THEOREM D1. Let $P$ be a $\lor$-semilattice with a partial $\land$ taking $\forall$-b, as values such that the existence of $a \land b$ implies $(a \lor \forall) \land (b \lor \forall) = (a \land b) \lor \forall$ for every $\forall$. Then $P$ is embedded in the distributive lattice $P$ if freely generates, and every congruence (for both the total and partial operation) of $P$ is the restriction of a lattice congruence on $P$.

This furnishes in particular a solution for the distributive case of Problem 29 in [G].

References


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Topologically nondegenerate functions

by

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Abstract. Let $M_\alpha$ be a compact, connected topological manifold and $F$ a continuous real valued function on $M_\alpha$ that is topologically nondegenerate in the sense of Morse [12]. Let $c$ be an arbitrary value of $F$ and set

$$F_\alpha = \{ \phi \in M_\alpha : F_\phi \leq c \}.$$

The “topological critical points” of $F$ on $F_\alpha$ are finite in number and can be related to the invariants of the homology groups of $F_\alpha$ as in the differentiable case (Morris and Cairns [14]). $F$-deformations and $F$-tractions make this possible. $F$-tractions are our extensions of retracting deformations of Borsuk [1]. Kirby and Siebenmann in [7] have affirmed the existence of topologically nondegenerate functions on $M_\alpha$ when $n \neq 4$ or 5.


Introduction. This paper is concerned with continuous, real-valued, topologically nondegenerate functions $F$, as distinguished from differentiably nondegenerate functions. (See § 1 for definitions.) The domain of $F$ is taken as a compact topological manifold $M_\alpha$. The paper [14] of Morse and Cairns is here extended from the differentiable case to the topological case. A brief abstract of this paper is found in [15].

Singular homology theory is used of the type first introduced by Eilenberg in 1944. See reference [6]. No “triangulations” are needed. Deformations termed “tractions”, are fundamental; they relax the conditions on “retracting deformations” as commonly defined. For original concepts see Borsuk [10]. The theorem of Kirby and Siebenmann on the existence of topologically nondegenerate functions, when $n \neq 4$ or 5, is a starting point. This paper draws heavily on Morse [13] in which topologically nondegenerate functions were first defined. Paper [16] recognizes the classical group structure for use in the necessary homology theory.

To avoid complexity in a first treatment this study has been subjected to many restrictions that can be readily removed. In particular, one could greatly lighten the condition that the manifold be compact. One could also remove the condition that the topological critical values be of simple type in the sense of § 0.
Part II. The (*) FG of the homology groups $H^q(F_\gamma, Z)$. Part II establishes the basic theorem, that if $e$ is an arbitrary value of $F$, the homology groups $H^q(F_\gamma, Z)$ are FG. The singleton critical values of $F$ are listed as a sequence

$$a_0 < a_1 < a_2 < ... < a_n.$$  

It is found that as $e$ increases from $a_0$ to $a_1$, the groups $H^q(F_\gamma, Z)$ remain isomorphic, except at most when $e$ increases from a value in an interval $[a_{i-1}, a_i]$ to $a_i$. In Part II we show that as $e$ increases, $H^q(F_\gamma, Z)$ remains FG without exception. One shows, by an inductive proof, that each of the groups $H^q(F_\gamma, Z)$ is FG.

Part III. Critical invariants of $T$-critical points $p_\alpha$ of $F$. Part III is concerned with the changes in the integral invariants (3) of the groups $H^q(F_\gamma, Z)$ as $e$ changes from $a_0$ to $a_1$. We shall associate a finite set of integers with each singleton critical value $a$ in the list (0.4) and term these integers critical invariants of $p_\alpha$ and $F_\gamma$. Foremost among these critical invariants of $p_\alpha$ are its T-index $\kappa_a$, its "free index" $\kappa_a^F$, and its positive "torsion (4) index" $\kappa_a^T$, defined only when $\kappa_a^F = 0$ and $\kappa_a^T > 0$. Free indices and torsion indices are defined in (16). Paper (16) gives the group theoretic background of this paper, while (12) gives its more geometric background.

Given an arbitrary value $e$ of $F$, the theorems of Part III show that the integral invariants of the groups $H^q(F_\gamma, Z)$ are determined by the critical invariants of the largest of the critical values $e = a_0 < a_1$ in the list (0.4) and the integral invariants of $H^q(F_\gamma, Z)$ where $\tilde{F}_\gamma = F_\gamma$.

The invariance of critical invariants. Let $h$ be a homeomorphism $\tilde{F} \rightarrow h(p)$ of the manifold $M_\gamma$ onto a manifold $M_\gamma$. Let $\tilde{F}$ be a continuous mapping of $M_\gamma$ into $F$ such that $F_\gamma(p) = F(h(p))$ for $p \in M_\gamma$. Our definitions will show the following. $\tilde{F}$ is TND if and only if $F$ is TND. If $p_\alpha$ is a T-critical or T-ordinary point of $F$, then $h(p_\alpha)$ will be, respectively, a T-critical or T-ordinary point of $\tilde{F}$. Each critical invariant $N$ of $p_\alpha$ will be a critical invariant of $h(p_\alpha)$.

The existence of TND functions on $M_\gamma$. If $M_\gamma$ is a compact differentiable manifold of class at least $C^r$, the existence of a TND function on $M_\gamma$ was made clear by Morse in 1927. See (16) and § 6 of (15). Eells and Kuiper in (4) have gone beyond the differentiable case and established the existence of TND functions on combinatorial manifolds. More generally Kirby and Siebenmann have affirmed the existence of TND functions.

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(*) FG abbreviates "finite generation" or "finitely generated."

(3) The integral invariants of a homology group are meant its Betti number and torsion coefficients.

(4) The torsion index $\kappa_a^T$ of a T-critical point $p_\alpha$ is not to be confused with a torsion coefficient of $H^q(F_\gamma, Z)$ or $H^q(\tilde{F}_\gamma, Z)$.

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(5) T abbreviates topological; ND abbreviates nondegenerate.
on each topological manifold $M_n$ for which $n$ is neither 4 nor 5. This
affirmation is found in a deep study [7] of the classification theorems
of metrizable topological manifolds of finite dimension. TND functions
on a compact topological manifold were first defined and studied in [12].
We here continue this study, seeking to make clear the implications
of the existence of TND functions in homotopy and homology theory.
The general concepts and theorems of Cerf [3] are part of this evolving theory.

Noncompact manifolds. The restriction of this paper to compact
manifolds is for simplicity only. One could obtain similar results on
a connected, noncompact manifold provided with a TND function $F$
such that for each value $x$ of $F$, $F_x$ is compact. Cf. Theorem 23.5 of [15]
for the differentiable case. In this more general case there may be a countably
infinite set of $T$-critical points.

Part I. $F$-tractions on $M_n$

§ 1. TND functions $F$ on $M_n$. We shall recall the definition of a TND
function $F$ on $M_n$, as given in [12].

Let $x_1, \ldots, x_n$ be coordinates of a point $x \in M^n$. Let $Z_{x}$ be an origin-centered
closed $n$-ball in $M^n$ of center $x$, and $Z_{x}$ its interior. Let $0$ be the null
$n$-tuple $(0, \ldots, 0)$.

**Definition 1.1.** A point $q \in M_n$ will be called a $T$-ordinary point
of $F$, if there exists an injective homeomorphism

$$ z \to \Phi(q)(z); \quad Z_0 \to M_n $$

such that $\Phi(q)(0) = q$ and for some sufficiently small scalar $\delta > 0$

$$ F(\Phi(q)(z)) = F(q) + \delta \cdot 0 \quad (z \in Z_0) $$

where $z_0$ is the $s$-th of the coordinates $x_1, \ldots, x_n$ of $x$.

The condition (1.2) implies that the level sets of $F$ in the neighborhood
$\Phi(Z_{x})$ of $q$ are images under $\Phi$ of open sets on a set of parallel
$(n-1)$-planes.

**Definition 1.2.** $T$-critical points of $F$. A point $c$ of $M_n$ which is
not a $T$-ordinary point of $F$ will be called a $T$-critical point of $F$.

Let $k$ be an integer on the range 0, 1, ..., $n$. Set

$$ -s_1 - s_2 - s_3 + s_4 + \ldots + s_n = Q(c) \quad (z \in M^n). $$

A $T$-critical point $c$ of $F$ will be said to have a $T$-index $k$ if there exists
a homeomorphism (termed canonical) of form

$$ z \to \Phi(c)(z); \quad Z_0 \to M_n $$

(1) Our reference to Kirby and Siebenmann is to one of a series of papers by these
authors on topological manifolds. It was obtained by writing to Kirby and Siebenmann.

into $M_n$, such that $\Phi_c(0) = c$ and for fixed $c$ and some sufficiently small scalar $\delta_c$

$$ F(\Phi_c(\delta); F(c)) = Q(c) \Phi_c(\delta) \quad (\delta \in Z_0). $$

We set

$$ \Phi_c(Z_0) = N_x, \quad \Phi_c(Z_0) = N_x^* $$

as $1$ in [12] and term $N_x$ and $N_x^*$ inner and outer canonical neighborhoods
of $c$ on $M_n$.

When $k = 0$, (1.5) shows that $F(p) \not\geq F(c)$ for $p \in N_x^*$. When $k = n,$
$F(p) \not\leq F(c)$ for $p \in N_x^-$.\n
**Definition 1.3.** TND functions $F$. The function $F$ will be termed
TND, if corresponding to each $T$-critical point $c$ of $F$, there exists
a canonical homeomorphism $\Phi_c$ of $Z_0$ into $M_n$, conditioning $F$ as in (1.5)
and defining a $T$-index $k$ of $c$.

The condition (1.5) requires that the level sets of $F$ in the neighborhood
$\Phi_c(Z_0)$ of $c$ be the $T$-images of level sets of the quadratic form $Q_c$
in a neighborhood of the origin in $M^n$.

We assume that $F$ is TND.

