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INSTITUTE FOR ADVANCED STUDY
Princeton, N. J.

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P-ideals and F-ideals in rings of continuous functions

by

David Rudd (Norfolk, Virginia)

Abstract. A ring of continuous functions is a ring of the form $C(X)$, the ring of all continuous real-valued functions on a completely-regular Hausdorff space X.

The author defines two classes of ideals in $C(X)$, P-ideals and F-ideals, which are analogs of P-spaces and F-spaces. He then discusses properties of these ideals, such as their structure spaces and zero-sets of their members, and characterizes those spaces X for which there exist P-ideals (or F-ideals) in $C(X)$.

Introduction. If X is a space so that every prime ideal in $C(X)$ is maximal, then X is said to be a P-*space*. We extend this concept to ideals in rings of continuous functions by defining a non-zero ideal I to be a P-ideal if every proper prime ideal in I is a maximal ideal in I. It is known [2, 14.29] that $C(X)$ is a P-ideal, i.e. X is a P-space, if and only if its real structure space (vX) is a P-space. We show that a modified version of this theorem holds for P-ideals. We also characterize those spaces whose rings of continuous functions possess a P-ideal.

If X is a space so that mM $(= \{f \mid f \in fM\})$ is prime for every maximal ideal M in $C(X)$, then X is said to be an F-*space*. We extend this concept also to ideals, by defining a non-zero ideal I to be an F-ideal if mM is prime whenever $M \not\supseteq I$ and M is a maximal ideal in $C(X)$. We are then able to show that I is an F-ideal if and only if its structure space is an F'-space, an analog to the theorem that X is an F-space if and only if βX is an F-space. We are also able to characterize those spaces whose rings of continuous functions possess an F-ideal.

Preliminaries and notations. The reader is referred to section 2 in [4] for most of the preliminaries. Familiarity with [2] is also assumed.

If $f \in C(X)$, then $Z(f) = \{x \mid f(x) = 0\}$, $\operatorname{pos} f = \{x \mid f(x) > 0\}$, and $\operatorname{neg} f = \{x \mid f(x) < 0\}$. If $f \in C^*(X)$ (i.e. f is bounded), then $\hat{f}$ denotes the extension of f to βX. In general $Z(\hat{f}) \supseteq Z(f)^\beta$ $\left(= \operatorname{cl}_{\beta X} Z(f)\right)$ but $\operatorname{int}_{\beta X} Z(\hat{f}) = \operatorname{int}_{\beta X} Z(f)^\beta$.

We shall use the letter M for maximal ideals of $C(X)$, and $M_x = \{f \mid f(x) = 0\}$.

We regard βX as the structure space of $C(X)$. Thus if U is open in βX, $U = \sim\{M \mid M \supseteq I\}$ for some ideal I in $C(X)$.

For any ideal I in $C(X)$, the maximal ideals of I are precisely those ideals of the form $I \cap M$ for $M \not\supseteq I$. These will be denoted by I_M. The structure space of real ideals of I is denoted by ϱI, and the structure space of all maximal ideals of I is denoted by μI. The space μI can be identified with an open subset of βX. (See [4, 3.9].)

A point $x \in X$ is said to be a *P-point* of X if $mM_x = M_x$. (See [2, 4L].)

A space X is said to be an F'-space if mM_x is prime for every $x \in X$. (See [1, 8.13].)

Two non-empty subsets A and B of X are said to be *completely separated* in X if there exists $f \in C(X)$ with $f(A) = \{0\}$ and $f(B) = \{1\}$.

1. P-ideals.

1.1. Definition. A non-zero ideal I is said to be a *P-ideal* of $C(X)$ if every proper prime ideal of I is maximal in I.

1.2. Lemma. *I is a P-ideal if and only if every prime ideal of $C(X)$ which doesn't contain I is maximal in $C(X)$.*

Proof. Suppose I is a P-ideal and P is prime in $C(X)$ with $P \not\supseteq I$. Then $I \cap P$ is maximal in I, whence by [4, 3.6], $I \cap P = I \cap M$ for some maximal ideal M in $C(X)$. It follows that $P = M$. The converse follows easily.

1.3. Remark. It follows easily from Lemma 1.2 that arbitrary sums and products of P-ideals are P-ideals.

1.4. Lemma. *If I is a P-ideal, then $I = mI$.*

Proof. Let $f \in I$, and assume $f \notin mI$. Since mI is semiprime in $C(X)$, there exists P prime in $C(X)$ with $P \supseteq mI$ and $f \notin P$. But then $P \not\supseteq I$, whence P is maximal in $C(X)$. Since $\overline{P} \supseteq mI$, we have a contradiction.

