense of Gateaux at the point $x_r \in E$, if and only if $\delta(x_r)$ is a single-point set. Thus, in this case, Theorem (2.7) can be rewritten in the following way:

(2.9) COROLLARY (Mazur [6]). Let $\phi: E \to R$ be a continuous convex function on the separable Banach space. Then $\phi$ is almost everywhere differentiable in the sense of Gateaux, i.e., the set of points, at which $\phi$ is not differentiable in the sense of Gateaux, is of the first category in $E$.

References