The set of $T$-ordinary points of $F$ in $M_n$ is clearly open. The set of
$T$-critical points of $F$ is accordingly closed in $M_n$ and hence compact,
since $M_n$ is compact. If $c$ is a $T$-critical point of $F$, one shows readily
that each point in $N_x^*$ other than $c$, is $T$-ordinary. As isolated points in
the compact set of $T$-critical points of $F$, $T$-critical points of $F$ are finite
in number.

A metric on $M_n$. As a compact topological manifold, $M_n$ admits
a metric which induces the topology with which $M_n$ is given.

$T$-critical values. If $c$ is a $T$-critical point of $F$, $F(c)$ will be called
a $T$-critical value of $F$. A value of $F$ which is not $T$-critical will be called
$T$-ordinary. If $c$ is a $T$-ordinary value of $F$, the level manifold $F^{-1}$ is a compact
$(n-1)$-manifold without boundary. If $c$ is a $T$-critical value
of $F$, the deletion of $T$-critical points of $F$ on $F^{-1}$ from $F^n$ will yield an
open $(n-1)$-manifold on $M_n$.

Singleton $T$-critical points and values of $F$ have been defined in § 0.
A singleton critical value $c$ will be assigned a $T$-index $k = k_c$ equal to
the $T$-index of the unique $T$-critical point $p_c$ for which $F(p_c) = c$. We
shall prove the following.

**Theorem 1.1.** If there exists a TND function $F$ on $M_n$ there exists
a TND function $F$ on $M_n$ of all whose $T$-critical values is a singleton
$T$-critical value.

(1) Conditions (1.2) and (1.4) in reference [12], satisfied on the right by scalars $q_0$
and $q_0$, respectively, should hold for some choice of these scales as positive numbers.
Moreover, if $F$ is given, $\hat{F}$ can be defined so as to have the same $T$-critical points as $F$ with the same $T$-indices and with singleton critical values which differ arbitrarily little from the $T$-critical values which they replace. The following lemma implies this. In this lemma, as elsewhere, $C\hat{X}$ will denote the complement in $\hat{M}_a$ of a subset $X$ of $M_a$.

**Lemma 1.1.** Let $F$ be a TND-function on $M_a$ and $a$ a $T$-critical point of $F$ for which $F(a)$ is not a singleton $T$-critical value of $F$. Let $e$ be a pre-scribed positive constant.

It is possible to redefine $F$ in a closed subneighborhood $H$ of the neighborhood $N^*_a$ of $a$, leaving $F(p)$ unchanged for $p \in CH$, so as to replace $F$ by a TND-function $\hat{G}$ on $\hat{M}_a$ such that

(i) $a$ is the only $T$-critical point of $\hat{G}$ in $H$.

(ii) The $T$-index $k$ of $\hat{G}$, relative to $\hat{F}$, is the $T$-index of $a$ relative to $G$.

(iii) $G(a)$ is a singleton critical value of $G$ such that

$$0 < |G(a) - F(a)| < e.$$  \hspace{1cm} (1.7)

Proof. One readily defines a $C^\omega$-mapping

$$z \mapsto \lambda(z): R^n \to R$$  \hspace{1cm} (1.8)

such that $\lambda(z) = 1$ for $|z| < 1$ and $\lambda(z) = 0$ for $|z| > \frac{1}{2}$. If $e > 0$ is sufficiently small the mapping $z \mapsto \mu(z)$ with values

$$\mu(z) = -z_1 - \ldots - z_{k-1} + z_{k+1} + \ldots + z_n + e\lambda(z) \quad (z \in R^n)$$  \hspace{1cm} (1.9)

has no critical point other than $z = 0$.

Definition of $\hat{G}$. We refer to the homeomorphism $\Phi_a$ of $Z_a$ onto $N^*_a$ introduced in (1.4) and set

$$H = \Phi_a(Z_a) \quad (q = \frac{1}{2}).$$  \hspace{1cm} (1.10)

Then $H$ is a compact subset of $N^*_a$ and $CH$ is open in $M_a$. Set

$$G(p) = F(p) \quad (p \in CH),$$  \hspace{1cm} (1.11)

$$G(p) = F(p) + \mu(q) \quad (p \in N^*_a)$$  \hspace{1cm} (1.12)

subject to the condition, $p = \Phi_a(z)$ for $|z| < 2$. The sets $CH$ and $N^*_a$ are open subsets of $M_a$ whose union is $M_a$ and whose intersection is $N^*_a - H$.

Both (1.11) and (1.12) define $G(p)$ on this intersection, but consistently (1).

From (1.12) and (1.5) we infer that

$$G(p) = F(p) + e\lambda(z), \quad (p \in N^*_a),$$  \hspace{1cm} (1.13)

taking account of the fact that $\lambda(z) = 1$ in (1.9) when $|z| < 1$.

We continue with a proof of the following.

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$(a)$ The mapping $G$ is TND on $M_a$.

To verify $(a)$ set $N^*_a - a = N^*_a$. Note that the subsets

$$N^*_a, CH, \hat{N}^*_a$$

of $M_a$ are open and have the union $M_a$. We shall show that the restrictions of $G$ to each of these sets is TND. This term is defined on open subsets of $M_a$ as on $M_a$. That $G|CH$ is TND follows from (1.11). That $G|CH$ is TND follows from (1.11). That each point of $H^*$ is $T$-ordinary relative to $G$ follows from (1.12), subject to the condition $p = \Phi_a(z)$, and from the fact that the mapping $x \mapsto \mu(x)$ of $R^n$ into $H$ has no critical point other than $x = 0$.

Thus $(a)$ is true.

Statement $(i)$ now follows from (1.12), while $(ii)$ follows from (1.13).

Statement $(iii)$ will be true, in accord with (1.13), if $e$ is sufficiently small.

This completes the proof of Lemma 1.1. Theorem 1.1 follows.

Singleton notation. It is assumed that $F$ is a TND-function on $M_a$ whose $T$-critical values $a$ are singleton values, and are listed in (0.4).

With each such value $a$ there is associated the unique $T$-critical point $p_a$ at the $T$-level $a$. The $T$-index of $p_a$ will be denoted by $k = k_a$ and will be termed the $T$-index of $a$, as well as the $T$-index of $p_a$. If $a = p_a$, the canonical homeomorphism $\Phi_a$ associated with $a$ as in (1.5), when $k = k_a$ will be denoted by $\Phi_a$. The canonical neighborhoods $N_a$ and $N^*_a$ of $a$ introduced in (1.6), will be denoted by $N^*_a$ and $N^*_a$, respectively.

§ 2. $F^*$-deformations. $F^*$-deformations will be defined after we have defined deformations of a more general character.

Let $t$ be a real variable termed the time. Let $I = [0, 1]$ denote an interval for $t$. With us a deformation of a subset $A$ of a topological space $X$ is a continuous mapping

$$\chi: I \to A \times X$$

such that $D(p, 0) = p$ for each $p \in A$. We shall denote $D(p, t)$ by $p_t$.

Given $p \in A$ we say that $p_t$ replaces $p$ under $D$ at the time $t$. We set $D(p, 2) = D_2(p)$ and term $D_2(p)$ the final image of $p$ under $D$.

By the carrier $|D|$ of $D$ is meant the image of $A \times I$ under $D$. We say that $D$ deforms $A$ on a subset $X$ of $\chi$ if $|D| \subset X$.

**Retracting deformations.** A deformation $D$ of $A$ is said to be a deformation retracting $A$ onto $D_1(A)$, if $D$ deforms $A$ on $A$ and leaves each point of $D_1(A)$ fixed.

In case $\chi$ is a differentiable manifold $M_a$, retracting deformations which are adequate for our purposes are relatively easy to define. When $M_a$ is no longer differentiable, retracting deformations of the
desired types may fail to exist. There is, however, a larger class of deformations which serve our purposes and will now be defined.

**Definition 2.1.** Tractions. A deformation $D$ of a subset $A$ of $x$ will be termed a traction of $A$ into a subset $B$ of $A$ (possibly $A$) if $D$ deforms $A$ on $B$ and deforms $B$ on $B$.

Each “deformation retracting $A$ onto $B$” is a “traction of $A$ into $B$” but a traction of $A$ into $B$ is not, in general, a deformation retracting $A$ onto $B$. However, a “traction of $A$ into $B$” shares with a “deformation retracting $A$ onto $B$” a fundamental property: there exist isomorphisms (1)

$$H_q(A, Z) \cong H_q(B, Z) \quad (q = 0, 1, \ldots, n).$$

See Theorem 3.3.

The following definition is given in [12], p. 192. A related definition is given in [11], p. 30.

**Definition 2.2.** An $F$-deformation of $A$ on $M_a$. A deformation of $A$ on $M_a$ which replaces each point $p \in A$ by a point $p_t$ at the time $t$ is called an $F$-deformation, if $F(p) \not= F(p_t)$ for each $t \in [0, 1]$ and $p \in A$, and is called a proper $F$-deformation of $A$, if in addition

$$(2.2) \quad F(p) < F(p_t) \quad \text{(whenever } p_t \not= p).$$

An $F$-deformation which is a traction is called an $F$-traction.

Recall that a $T$-critical value $a$ of $F$ is, by hypothesis, a “singleton value”, assumed at just one $T$-critical point of $F$ denoted by $a$. See singleton notation at end of §1.

**Definition 2.3.** The $T$-$(k$-disc$)$, $K^a$. For each integer $k \leq n$, we introduce the open $(k$-disc$)$

$$c_k^a = \{a \in R^n \mid x_1^a + \ldots + x_k^a < 4; x_{k+1} = \ldots = x_n = 0\},$$

in $R^n$. Note that $c_0^a = 0$ and that $c_{k+1}^a$ is an origin-centered $k$-ball in $R^n$ on which $\|a\| < 2$. Let $a$ be a $T$-critical value and $\sigma = p_a$, the $T$-critical point on the $k$-level $a$. Let $\Phi_a^k$ be the canonical homeomorphism $\Phi_a$ of $Z_a$ onto $N_a^k$, of Definition (1.6) when $\sigma = p_a$ and $a = a_a$, and denote this set $N_a^k$ by $N_a^{k}$. Using the same homeomorphism $\Phi_a^k$, we introduce a $T$-$(k$-disc$)$, the subset

$$K^a = \Phi_a^k(c_k^a) \quad (k = a_a)$$

of $N_a^k$. It is clear that $N_a^k$ and its subset $K^a$ satisfy the relation

$$(2.5) \quad K^a \subset N_a^k \cap F_a.$$

(1) With us an “isomorphism” is understood, a priori, to be surjective unless the contrary is noted.