1.5. Theorem. *Let I be an ideal of $C(X)$. Then the following are equivalent.*

(1) *I is a P-ideal.*

(2) *For all ideals A in I, $mA = A = \bar{A}$. ($\bar{A}$ denotes the closure of A in the relative m-topology on I.)*

(3) *For all $f \in I$, $Z(f)$ is open.*

Proof. (1) ⇒ (2) Let A be a proper ideal in I. Since $I = mI$, A is contained in at least one maximal ideal of I by [4, 3.7]. We have

$$\begin{aligned} mA &= \bigcap \{P \mid P \text{ is prime in } I \text{ and } P \supseteq mA\} \\ &= \bigcap \{K \mid K \text{ is maximal in } I \text{ and } K \supseteq mA\} \\ &= \bigcap \{M \mid M \text{ is maximal in } C(X) \text{ and } M \supseteq mA\} \quad (\text{by } [4]) \\ &= m\bar{A} \quad (\text{by } [2], 7Q) \\ &= \bar{A} \quad (\text{by } [4], 2.5). \end{aligned}$$

Since $mA \subseteq A \subseteq \bar{A}$, we have equality.

(2) ⇒ (3) Let $f \in I$ and form $A = \{nf + if \mid i \in I \text{ and } n = 0, \pm 1, \pm 2, \ldots\}$, an ideal of I. Since $f \in A$, $f = fa$ for some $a \in A$, so $f = f(nf + if)$. It follows that $Z(f)$ is open.

(3) ⇒ (1) Let P be prime with $P \not\supseteq I$, and let M be the maximal ideal of $C(X)$ which contains P. Let $f \in I \cap M$, and define $i = 0$ on $Z(f)$ and $i = 1/f$ on $\sim Z(f)$. Then $f = if^2 \in mI \cap mM \subseteq P$. Thus $I \cap M = I \cap P$, and it follows that $P = M$.

We remark that several other statements similar to the ones in [2, 14.29] can be found which are equivalent to I being a P-ideal.

It is evident that every ideal in a P-space is a P-ideal. It is also clear that if a maximal ideal M is a P-ideal in $C(X)$, then X must be a P-space. Below we list some easy examples of P-ideals in spaces which are not P-spaces.

1.6. Example. Let $X = N^*$, the one-point compactification of the discrete space of counting numbers N. Let I be the ideal of functions which are eventually zero. Then I is a P-ideal and X is not a P-space. (Nor is X an F-space.)

1.7. Example. The ideal $O_\sigma = mM_\sigma$ in the space Σ [2, 4M] is a prime P-ideal, and Σ is not a P-space. (It is easily seen that $Z(f)$ is open for any $f \in mM_\sigma$.)

We shall use the notation ΔI for the (possibly empty) set of all x with $M_x \supseteq I$. We then have

1.8. Lemma. *If I is a P-ideal and $x \in X \setminus \Delta I$, then x is a P-point of X. In particular, if I is a free P-ideal, then X is a P-space.*

Proof. Trivial.

It is evident that the natural isomorphism of $C(X)$ onto $C(vX)$ preserves P-ideals. Thus if I is a P-ideal of $C(X)$, then I^v (its image under the natural isomorphism) is a P-ideal in $C(vX)$. The structure space of real ideals of I^v is a collection of fixed ideals each of which can be identified (by 1.8) with a P-point of vX.

1.9. Theorem. *If I is a P-ideal then its structure space of real ideals (ϱI) is a P-space. Conversely, if ϱI is a P-space, then mI is a P-ideal.*

Proof. The first part of the theorem follows from the above discussion.

For the second part of the theorem, we may assume without loss of generality that X is realcompact. Consider $f \in mI$. Then $Z(f) \supseteq X \setminus Z(h) \supseteq Z(i)$ for some $h \in C(X)$ and $i \in I$. If $x \in Z(i)$, then $x \in \operatorname{int} Z(f)$. If $x \notin Z(i)$, then $M_x \not\supseteq I$, from which it follows that x is a P-point and $x \in \operatorname{int} Z(f)$. We have shown that $Z(f)$ is open, and hence the result follows by 1.5.

1.10. Remark. The ideal I could be taken to be all of $C(X)$ in the above theorem, and we would then have that X is a P-space if and only if vX is a P-space. Since $vX = \varrho(C(X))$, we have a partial generalization of this result. It is easy to obtain an example of an ideal I which is not a P-ideal but whose structure space of real ideals is a P-space. (Let $I = M_\infty$ in $C(N^*)$ where N^* is the one-point compactification of N.)

We now characterize those spaces whose rings of continuous functions have P-ideals.