(2) That a 0-disc 0 is open is a convention.

We say that $K^a$ and $N_a^k$ are related by $\Phi_a^k$. Note that $K^a$ is a $T$-$(k$-disc$)$ included in $N_a^k$, meeting $F_a$ in just one point $a$.

The following lemma is basic. Its formulation differs trivially from the formulation of Lemma (1) 3.1 of [12]. It is proved in [12].

**Lemma 2.1.** Let $\sigma = p_a$ be an arbitrary $T$-critical point of $F$ with $T$-index $k = a_a$. Let $N_a^k, N_a^{k-1}$ be canonical neighborhoods of $\sigma$, defined as in (1.6) when $\sigma = p_a$ and $a = a_a$. Corresponding to $\sigma$ there exists a proper $F$-deformation $A_\sigma$ of $M_a$ with the following properties.

I. $A_\sigma$ leaves $\sigma$ and $CN_a^\sigma$ pointwise fixed.

II. $A_\sigma$ displaces each point of $N_a^{k-1}$.

III. $A_\sigma$ deforms $N_a^k$ on $N_a^{k-1}$ into the $T$-$(k$-disc$)$ $K^a$ defined in (2.4).

IV. $A_\sigma$ deforms $K^a$ on $N_a^{k-1}$ onto $K^a$.

The $F$-level sections of $K^a$, when $k = a_a$. The definitions of $N_a^k$ in (2.3), and of $K^a$ in (2.4), show the following. If $1 < k < n$ and $c$ is a value of $F$ on $K^a$, the section of $K^a$ at the $F$-level $c$ is a topological $(k-1)$-sphere. If $k = n$, this section is a pair of distinct points. These sections of $K^a$ vary continuously with $c$ and shrink to $\sigma$ as $c$ increases.

When $k = 0$, $K^a = \sigma$.

In §3 of [12] the following Lemma is established as Lemma 3.1.

**Lemma 2.2.** There exists a proper $F$-deformation $A_\sigma$ of $M_a$ onto $M_a$ which leaves each $T$-critical point of $F$ fixed and displaces each other point of $M_a$.

The following lemma refers to a $T$-critical value $a$ of $F$ and to the subset

$$(2.6) \quad F_{\sigma} = \{p \in M_a \mid F(p) < a\}$$

of $M_a$. Theorem 3.1 of [12] is established with the aid of the deformation $A_\sigma$ of Lemma 2.2 and implies the following.

**Lemma 2.3.** Let $a$ be a $T$-critical value of $F$ and $(a, b]$ an interval free of $T$-critical values of $F$. If $\sigma = p_a$, there exists an $F$-traction $D$ of $F_a$ into $\sigma \cup F_{\sigma}$. ($\sigma$).

Let $N_a^{k-1}$ be a canonical neighborhood $N_a^k$, of $\sigma = p_a$, introduced in §1. Lemma 2.3 has the following corollary.

(1) In ref. [12], $T$ abbreviates topological. The following table of notational errors of (12) is appended:

| Page | 189, 190, 195, 192, 194, 195, 197 |
| Line | 13, 15, 14, 18, 63 |
| Symbol | $\Phi_a, \Phi_a$, $\sigma_a, Z, p \in N_a^{k-1}, F$ |

(2) In [12] $p_a$ is denoted by $\delta_{\sigma}$.

(3) $F_{\sigma}$ is empty when $a = a_a$. 
COROLLARY 2.1. Under the conditions of Lemma 2.3 the following is true. If \( a \in E \) is such that \( a - c \) is sufficiently small and positive, then \( D \) is an \( F \)-deformation of \( F_a \) into \( N_a^2 \cup F_a \) if \( k_a > 0 \), and into \( a \cup F_a \) if \( k_a = 0 \). If \( a = a_n \), \( F_a \) is empty.

The \( F \)-deformation \( D \) of Lemma 2.3 of \( F_a \) is into a final image \( D_t(F_a) \subset a \cup F_a \). This final image is closed and is accordingly included in \( N_a^2 \cup F_a \) for a suitable choice of \( c \), with \( a - c \) sufficiently small and positive.

When \( k_a = 0 \), \( N_a^2 \cup F_a = a \); hence \( D_t(F_a) \subset a \cup F_a \) when \( k_a = 0 \). (Cf. (1.5) and (1.6)).

Preparation for Theorem 2.1. The formulation of Theorem 2.1 requires the definition of a \( T \)-saddle of \( F \) at each \( T \)-critical point \( p_a \) for which \( k_a > 0 \). The set \( X^{\alpha} \), defined in (2.4), is on \( F_a \), and below the \( F \)-level \( a \), except for the point \( a \). \( X^{\alpha} \) will serve our purposes as a \( T \)-saddle of \( p_a \) if we cut off from \( X^{\alpha} \) all points definitely below a suitably chosen \( F \)-level \( c < a \).

DEFINITION 2.4. A \( T \)-saddle \( L^{\alpha} \) of \( F \) at \( p_a \). Our \( T \)-saddle is defined only when \( k = k_a > 0 \). Let \( c \) be a value of \( F \) on \( N_a^2 \) such that \( a > c > a \), where \( a \) is the \( T \)-critical value next below \( a \). We set

\[
L^{\alpha} = \{ p \in X^{\alpha} \mid a > F(p) > c \}
\]

and term \( L^{\alpha} \) a \( T \)-saddle of \( F \) at \( c = p_a \).

It is a consequence of (2.4), (2.7)', and the choice of \( c \) that

\[
L^{\alpha} \subset N_a^2 \cup F_a
\]

and that \( L^{\alpha} \) meets \( F_a \) only in \( a = p_a \).

Corollary 2.1 leads to the following basic theorem.

THEOREM 2.1. Let \( [a, \beta] \) be an interval in which \( a \) is the only \( T \)-critical value. Let \( \gamma \) be a value in \([a, \beta]\). If \( a > \gamma > a \), there exists an \( F \)-deformation \( \delta \) of \( F_a \) into \( L^{\alpha} \cup F_a \) when \( k_a > 0 \), and into \( a \cup F_a \) when \( k_a = 0 \).

Proof. By Corollary 2.1, there exists an \( F \)-deformation \( D \) of \( F_a \) into \( N_a^2 \cup F_a \) if \( k = k_a > 0 \), and if \( a > c \) is sufficiently small and positive. If \( X^{\alpha} \) is the \( F \)-deformation \( A_\delta \) of Lemma 2.1 when \( c = p_a \), then by (2.4) the product \( F \)-deformation \( \delta = A_\delta D \) will satisfy Theorem 2.1 when \( k > 0 \) and \( a - c \) is sufficiently small.

In case \( k = 0 \), \( D \) of Corollary 2.1 will serve as \( \delta \) of Theorem 2.1.

Thus Theorem 2.1 is true.

Permanent notation. If \( a = p_a \) is a \( T \)-critical point with \( T \)-index \( k = k_a > 0 \), a deletion of \( c \) from \( F_a \), or from the sets \( X^{\alpha} \) and \( L^{\alpha} \) included in \( F_a \), will yield sets to be denoted by

\[
\hat{F}_a, \hat{X}^{\alpha}, \hat{L}^{\alpha},
\]

respectively. \( \hat{F}_a \) is similarly defined and non-empty when \( k_a = 0 \), provided \( a \) is not the minimum \( T \)-critical value \( a_n \).

The following theorem supplements Theorem 2.1.

THEOREM 2.2. Let \( a \) and \( b \), with \( a < b \), be \( T \)-critical values of \( F \) such that \( (a, b) \) is an interval of \( T \)-ordinary values of \( F \). Then there exists an \( F \)-deformation \( \delta \) of \( F_b \) into \( F_a \).

The proof of Theorem 2.2 is begun by verifying the following statement.

(i) If \( b \) is given as in Theorem 2.2 and if \( c \) is such that \( b - a \) is sufficiently small and positive, then there exists an \( F \)-deformation \( \delta \) of \( \hat{F}_b \) into \( F_a \).

Proof of (i). Let \( k = k_a \) be the \( T \)-index of \( c = p_a \). Suppose that \( k = k_a > 0 \). It follows from Theorem 2.1, with \( a \) and \( \gamma \) of Theorem 2.1 both taken as \( b \) of Theorem 2.2, that if \( b - a \) is sufficiently small and positive, there exists an \( F \)-deformation \( D \) of \( F_b \) into a set of form \( N_b^2 \cup F_b \). It is trivial that there exists an \( F \)-deformation \( D \) of \( \hat{F}_b \) into \( F_b \), retracting \( N_b^2 \cup F_b \) onto \( F_a \), so that \( D \) is an \( F \)-deformation of \( \hat{F}_b \) into \( F_a \). Thus (i) is true when \( k_a = 0 \).

When \( k = k_a = 0 \), statement (i) follows from Lemma 2.2, since \( \hat{F}_b \) is compact when \( k_a = 0 \) and contains no \( T \)-critical point at the \( F \)-level \( b \).

It follows from Theorem 2.1 with \( a, b \) of Theorem 2.1 taken as \( a, c \), where \( a - c < b \), that there exists an \( F \)-deformation \( D' \) of \( \hat{F}_b \) into \( F_a \).

Hence the product deformation \( D'd \) is an \( F \)-deformation of \( \hat{F}_b \) into \( F_a \).

Thus Theorem 2.2 is true.

Introduction to Lemma 2.4. Let \( a = p_a \) be a \( T \)-critical point of \( F \) with a positive \( T \)-index \( k = k_a \). We seek a neighborhood \( X \) of \( a \) relative to \( F_a \), such that there exists an \( F \)-deformation of \( X \) which retracts \( X \) into a \( T \)-saddle \( L^{\alpha} \) of \( F_a \). Taking account of the definition of \( L^{\alpha} \) in (2.7)' and of \( X^{\alpha} \) in (2.4), one sees that a neighborhood of \( p_a \) relative to \( F_a \), with the desired property, is defined by the union \( X^{\alpha} \) of all sections of \( X^{\alpha} \) with \( F \)-levels in the interval \([c, a]\). We suppose that \( a - c < 1 \) and note that

\[
X^{\alpha} = N_a^2 \cup (F_a - F_{k_a}) = \Phi_k^a(A) \quad \text{(cf. (2.4))}
\]

where

\[
A = \{ x \in Z_a \mid 0 \geq LQ_a(x) \geq a - c \} \quad \text{(cf. (1.5))}
\]

LEMMA 2.4. When \( k = k_a > 0 \), there exists an \( F \)-deformation \( \delta \) of \( X^{\alpha} \), retracting \( X^{\alpha} \) onto the \( T \)-saddle \( L^{\alpha} \) of \( F \) at \( a = p_a \).

Let \( B \) be the subset of \( A \) of points \((x_1, \ldots, x_k, 0, \ldots, 0) \in A \). Then

\[
X^{\alpha} = \Phi_k^a(A); L^{\alpha} = \Phi_k^a(B) \quad \text{(cf. (2.7))}
\]

One readily shows that there exists an \( F \)-deformation retracting \( A \) onto \( B \), in which a point \( x \in A \) is 'replaced' by a point \( z(x) \in A \), as \( t \) increases.
from 0 to 1, with \( Q(s) \geq Q(0) \). Lemma 2.4 is satisfied by a deformation \( d \) in which, as \( t \) increases from 0 to 1, each point \( \Phi(t) \) in \( X^\infty \) is replaced by \( \Phi(t) \) as \( t \) ranges over \( A \).

§ 3. Relevant theorems in singular homology theory. We shall be concerned with singular homology on a topological space \( X \). Use will be made of Eilenberg’s definition in [9] of singular \( r \)-cells. No triangulations of \( X \) are presupposed. See also § 26 of [15].

Given \( X \) and the ring \( Z \) of rational integers, the singular \( r \)-cells on \( X \) are combined linearly, with coefficients in \( Z \), to define a \( Z \)-module, denoted by \( C_r(X, Z) \). The singular \( r \)-cells on \( X \) form a basis for \( C_r(X, Z) \) in the sense of Bourbaki [3], p. 11. The elements of \( C_r(X, Z) \) are termed \( r \)-chains. The homology groups \( H_r(X, Z) \) are well-defined for each rational integer \( r \). They are trivial if \( r < 0 \).

The following notational innovation is very useful.

**Definition 3.0.** \( \langle u_0', \ldots, u_r' \rangle \). If \( u' \) is an \( r \)-cycle of \( C_r(X, Z) \) then \( \langle u_0', \ldots, u_r' \rangle \rangle \) shall denote the subset of \( r \)-cycles in \( C_r(X, Z) \) which are homologous to \( u' \) on \( X \). One can regard \( \langle u_0', \ldots, u_r' \rangle \rangle \) as an element in \( H_r(X, Z) \).

**Definition 3.1.** The chain-transformation \( \tilde{\varphi} \) (Eilenberg). Let there be given a continuous mapping \( \varphi : X \to Y \) of a topological space \( X \) into a topological space \( Y \). A singular \( q \)-cell \( \sigma' \) on \( X \) is defined by the class of \( \varphi \)-equivariant mappings \( r \) of vertex-ordered simplicial \( q \)-simplices into \( Y \). Cf. § 26 of [15]. In a chain-transformation,

\[
\tilde{\varphi} : C_q(X, Z) \to C_q(Y, Z) \quad (q = 0, 1, 2, \ldots),
\]

\( \tilde{\varphi} \) is defined by \( \varphi' \), the image \( \varphi' \sigma \) of \( \sigma' \) on \( Y \) is defined by the compositions \( \varphi \tau \) of \( \tau \) with \( \varphi \) of the equivalent mappings \( r \) of \( Y \) which define \( \sigma' \). The mappings \( \tilde{\varphi}' \), so defined for cells \( \sigma' \), are extended linearly over \( Z \) to define the mappings (3.0). Eilenberg shows that \( \tilde{\varphi} \) is permutable with the boundary operator \( \partial \). Natural homomorphisms

\[
\tilde{\varphi}^* : H_q(X, Z) \to H_q(Y, Z) \quad (q = 0, 1, \ldots)
\]

are induced by \( \tilde{\varphi} \).

Let \( s \) be a \( q \)-cycle (over \( Z \)) on \( X \) and \( d \) a deformation of \( \varphi \) on \( X \). If \( d_i \) is the terminal mapping of \( d_i \), the homology \( s - d_i s \) is valid on the image under \( d \) of any carrier \( |s| \) of \( s \). By a carrier \( |s| \) of \( s \) is understood any subset of \( X \) on which the cycle \( s \) is well-defined. Cf. Corollary 3.7.1 of [15].

The \( F \)-transformations defined in § 2 induce isomorphisms as follows.

**Theorem 3.1.** Let \( \varphi \) and \( \varphi' \) be topological spaces, with \( \varphi' \) a subspace of \( \varphi \), and let \( d \) be a deformation of \( \varphi \) into \( \varphi' \). Isomorphisms

\[
\tilde{\varphi} : H_q(X, Z) \to H_q(Y, Z) \quad (q = 0, 1, \ldots)
\]

are induced in which, for each \( q \)-cycle \( s \) on \( X \), \( \langle \varphi(s) \rangle \langle s \rangle \) is mapped onto \( \langle \varphi(s) \rangle \langle s \rangle \).

The homomorphism \( G_q \) of Theorem 3.1 which we have affirmed to be an isomorphism of \( H_q(X', Z) \) onto \( H_q(X, Z) \) is induced by the inclusion mapping \( i \) of \( X' \) into \( X \). It is clearly an isomorphism if the following is true.

(a) Each \( q \)-cycle on \( X \) is homologous on \( X \) to a \( q \)-cycle on \( X' \).

(b) Each \( q \)-cycle on \( X' \) which is bounding on \( X \) is bounding on \( X' \). In fact \( d_i \) is surjective if (a) holds and has a null kernel if (b) holds.

**Proof of (a).** Since \( d \) deforms \( X \) on \( X' \), (a) is clearly true.

**Proof of (b).** Let \( d_i \) be the terminal mapping of \( d \). As is well-known, one can associate with \( d \) a linear homomorphism (see § 27 of [15])

\[
d : C_q(X, Z) \to C_{q+1}(X, Z) \quad (r = 0, 1, \ldots)
\]

such that for each \( r \)-chain \( s \) on \( X \),

\[
\partial d s = d \partial s - d s
\]

Moreover, the definition of \( d \) in [15] is such that a carrier \( \partial s \) exists on a subset \( X' \) of \( X \) if \( d \) deforms \( i |s| \) on \( X \). (Lemma 3.1 of [15].)

As in (b), let \( u' \) be a \( q \)-cycle on \( X \) such that

\[
\partial u' = \partial u',
\]

where \( u' \) is a \( (q+1) \)-chain on \( X \). By virtue of (3.3)

\[
d u' = \partial u' - \partial u',
\]

On applying \( \partial \) to the members of (3.4)" and making use of (3.4)" we find that

\[
0 = \partial \partial u' - \partial u' - \partial u',
\]

By hypothesis both \( \partial \) and \( \partial u \) are on \( X' \), so that (3.4)" implies that \( \partial u' = 0 \) on \( X' \), confirming (b).

**Theorem 3.1 follows from (a) and (b).**

**Relative homologies over \( Z \).** Relative homologies were introduced by Lefschetz. Suitably modified, relative homologies will serve in § 4 to characterize the effect on a group \( H_q(X, Z) \) of replacing \( X \) by \( X' \). Here \( a \) is a \( T \)-critical value of \( F \), with \( T \)-index \( \chi = k > 0 \).

Given a topological space \( X \), a subspace \( A \) of \( X \) is taken as a "modulus" and the pair \( (X, A) \) termed admissible. The \( \gamma \)-th relative homology group is denoted by \( H_q(X, A, Z) \). On passing from a field \( F \) to the ring \( Z \), Theorem 28.4 of [15] leads to the following theorem. The proof is similar to that in [15].

**Theorem 3.2.** Let \( (X, A) \) and \( (X', A') \) be admissible pairs with \( X' \subset X \) and \( A' \subset A \). Let \( d \) be a deformation retracting \( X \) into \( X' \) and \( A \) onto \( A' \), with \( d \), the terminal mapping of \( d \).
Corresponding to the inclusion mapping \( i \) of \((\chi', A') \) into \((\chi, A)\) the \( \text{chain transformation} \bar{\gamma} \) induces isomorphisms
\[
H_q(\chi', A', Z) \xrightarrow{\bar{\gamma}} H_q(\chi, A, Z) \quad (q = 0, 1, \ldots)
\]
under whose respective inverses the rel. homology class on \( \chi \) of a \( q \)-cycle \( \zeta_q \mod A \), corresponds to the rel. homology class on \( \chi' \) of \( \bar{\zeta}_q \mod A' \).

The proof of Theorem 28.4 in [15] shows the following:

(a) A \( q \)-cycle \( \zeta_q \mod A \) is homologous on \( \chi \mod A \) to the \( q \)-cycle \( \bar{\zeta}_q \mod A' \) on \( \chi' \mod A' \).

(b) A \( q \)-cycle \( \alpha' \) on \( \chi' \mod A' \) which is bounding on \( \chi \mod A \) is bonding on \( \chi' \mod A' \).

Statement (a) implies that the inclusion induced homomorphisms \( \bar{\gamma}_n \) are surjective, while (b) implies that the kernels of these mappings are null.

Theorem 3.3 follows.