1.11. THEOREM. *$C(X)$ has a P-ideal if and only if the set of P-points in vX has non-empty interior in vX.*

Proof. Assume I is a P-ideal in $C(X)$. Without loss of generality, we may assume that X is realcompact. If $\Delta I = \emptyset$, then X is a P-space, so assume $\Delta I \neq \emptyset$. Then $X \setminus \Delta I$ is contained in the set of all P-points of X, and its complement ΔI can not equal X.

Conversely, suppose that the set of P-points in vX contains a non-empty open set U in vX. Let $I^v = \{g^v|\ Z(g^v) \supseteq vX \setminus U\}$ and consider the ideal mI^v in $C(vX)$. Clearly, I^v is not the zero ideal. Arguing as in 1.9, it follows that mI^v is a P-ideal, and hence mI is a P-ideal in $C(X)$.

2. F-ideals. It is natural to attempt to extend the notion of F-spaces to ideals just as it was done in the previous section with P-spaces. The situation is somewhat more complicated, however.

2.1. DEFINITION. A non-zero ideal I of $C(X)$ is called an *F_1-ideal* in $C(X)$ if mK is prime in I for every maximal ideal K in I.

The above definition would seem to be the natural extension, but unfortunately not every ideal in an F-space will be an F_1-ideal.

2.2. LEMMA. *If mI_M is prime in I, then $mI_M = I \cap (mM)$ and mM is prime in $C(X)$.*

Proof. Since mI_M is prime in I, there exists P prime in $C(X)$ with $mI_M = (mI) \cap (mM) = I \cap P$. Since $M \not\supseteq I$, there exist $m \in mM$ and $i \in mI$ with $m+i = 1$. Thus for any $p \in P$, $p = pm + pi \in mM$, and it follows that $P = mM$.

2.3. LEMMA. *If mI_M is prime in I for some maximal ideal I_M in I, then $I = mI$.*

Proof. There exist $m \in mM$ and $i \in mI$ so that $m+i = 1$. For any $g \in I$, $g = gm + gi \in mI$, since $I \cap (mM) = (mI) \cap (mM)$.

We note that it is possible for mM to be prime in $C(X)$ with mI_M not prime in I. For example, choose $I \neq mI$ in $C(\beta N)$, where I is a free ideal. Then for any maximal ideal $M \not\supseteq I$, mM is prime but $m(I_M)$ is not prime in I by 2.3.

Since it is necessary that $I = mI$ for I to be an F_1-ideal, not every ideal in an F-space will be an F_1-ideal.

To remedy this situation, we modify our definition, guided by Lemma 1.2 in the previous section.

2.4. DEFINITION. A non-zero ideal I in $C(X)$ is said to be an *F-ideal* if mM is prime whenever M does not contain I.

Trivially, every ideal in an F-space is an F-ideal, and every F_1-ideal is an F-ideal. Also sums and products of F-ideals are F-ideals.

2.5. LEMMA. *For any $f \in C(X)$,* pos$f$ *and* neg$f$ *are completely separated if and only if there is a function $g \in C(X)$ with $f = g|f|$.*

Proof. See [2, 14.22].

2.6. THEOREM. *An ideal I in $C(X)$ is an F-ideal if and only if* pos$f$ *and* neg$f$ *are completely separated for every $f \in mI$.*

Proof. Suppose I is an F-ideal and $f \in mI$. We form the ideals $K = \{k \in C(X)|\ \text{pos} f \subseteq Z(k)\}$ and $J = \{j \in C(X)|\ \text{neg} f \subseteq Z(j)\}$. Assume $K+J$ is contained in a maximal ideal M in $C(X)$. We claim that $M \supseteq I$. To see this, assume $M \not\supseteq I$. By hypothesis, either $f \vee 0 \in mM$ or $f \wedge 0 \in mM$. If $f \vee 0 = (f \vee 0)m$ for some $m \in M$, then $1 - m \in K \subseteq M$, a contradiction. Similarly, we can not have $f \wedge 0 \in mM$, and hence $M \supseteq I$. Thus the ideal $I \cap (K+J)$ is contained in no maximal ideal of I, from which it follows by [4, 3.7] that $mI \subseteq I \cap (K+J)$. If $f = fi$ where $i \in mI$, then $i = k+j$ for some $k \in K$ and $j \in J$. It is easily verified that $f = (j-k)|f|$.