Excision. Among the axioms of Eilenberg and Steenrod, formulated on page 11 of [6], is found the Excision Axiom. Our next theorem formulates a simplified version of the Excision Axiom adequate for our purposes.

**Theorem 3.3.** Excision. Let \( \chi \) be a metric space, \( A \) a proper subspace of \( \chi \) and \( A^* \) a subspace of \( A \) such that for some positive \( c \)
\[
(\chi - A) \supset C > 0 \quad (3.6)
\]
where \((\chi - A)\) is the open \( c \)-neighborhood of \( \chi - A \), relative to \( \chi \).

There are exist isomorphisms,
\[
H_q(\chi - A^*, A - A^*) : H_q(\chi, A, Z) \quad (q = 0, 1, \ldots) \quad (3.7)
\]
induced by the inclusion mapping
\[
i: (\chi - A^*, A - A^*) \rightarrow (\chi, A). \quad (3.8)
\]

The proof of Theorem 28.3 of [15] yields a proof of the above Excision Theorem, provided, of course, the field \( \mathbb{K} \) of [15] is replaced by the ring \( Z \) of integers.

**Definition 3.2.** \( \# \)-mappings \( J_q^\# \). The mappings of homology groups into homology groups which have been introduced in this section have all been isomorphisms. We shall now define an "inclusion induced" homomorphism
\[
J_q^\#: H_q(\hat{A}, Z) \rightarrow H_q(A, Z) \quad (3.9)
\]
which may or may not be an isomorphism, depending on the value of \( q \), on the \( T \)-index \( k_a \) of \( a \) and other integral critical invariants to be introduced.

Let \( i_* \) be the inclusion mapping of \( \hat{A} \) into \( A \) and \( \bar{\gamma}_n \) the corresponding chain-transformation
\[
\bar{\gamma}_n: \mathbb{C}_q(\hat{A}, Z) \rightarrow \mathbb{C}_q(A, Z) \quad (q = 0, 1, \ldots)
\]
For each \( q \) there is thereby induced a homomorphism of the form (3.8).

One sets (3.6) equal to \( J_q^\# \). We term \( J_q^\# \) a \( \# \)-mapping. Whether \( J_q^\# \) is an isomorphism or not, it is of basic importance. If \( \hat{A} \) is a \( q \)-cycle in \( \mathbb{C}_q(\hat{A}, Z) \), then in the notation of Definition 3.6
\[
J_q^\#([\langle \zeta_q, \hat{A} \rangle]) = [\langle \zeta_q, A \rangle]
\]
in accord with the definition of \( \bar{\gamma}_n \).

**Part II. The finite generation of the groups \( H_q(F_e, Z) \)**

§ 4. The homology groups \( H_q(F_{k_e}, Z) \). We shall characterize the groups \( H_q(F_{k_e}, \hat{A}, Z) \) when \( a > a_0 \). It will then be relatively easy to give an inductive proof that for each \( T \)-critical value \( a > a_0 \) of \( F_e \) in the list (0.4), \( H_q(F_e, Z) \) is FG and to conclude that \( H_q(F, Z) \) is FG for each value \( c \) of \( F \) on \( M_g \).

The neighborhoods \( X^{k_e} \) of \( \sigma = a \) relative to \( F_e \) when \( k = k_a > 0 \). \( X^{k_e} \) was defined in (2.9). Note the inclusions
\[
F_a \subset X^{k_e} \subset L^{k_e} \quad \text{(cf. (2.9)).} \quad (4.0)
\]
Deleting \( c \) from each of these sets one finds that
\[
\hat{F} \subset X^{k_e} \subset L^{k_e}. \quad (4.1)
\]

**Preparation for Theorem 4.1.**

**Lemma 4.1.** If \( i \) is the inclusion mapping,
\[
i: \langle X^{k_e}, X^{k_e} \rangle \rightarrow \langle F_a, F_e \rangle \quad (k = k_a > 0)
\]
then \( \bar{\gamma} \) induces the isomorphisms,
\[
\bar{\gamma}_n: H_q(X^{k_e}, X^{k_e}, Z) \rightarrow H_q(F_a, F_e, Z) \quad (q = 0, 1, \ldots). \quad (4.2)
\]

That the homomorphism \( \bar{\gamma}_n \) induced by the chain-transformation \( \bar{\gamma} \) is an isomorphism is a consequence of Excision Theorem 3.3. One identifies \((\chi, A)\) of the Excision Theorem with \((F_a, F_e)\) and sets \( F_a - X^{k_e} = A^* \).

With this understood
\[
A^* \subset A, \quad X^{k_e} = \chi - A^*, \quad X^{k_e} = A - A^*. \quad (4.3)
\]
The excision condition (3.6) is satisfied, since a sufficiently small \( c \)-neighborhood, relative to \( F_a \) of \( \chi - A = p_a \), is included in \( \chi - A^* = X^{k_e} \) when \( k = k_a > 0 \).
The isomorphism (4.3) follows from Theorem 3.3. According to Lemma 2.4 there exists an $F$-deformation retracting $X^{n_0}$ onto the $T$-saddle $L^{n_0}$, holding $\sigma = p_a$ fast. It follows from Theorem 3.3 that if $i$ is the inclusion mapping

$$i: (L^{n_0}, \tilde{X}^{n_0}) \rightarrow (\tilde{X}^{n_0}, X^{n_0}) \quad (k_a > 0)$$

the chain transformation $\tilde{\theta}$ induces isomorphisms,

$$H_d(L^{n_0}, \tilde{L}^{n_0}, Z) \cong H_d(X^{n_0}, \tilde{X}^{n_0}, Z) \quad (q = 0, 1, \ldots)$$

From this result and from Lemma 4.1 we infer the following.

**Theorem 4.1.** Let $\sigma = p_a$ be a $T$-critical point with positive $T$-index $k = k_a$. If $J$ is the inclusion mapping

$$J: (L^{n_0}, \tilde{L}^{n_0}) \rightarrow (F_a, \tilde{F}_a) \quad (k_a > 0)$$

the chain transformation $\tilde{J}$ induces isomorphisms,

$$H_d(L^{n_0}, \tilde{L}^{n_0}, Z) \cong H_d(F_a, \tilde{F}_a, Z) \quad (q = 0, 1, \ldots)$$

**Preparation for Theorem 4.2.** Theorem 4.2 will make clear what are the invariants of the Abelian groups (4.5)'s. By virtue of Definition 2.4 of the $T$-saddle $L^{n_0}$, one sees that there exists a homeomorphism,

$$\Theta_k: L^{n_0} \rightarrow \tilde{A}_k \quad (k = k_a > 0)$$

of $L^{n_0}$ onto an origin-centered $k$-ball $A_0$ in $R^n$, with $\Theta_k(p_a) = 0$. Set $\tilde{A}_k = A_0$. Under $\Theta_k$, $L^{n_0}$ is mapped homeomorphically onto $\tilde{A}_k$. A classical theorem then implies the following. (Cf. Theorem 21.3 of [15].)

**Lemma 4.2.** The chain-transformation $\Theta_k$ induces isomorphisms,

$$H_d(L^{n_0}, \tilde{L}^{n_0}, Z) \cong H_d(\tilde{A}_k, \tilde{A}_k, Z) \quad (q = 0, 1, \ldots, k = k_a > 0)$$

The Abelian group $H_d(\tilde{A}_k, \tilde{A}_k, Z)$, $k > 0$, is free, as is readily shown, and has a base of dimension $d_0$. This group is trivial except when $q = k$, and when $q = k$ has a base which consists of a single element of infinite order. Because of the isomorphisms (4.5)'s and (4.6) we infer the following.

**Theorem 4.2.** The homology groups (4.5)'s are free. They are trivial except when $q = k$. When $q = k > 0$, a base for these groups consists of a single element of infinite order.

**Preparation for Theorem 4.3.** According to Theorem 4.2 when the $T$-index of $\sigma = p_a$ is a positive integer $k = k_a$, a base of the free group $H_0(F_a, \tilde{F}_a, Z)$ consists of a single element of infinite order. Such an element is the homology class of special $k$-cycles $\mu_a$ on $F_a \mod \tilde{F}_a$ which we shall term a saddle $k$-cycle of $\sigma = p_a$, and shall now characterize.

Three conditions are required.

**Definition 4.1.** A prebase for a relative homology group. Given a relative homology group $H_d(X, \tilde{X}, Z)$ which is free and has a finite base, a set of non-trivial relative $q$-cycles, one from each relative homology class in a base for $H_d(X, \tilde{X}, Z)$, will be called a prebase for $H_d$. A prebase may be empty.

We seek a prebase for the group $H_d(F_a, \tilde{F}_a, Z)$, $k > 0$. According to Theorem 4.2 it will consist of one $k$-cycle on $F_a \mod \tilde{F}_a$. We shall define a prebase which is given by a single singular $k$-cell (taken $\mod F_a$) which is simply carried by $F_a$ in the sense of the following definition.

**Definition 4.2.** Simply-carried singular $q$-cell. A singular $q$-cell on $M_a$ is defined by an equivalence class (Hilbert's) of mappings $\tau: x \rightarrow M_a$ of vertex-ordered $q$-simplices $x$ into $M_a$. If the mappings $\tau$ are homoeomorphisms of their domains $x$ onto their images $\tau(x)$, the resultant singular $q$-cell on $M_a$ will be said to be simply-carried.

We give a fundamental definition.

**Definition 4.3.** A saddle $k$-cell $x_k$. If $k = k_a > 0$, a singular $k$-cell which is "simply-carried" on some $T$-saddle $L^{n_0}$ of $p_a$ with $p_a$ an interior point of the carrier of $x_k$ will be called a saddle $k$-cell of $p_a$. Taken $\mod L^{n_0}$, $x_k$ will be called a saddle $k$-cycle on $L^{n_0} \mod L^{n_0}$.

**Theorem 4.3.** If a $T$-critical point $\sigma = p_a$ has a positive $T$-index $k = k_a$ the following is true.

(i) A saddle $k$-cell $x_k$ is simply-carried by a $T$-saddle $L^{n_0}$ of $p_a$ in $\mod L^{n_0}$ of $H_d(F_a, \tilde{F}_a, Z)$.

(ii) Such a saddle $k$-cell is taken $\mod F_a$ is a prebase of $H_d(F_a, \tilde{F}_a, Z)$.

Proof of (i). To prove (ii) use will be made of the isomorphism (4.6), supplemented by the following affirmation.

(a) Let $g_0, k > 0$ be a singular $k$-cell simply-carried by $\tilde{A}_k$ with the center $0$ of $\tilde{A}_k$ in the interior of the carrier $[g_0]$ of $g_0$. Taken $\mod \tilde{A}_k$, $g_0$ is a $k$-cycle on $L^{n_0} \mod L^{n_0}$. which is a prebase of $H_0(\tilde{A}_k, \tilde{A}_k, Z)$.

The proof of (a) is elementary and will be left to the reader. One should note that the carrier $[g_0]$ is a topological $(k-1)$-sphere (i) on $A_k$ whose "Jordan" interior on $\tilde{A}_k$ contains $0$. The reader will find Lemma 29.0 of [15] useful in proving (a).

Grunting the truth of (a), Theorem 4.3 (i) follows from Lemma 4.2. For the isomorphism (4.6) is induced by the homeomorphism $\Theta_0$ of $L^{n_0}$ onto $\tilde{A}_k$. Under the inverse of $\Theta_0$ the cycle $g_0$ on $L^{n_0} \mod L^{n_0}$ of (a) goes into a saddle $k$-cycle $x_k$ of $p_a$ on $L^{n_0} \mod L^{n_0}$, which is a prebase of $H_0(L^{n_0}, \tilde{L}^{n_0}, Z)$.

Proof of (ii). If $J$ is the inclusion mapping (4.5)'s and $x_k$ a saddle $k$-cell on $L^{n_0}$, which, taken $\mod L^{n_0}$, is a prebase of $H_0(L^{n_0}, \tilde{L}^{n_0}, Z)$,

(1) A topological $0$-sphere is understood to be a pair of points.
then by Theorem 4.1, $\tilde{F}_a$ will be a prebase of $H_0(\tilde{F}_a, \tilde{F}_a, Z)$, Statement (ii) follows, since $F_a = a_n \mod \tilde{F}_a$.

$T$-saddles $L^a$ are a means to an end, the definition of saddle $k$-cells of $p_a$ on $F_a$. In the following corollary of Theorem 4.3 the ends rather than the means come to the fore.

**COROLLARY 4.1.** If $a_0(1)$ and $a_0(2)$ are two rel. saddle $k$-cycles on $F_a$ of the same $T$-critical point $a = p_a$ with $T$-index $h = h_0$, then for some choice of $e$ as $1$ or $-1$

$$a_0(1) \sim e a_0(2) \quad (on \, F_a \, mod \, \tilde{F}_a)$$

and consequently,

$$e a_0(1) \sim e a_0(2) \quad (on \, \tilde{F}_a) 

\text{(4.7)}$$

Proof of (4.7). According to Theorem 4.3 both $a_0(1)$ and $a_0(2)$ are prebases of the free Abelian group $H_0(\tilde{F}_a, \tilde{F}_a, Z)$. The relative homology (4.7) is implied.

Proof of (4.8). The homology (4.7) implies that

$$a_0(1) - e a_0(2) = e a_0^{k+1} + e a_0^k \quad (k = h_0) \text{(4.9)}$$

where $a_0^{k+1}$ and $a_0^k$ are integral chains on $F_a$ and $\tilde{F}_a$ respectively. The application of $e$ to the members of (4.9) yields (4.8).

A critical homology class $\langle (\tilde{a}_0, \tilde{F}_0) \rangle$. Corresponding to a saddle $k$-cell $a_0$ of a $T$-critical point $p_a$ of positive $T$-index $h = h_0$, the homology class of $\tilde{a}_0$ on $\tilde{F}_0$ is denoted by $\langle (\tilde{a}_0, \tilde{F}_0) \rangle$ and termed a critical homology class of $\tilde{F}_0$. (Cf. Definition 3.3). It may be regarded as an element in $H_{-1}(\tilde{F}_0, Z)$. According to (4.8) any other critical homology class of $\tilde{F}_0$ has the form $e \langle (\tilde{a}_0, \tilde{F}_0) \rangle$ where $e = 1$. As an element of $H_{-1}(\tilde{F}_0, Z)$, the order of $\langle (\tilde{a}_0, \tilde{F}_0) \rangle$ may be finite or infinite. We now define a basic invariant $t^r$.

**DEFINITION 4.4.** $t^r$. The torsion index $t^r$ of $p_a$ when $h_a > 0$. The order of $\langle (\tilde{a}_0, \tilde{F}_0) \rangle$ in $H_{-1}(\tilde{F}_0, Z)$, when finite, will be denoted by $t^r$ and termed the torsion $t^r$ index of $p_a$. No definition of $t^r$ is given when the order of $\langle (\tilde{a}_0, \tilde{F}_0) \rangle$ in $H_{-1}(\tilde{F}_0, Z)$ is infinite.

If $t^r$ exists, it is positive and for each saddle $k$-cell $a_0$ of $p_a$

$$t^r = a_0 \quad (on \, \tilde{F}_a) \text{(4.10)}$$

If $\mu < 0$ is an integer such that

$$\mu a_0 = 0 \quad (on \, \tilde{F}_a) \text{(4.11)}$$

then $t^r$ exists and $\mu = m t^r$ for some integer $m \neq 0$. This is an elementary result in the theory of cyclic groups.

**DEFINITION 4.5.** $\lambda_a$. A $k$-cycle on $F_a$, $t^r$-fold linking. When $k = h_a > 0$ and $t^r$ exists, (4.10) holds and there accordingly exists a $k$-chain $c^k_0$ on $\tilde{F}_a$ such that

$$t^r = a_0 \quad (on \, \tilde{F}_a)$$

and hence a $k$-cycle

$$c^k_0 = a_0 \quad (on \, F_a) \text{(4.12)}$$

We term $\lambda_a$ a $k$-cycle which is $t^r$-fold linking on $F_a$, which belongs to $p_a$ and is associated with $a_0$.

The following lemma is a consequence of Corollary 4.1.

**LEMMA 4.3.** (i) Any two $t^r$-fold linking $k$-cycles $\lambda_a(1)$ and $\lambda_a(2)$ on $F_a$ satisfy a relative homology

$$\lambda_a(1) \sim \lambda_a(2) \quad (on \, F_a \, mod \, \tilde{F}_a) \text{(4.14)}$$

where $e$ has one of the values $\pm 1$.

(ii) If $\lambda_a$ is a $k$-fold linking $k$-cycle on $F_a$, then $\mu a_0 \sim 0$ on $F_a \, mod \, \tilde{F}_a$ for no nonnull integer $\mu$.

Proof of (i). The relative homology (4.14) follows from (4.7) and (4.13).

Proof of (ii). It follows from (4.13) that

$$\lambda_a \sim t^r a_0 \quad (on \, F_a \, mod \, \tilde{F}_a) \text{(4.15)}$$

Moreover, $a_0$, taken $\mod \tilde{F}_a$, is a prebase of the free group $H_0(\tilde{F}_a, \tilde{F}_a, Z)$ in accord with Theorem 4.3 (ii), so that $\mu a_0 \sim 0$ on $F_a \, mod \, \tilde{F}_a$ for no nonnull integer $\mu$. Reference to (4.15) shows that (ii) is true.

This completes the proof of Lemma 4.3.

Theorem 4.4 distinguishes between the cases in which a torsion index $t^r$ of $p_a$ exists or does not exist.

**THEOREM 4.4.** If the $T$-index $h = h_0$ of a $T$-critical point $p_a$ is positive, the following is true.

(i) In case $p_a$ has a torsion index $t^r$ and $\lambda_a$ is a $t^r$-fold linking $k$-cycle on $F_a$ associated with $p_a$, then if $c^k_0$ is a $k$-cycle on $F_a$

$$c^k_0 = \mu a_0 \quad (on \, F_a \, mod \, \tilde{F}_a) \text{(4.16)}$$

for some integer $m$ (possibly zero).

(ii) If no torsion index of $p_a$ exists, then if $c^k_0$ is a $k$-cycle on $F_a$

$$c^k_0 \sim 0 \quad (on \, F_a \, mod \, \tilde{F}_a) \text{(4.17)}$$

In both cases (i) and (ii), Theorem 4.3 (ii) implies that for some integer $\mu$ (possibly zero)

$$c^k_0 = \mu a_0 + e a_0^{k+1} \quad (on \, F_a \, mod \, \tilde{F}_a) \text{(4.18)}$$

(*) The torsion index $t^r$ is not to be confused with a torsion coefficient of $H_0(\tilde{F}_a, Z)$. 

for a suitably chosen chain $c^q_{k+1}$ on $F_s$. From (4.18) we infer that in both cases (i) and (ii)
(4.19)  \hspace{1cm} \mu \delta c^q_k = 0 \hspace{1cm} \text{on} \hspace{1cm} \bar{F}_s.

Proof of (i). If $\mu = 0$ in (4.18), (4.18) implies (4.16) with $m = 0$. If $\mu \neq 0$, (4.19) implies that $\mu$ is an integer multiple of $r^q$, since $r^q$ is finite. In this case (4.16) follows from (4.18) and (4.15).

Proof of (ii). In the case of (ii), (4.18) implies that $\mu = 0$ in (4.19). Otherwise $r^q$ would exist contrary to the hypothesis of (ii). When $\mu = 0$, (4.18) implies (4.17).

Thus Theorem 4.4 is true.

We shall now establish a corollary of Theorem 4.2 which will be useful both in Part II and Part III. Here the $T$-index $k_s > 0$.

Corollary 4.2. Concerning the $\#-$mapping $J^q_s$, $a > a_s$, of Definition 3.2, the following is true.
(i) $Ker J^q_s = 0$ when $q \neq k_s - 1$.
(ii) $J^q_s$ is surjective when $q \neq k_s$.
(iii) For $q$ or $k_s$ or $k_s - 1$, $J^q_s$ is an isomorphism of $H_0(F_s, Z)$ over $H_0(F_s, Z)$.