Conversely, suppose pos$f$ and neg$f$ are completely separated for any $f \in mI$, and consider $M \not\supseteq I$. It suffices to show that mM is pseudo-prime. To this end, suppose $g \cdot h = 0$ with g and h non-negative functions in $C(X)$. There exist $m \in mM$ and $i \in mI$ so that $m+i = 1$ with m and i non-negative. We let $f = gi - hi \in mI$, and form the ideals J and K as above. Since pos$f$ and neg$f$ are completely separated, it follows that there exist $j \in J$ and $k \in K$ with $j+k = 1$. If $j \notin M$, then for some $a \in C(X)$ and $m_1 \in M$, $aj + m_1 = 1$, and hence $Z(m_1) \subseteq X \setminus Z(j) \subseteq Z(f \vee 0) \subseteq Z(gi)$. Thus $gi \in mM$, from which it follows that $g = gm + gi \in mM$. Similarly, if $k \notin M$, we could infer that $h \in mM$.

We observe that I could be taken to be all of $C(X)$ for an F-space X in the above proof.

It is possible to have a function f in an F-ideal so that pos$f$ and neg$f$ are not completely separated.

2.7. EXAMPLE. Let $X = [-1, 1]$ with every point discrete except 0, and a neighborhood of 0 is a neighborhood in the usual topology. Then $M_0 = \{g \in C(X)|\ g(0) = 0\}$ is an F-ideal since pos$f$ and neg$f$ are completely separated for any $f \in mM_0$. (Indeed, if $f \in mM_0$, then $Z(f)$ is open.) Of course, the identity function i is in M_0, and pos$i$ and neg$i$ are not completely separated.

We now wish to obtain a structure-space characterization of F-ideals similar to 1.9. We recall that $C(X)$ is an F-ideal (i.e. X is an F-space) if and only if its structure space (βX) is an F-space.

We first define an analog of a P-point in a space.

2.8. Definition. Let $y \in Y$. Then y is an *F-point* in Y if for any $f \in C(Y)$ there exists a neighborhood U of y so that f does not change sign on U.

2.9. Lemma. *The following are equivalent for any $y \in Y$.*

(1) *y is an F-point of Y.*

(2) *If $f \cdot g = 0$, then $y \in \operatorname{int} Z(f)$ or $y \in \operatorname{int} Z(g)$.*

(3) *mM_y is prime.*

Proof. (1) ⇒ (2) Suppose $f \cdot g = 0$ and form $h = |f| - |g|$. Then there exists an open set U with $y \in U$ and $h(x) \geqslant 0$ say, for all $x \in U$. Then $g(x) = 0$ for all x in U.

(2) ⇒ (3) It suffices to show that mM_y is pseudoprime, so consider $f \cdot g = 0$. Suppose $y \in \operatorname{int} Z(f)$. Choosing $h \in C(X)$ so that $h(y) = 1$ and $h = 0$ on $\sim(\operatorname{int} Z(f))$, we have that $f = f(1-h) \in mM_y$.

(3) ⇒ (1) Given $f \in C(X)$, $(f \vee 0) \cdot (f \wedge 0) = 0$.

2.10. Remark. It is easily seen that X is an F'-space if and only if every point in X is an F-point of X. Thus, for an F-ideal I, if $x \in X \setminus \Delta I$, then x is an F-point of X. In particular, if I is a free F-ideal, then X is an F'-space. It is also not hard to show that mM^p is prime in $C(X)$ if and only if p is an F-point of βX, and hence X is an F-space if and only if every point in βX is an F-point of βX.

2.11. Theorem. *An ideal I is an F-ideal if and only if μI is an F'-space.*

Proof. Suppose I is an F-ideal and let $y \in \mu I$. Suppose $f \in C(\mu I)$ and consider an open set U in βX so that $y \in U \subseteq \overline{U}$ (in βX) $\subseteq \mu I$. There exists $h \in C(\beta X)$ with $h|\overline{U} = f$. Now y is an F-point of βX so there exists V open in βX with $y \in V$ so that h does not change sign on V. Clearly f does not change sign on $U \cap V$.

Conversely, suppose μI is an F'-space. Then for any $p \in \mu I$, p is an F-point of μI. To see that p is an F-point of βX, consider $\hat{f} \in C(\beta X)$. Then there exists U open in μI so that $\hat{f}|U$ does not change sign. Since U is open in βX, we have a neighborhood of p (in βX) on which f does not change sign. Since p is an F-point of βX, mM^p is prime, and we have that I is an F-ideal.

2.12. Theorem. *Let A denote the set of F-points of βX. Then $C(X)$ has an F-ideal if and only if A has non-empty interior in βX.*

Proof. If I is an F-ideal, then $\emptyset \neq \mu I \subseteq A$.

Conversely if $\operatorname{int} A \neq \emptyset$, consider $\operatorname{int} A = \sim\{M | M \supseteq I\}$ for some non-zero ideal I in $C(X)$. Clearly I is an F-ideal.

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