Proof of (i). If $\phi^q_s$ is a $q-$cycle on $F_s$ such that $\phi^q_s = 0$ on $F_s$, we shall show that $\phi^q_s = 0$ on $F_s$ when $q \neq k_s - 1$, implying thereby that Ker $J^q_s = 0$ when $q \neq k_s - 1$ (cf. Definition 3.2).

Suppose on the contrary that there exists a group $(g+1)$-chain $\phi^{g+1}_s$ such that $\phi^{g+1}_s$ is a $q-$cycle on $F_s$ such that $\phi^{g+1}_s = 0$ on $F_s$. Since $q + 1 \neq k_s$ by hypothesis of (i), Theorem 4.2 implies that $\phi^{g+1}_s = 0$ on $F_s$ and $\phi^{g+1}_s$ is equivalent to $\phi^{g+1}_s = 0$ on $F_s$.

(4.20)
\[ \phi^{g+1}_s = \delta \phi^{g+1}_s + \phi^{g+1}_s \]

Since $\delta \phi^{g+1}_s = \phi^{g+1}_s$, by hypothesis of this paragraph, (4.20) implies that $\phi^{g+1}_s = 0$ on $F_s$.

We infer that (i) is true.

Proof of (ii). It is sufficient to show that $\phi^{g+1}_s$ is a $q-$cycle on $F_s$, if $q \neq k_s$ and $a > a_s$, then for some $q-$cycle $\phi^q_s$ on $F_s$,

(4.21)
\[ \phi^q_s \sim \phi^{g+1}_s \hspace{1cm} \text{on} \hspace{1cm} F_s. \]

The homology is trivial when $k_s = 0$ and $a > a_s$, since $F_s$ is then the union of sets $s$ $F_s$ and $p_s$ whose closures are disjoint.

When $k_s = k_s = 0$ and $q \neq k_s$ it follows from Theorem 4.2 that

(4.22)
\[ \phi^q_s = \delta \phi^{g+1}_s + \phi^q_s \hspace{1cm} \text{on} \hspace{1cm} F_s \]

(\# Subscript $m + \text{will indicate that the cycle or chain is on} \bar{F}_s$ or $F_s$ respectively.

(\* Subscript $m + \text{should be denoted by} (p_s) \text{as considered a set.}$

for suitable chains $\phi^{g+1}_s$ and $\phi^q_s$. An application of $\delta$ to the members of (4.22) shows that $\phi^q_s$ is a $q-$cycle. With this understood, (4.22) implies (4.21), on taking $\phi^q_s$ as $\phi^q_s$. Thus (ii) is true.

Proof of (iii). Statement (iii) follows immediately from (i) and (ii).

Thus Corollary 4.2 is true.

§ 5. Proof of finite generation. The principal theorem of this section follows.

Theorem 5.1. If $\gamma$ is any value of $F$ on $M_s$, the homology groups $H_0(F_s, Z)$ are $FG$.

The proof of Theorem 5.1 is inductive on $T$-critical values $a$ we shall set

(5.0')
\[ H_0(F_s, Z) = H_0^a \]

(5.0'')
\[ H_0(\bar{F}_s, Z) = H_0^a \hspace{1cm} (q = 0, 1, \ldots) \]

and corresponding to the listing (5.1) of the $T$-critical values $a$, of $E$, shall list the homology groups

(5.1)
\[ H_0^a, H_0^a, \ldots, H_0^a \hspace{1cm} (q = 0, 1, \ldots) \]

It is trivial that the groups $H_0^a$ are FG. We shall give an inductive proof of the following.

Theorem 5.2. Each homology group in the list (5.1) is $FG$.

Before coming to the proof of Theorem 5.2 note that Theorem 5.1 implies Theorem 5.1 by virtue of the following lemma.

Lemma 5.1. If $\gamma$ is an ordinary value of $F$ and if $a$ is the maximum of the $T$-critical values of $F$ less than $\gamma$, then if $s$ is the inclusion mapping of $F_s$ into $F_s$, the corresponding chain-transformation induces isomorphisms

(5.2)
\[ H_0(F_s, Z) \rightarrow H_0(F_s, Z) \hspace{1cm} (q = 0, 1, \ldots) \]

Proof of Lemma 5.1. It is a corollary of Theorem 5.1 that there exists an $F$-traction of $F_s$ into $F_s$, so that by Theorem 3.1, (5.2) holds as stated.

Proof of Theorem 5.2. It is sufficient to prove Lemmas 5.2 and 5.3.

Lemma 5.2. If $a_s$ is a $T$-critical value in the listing (5.1) with $a_s > a_s$, then if $H_0^{a_s}$ is FG, $H_0^{a_s}$ is FG.

Proof of Lemma 5.3. According to Theorem 2.2 there exists an $F$-traction of $\bar{F}_s$ into $F_a$. Hence by Theorem 3.1 there is an isomorphism

(5.3)
\[ H_0^{a_s} \rightarrow H_0^{a_s} \]

Hence $H_0^{a_s}$ is FG if $H_0^{a_s}$ is FG. Thus Lemma 5.2 is true.
LEMMA 5.3. If $a_\ell$ is a $T$-critical value in the list (5.4) with $a_\ell > a_\ell$, then if $H^\ell_\infty$ is FG, $H^{\ell_\infty}_\infty$ is FG.

Proof of Lemma 5.3. Set $a = a_\ell$. Let $k = k_\ell$, the $T$-index of $a$.

Three mutually exclusive cases arise:

Case I. $k = 0$,

Case II. $k > 0$, $q \neq k$,

Case III. $k > 0$, $q = k$.

Proof in Case I. In this case $F_\ell$ is the union of two disjoint closed sets $\sigma = \sigma_\ell$ and $\tilde{F}_\ell$. Lemma 5.2 follows trivially in Case I.

Proof in Case II. $J^\ell_\infty$ is a homomorphism. According to Corollary 4.2 (ii) $J^\ell_\infty$ maps $H^\ell_\infty$ onto $H^\ell_\infty$ when $q \neq k$. A finite base for $H^\ell_\infty$ accordingly goes under $J^\ell_\infty$ onto a set of generators of $H^\ell_\infty$. Thus $H^\ell_\infty$ is FG in Case II.

Case III. We shall make use of Theorem 4.4, distinguishing between the cases in which a torsion index $t^\ell$ of $p_\ell$ exists and does not exist.

According to (i) of Theorem 4.4, when $t^\ell$ exists, an arbitrary $k$-cycle $e^\ell_k$ on $F_\ell$ is such that

$$e^\ell_k \sim m_{\lambda_\ell} \mod (F_\ell \mod \tilde{F}_\ell)$$

for some integer $m$, or equivalently

$$e^\ell_k = m_{\lambda_\ell} + \delta e^\ell_{k+1} + e^\ell_k$$

for suitable chains $e^\ell_{k+1}$ and $e^\ell_k$ on $F_\ell$ and $\tilde{F}_\ell$, respectively. It follows from (5.5) that $e^\ell_k$ is a $k$-cycle on $F_\ell$. We draw the following conclusion from (5.5): if $y_1, ..., y_r$ is a probase for $H^\ell_\infty$ the homology classes (Def.3.0)

$$(y_1, F_\ell), (y_2, F_\ell), ..., (y_r, F_\ell)$$

generate $H^\ell_\infty$.

According to (ii) of Theorem 4.4 when $t^\ell$ does not exist, an arbitrary $k$-cycle $e^\ell_k$ on $F_\ell$ is such that

$$e^\ell_k \sim 0 \mod (F_\ell \mod \tilde{F}_\ell)$$

and one concludes again that $H^\ell_\infty$ is FG.

Thus Lemma 5.3 is true and, together with Lemma 5.2, implies Theorem 5.3. Theorem 5.1 follows, as stated previously.

We add a theorem which is related to the theorems of this section but which is not needed to prove Theorem 5.1.

THEOREM 5.4. If $c$ is any value of $F$ on $\mu_\infty$ and if $q > n$, the homology group $H^\ell_\infty(F_\mu, Z)$ is trivial.

It follows from Lemma 5.4 that Theorem 5.3 is true if and only if each homology group in the sequence (5.1) is trivial for $q > n$.

The isomorphism (5.3) is valid for any $T$-critical value $a_\ell > a_\ell$. Moreover, inclusion induced isomorphisms,

$$H^\ell_\infty \rightarrow H^\ell_\infty \rightarrow ... \rightarrow H^\ell_\infty \rightarrow H^\ell_\infty$$

are valid by (iii) of Corollary 4.2. For $q > n$ each homology group in the sequence

$$H^\ell_\infty; H^\ell_\infty; ...; H^\ell_\infty; H^\ell_\infty$$

is accordingly trivial, since $H^\ell_\infty$ is trivial.

Theorem 5.3 follows.

PART III. Critical invariants of $T$-critical points $p_\ell$

§ 6. Program for PART III. We are concerned with relations between successive groups in the sequence

$$H^\ell_\infty; H^\ell_\infty; H^\ell_\infty; ...; H^\ell_\infty; H^\ell_\infty$$

Here $q$ is the range $0, 1, ...$ Groups in this sequence, which are separated by ad colon, admit an inclusion induced isomorphism of form (5.3). Let $a$ be any one of the $T$-critical values $a_\ell$, ..., $a_\ell$. In Definition 3.2 we have introduced an inclusion induced homomorphism

$$J^\ell_\infty: H^\ell_\infty \rightarrow H^\ell_\infty$$

It follows from Corollary 4.2 (iii) that $J^\ell_\infty$ is an isomorphism if $q$ is neither $k_\ell$ nor $k_\ell - 1$. In § 7 our attention will be restricted to the case, $q = k_\ell$, while in § 9 we shall study the case, $q = k_\ell - 1$.

Each group $H^\ell_\infty$ is a direct sum

$$H^\ell_\infty = \Delta^\ell_\infty \oplus \Delta^\ell_\infty$$

of its torsion subgroup$^{26}$ $\Delta^\ell_\infty$ and a complementary free subgroup $\Delta^\ell_\infty$, termed a "Betti subgroup" of $H^\ell_\infty$. The decomposition (6.3) is possible, since the group $H^\ell_\infty$ has been shown to be FG. We shall similarly represent $H^\ell_\infty$ as a direct sum

$$H^\ell_\infty = \Delta^\ell_\infty \oplus \Delta^\ell_\infty$$

of its torsion subgroup and a complementary Betti subgroup $\Delta^\ell_\infty$.

Of particular interest is the sequence,

$$\Gamma^\ell_\infty; \Gamma^\ell_\infty; \Gamma^\ell_\infty; ...; \Gamma^\ell_\infty; \Gamma^\ell_\infty;$$

of Betti subgroups and the sequence,

$$\Delta^\ell_\infty; \Delta^\ell_\infty; \Delta^\ell_\infty; ...; \Delta^\ell_\infty; \Delta^\ell_\infty;$$

of torsion subgroups. Groups separated by a colon are isomorphic because
(5.3) holds. Groups in these sequences separated by a comma are isomorphic, except at most when \( q = k_a \) or \( k_a - 1 \), in accord with Corollary 4.2.

The dimension of a Betti group \( H^k_a \) is termed the \( k \)th Betti number of \( F_a \). Similarly the dimension of a Betti group \( H^k_a \) is termed the \( k \)th Betti number of \( F_a \). The torsion coefficients of the groups \( H^k_a \) and \( H^k_a \) are defined in the usual way and termed torsion coefficients of dimension \( q \) of \( F_a \) and \( F^* \), respectively. We shall show how these invariants change (if at all) as one passes from \( F_a \) to \( F_a \). The only dimensions \( q \) for which there can be a change as one passes from \( F_a \) to \( F_a \) are the dimensions \( q = k_a \) or \( k_a - 1 \).

Critical invariants. Program. The data needed to determine the changes in the torsion coefficients and Betti numbers as one passes from \( F_a \) to \( F_a \) are certain integers termed critical invariants. They are associated with each \( T \)-critical point \( p_a \).

If \( k = k_a > 0 \) and one compares \( H^k_a \) with \( H^k_a \) and \( H^k_a \) with \( H^k_a \) (as in § 7) the critical invariants are the \( T \)-index \( k = k_a \) and the torsion index \( t^* \) of \( p_a \) defined in Definition 4.4. If \( k_a = 0 \) a torsion index \( t^* \) is not defined.

If \( k = k_a > 0 \) and one compares \( H^k_a \) with \( H^k_a \) and \( H^k_a \) with \( H^k_a \) (as in § 9) the above critical invariants must be supplemented by other critical invariants, including integers \( s^* \) defined in § 9. As will be seen, these invariants are determined for each \( T \)-critical point \( p_a \) for which \( k = k_a > 0 \) by the critical homology classes \( \overline{H^k_a} = H^k_a \) in \( H^k_a \) defined in § 9. Here \( s^* \) is the “saddle \( k \)-cell” of Definition 4.4. The integers \( s^* \) were first defined in [14].

It follows from the definition of each of these so-called “critical invariants” that they are unchanged if \( M_a \) is mapped by a homeomorphism \( \phi \) onto another manifold \( M_a \) and \( F \) replaced by a mapping \( F^* \) of \( M_a \) into \( F \) such that \( F^* = F \) when \( k = k_a \).

§ 7. From \( H^k_a \) to \( H^k_a \) when \( k = k_a \). Use will be made of the inclusion-induced homomorphism

\[
J^k_a : H^k_a \rightarrow H^k_a \quad (\text{Definition 3.2})
\]

in case \( k = k_a \). As we shall see in Theorem 7.2, the homomorphism (7.1) may not be an isomorphism. However, the restriction \( J^k_a H^k_a \) is an isomorphism, as the following theorem explicitly affirms.

**Theorem 7.1.** If \( a > a^* \) and \( k = k_a \), then \( J^k_a H^k_a \) is an isomorphism, as the following theorem explicitly affirms.

\[
J^k_a H^k_a \rightarrow H^k_a
\]

If \( k_a = 0 \) we understand the theorem to assert that \( H^k_a \) is trivial.

Since \( J^k_a \) is a homomorphism, \( J^k_a \) is a homomorphism

\[
J^k_a : H^k_a \rightarrow H^k_a
\]

into \( H^k_a \). By hypothesis of this section \( k = k_a \) and \( a > a^* \). Hence \( \ker J^k_a = 0 \).
be a base for $\Sigma_2^a$. When $a = 0$, let $p_a$ be the 0-cycle with carrier $p_a$. When $k_a > 0$ and $a > a_0$, let

$$w_1, \ldots, w_k$$

be the images of the respective elements (7.6) under the $\otimes$-homomorphisms $J^k_2$ (7.1). If $k_a > 0$ and if a torsion index $r^a$ of $p_a$ exists, a $r^a$-fold linking $k_a$-cycle $\lambda_a$ exists in accord with Definition 4.5. Consider the homology class $[(\lambda_a, F_a)]$ of $\lambda_a$ on $F_a$. The second principal theorem of this section follows.

**Theorem 7.2.** (i) When $k = k_a > 0$ and a torsion index $r^a$ of $p_a$ fails to exist, the elements (7.7) give a base for a Betti group $\Sigma_2^a$ of $H^a_2$.

(ii) When $k = k_a > 0$ and a torsion index $r^a$ of $p_a$ exists, the elements

$$[(\lambda_a, F_a)], w_1, \ldots, w_k,$$

give a base for a Betti group $\Sigma_2^a$.

(iii) When $k = k_a = 0$ and $a > a_0$, the elements,

$$[(p_a, F_a)], w_1, \ldots, w_k,$$

give a base for a Betti group $\Sigma_2^a = \Sigma_2^a$.

**Proof of (i).** Statement (i) is a consequence of the following lemma.

**Lemma 7.1.** When $k = k_a > 0$ and no torsion index $r^a$ of $p_a$ exists, the homomorphism $J^k_2$ (7.1) is an isomorphism.

Under the hypotheses of Lemma 7.1, Theorem 4.4 (ii) implies that $J^k_2$ is surjective. Moreover $\ker J^k_2 = 0$ when $a > a_0$ and $k = k_a$ by Corollary 4.2 (i), so that $J^k_2$ is an isomorphism when $r^a$ fails to exist. The lemma follows and implies (i) of Theorem 7.2.

**Proof of (ii).** Let $\tilde{H}^a_2$ denote the image of $H^a_2$ under $J^k_2$. Since $\ker J^k_2 = 0$ the homomorphism (7.10)

$$J^k_2: \tilde{H}^a_2 \rightarrow H^a_2$$

is an isomorphism, so that the set of elements (7.7) is a base for a Betti subgroup $\Sigma_2^a$ of $H^a_2$.

Since a torsion index $r^a$ of $p_a$ exists by hypothesis of (ii), a $r^a$-fold linking $k_a$-cycle $\lambda_a$ exists. The relation

$$H^a_2 = \langle (\lambda_a, F_a), \tilde{H}^a_2 \rangle$$

follows from (4.16) and the isomorphism (7.10). That is, when $r^a$ exists, $[(\lambda_a, F_a)]$ and $\tilde{H}^a_2$ generate $H^a_2$.

---

(1) Strictly, not $J^k_2$, but a homomorphism, say $\tilde{J}^k_2$ induced by $J^k_2$.

(2) If an Abelian group $A$ is generated by the elements in subsets $A_1, \ldots, A_n$ of $A$, one writes $A = \langle A_1, \ldots, A_n \rangle$.

---

**Topologically non-degenerate functions**

Statement (ii) follows if $H^a_2$ is a direct sum,

$$H^a_2 = \langle (\lambda_a, F_a), \tilde{H}^a_2 \rangle \oplus \tilde{H}^a_2.$$ 

That (7.12) is true is a consequence of (7.11) and (ii) of Lemma 4.3, which implies that for no integer $\mu \neq 0$ is $\mu[(\lambda_a, F_a)]$ an element in $\tilde{H}^a_2$.

**Proof of (iii).** Statement (iii) is consequence of the fact that when $k = k_a = 0$ and $a > a_0$ $F_a$ is the union of disjoint closed sets, $p_a$ and $\tilde{F}_a$.

This completes the proof of Theorem 7.2.

**§ 8. Elementary group quotients $A/W$.** Certain general theorems on Abelian groups presented in [16] will be recalled and will be applied in § 9. For references to relevant books on groups see [16].

**Objectives of § 8.** There is given a FG Abelian group $A$, together with a cyclic subgroup $W = \{w\}$ of $A$ generated by an element $e^A$. One seeks to determine the Betti numbers and torsion coefficients of $A/W$ in terms of minimal data on $A$ and $W$. We shall describe such data.

Recall that Abelian group $A$ which is FG is a direct sum

$$A = 3 \oplus 2$$

of its uniquely determined torsion subgroup $3$ and a “complementary” free subgroup $2$ of $A$, termed a Betti subgroup of $A$. $3$ has a base

$$w_1, \ldots, w_9$$

consisting of $\beta$ elements of $A$, every non-trivial linear combination of which (over $Z$) has an infinite order in $A$. In general $3$ is not uniquely determined by $2$, nor the base (8.2) uniquely determined by $3$. However the number $\beta$ is independent of the choice of the free group complementary to $2$ and of the choice of a base of $3$. We term $\beta$ the Betti number of $A$.

The torsion coefficients of $2$. It is a classical theorem that a finite, non-trivial Abelian group, $3$, is a direct sum of a finite set of cyclic subgroups which can be canonically arranged so as to have orders $q_1, q_2, \ldots, q_n$ exceeding $1$, each of which, except $q_n$, is divisible by its successor. These integers are uniquely determined by $3$ and are termed the torsion coefficients (1) of $3$.

**Elementary divisors** $ED$ of $3$. It is known that a finite, non-trivial group, $3$, is a direct sum $g_1 \oplus \cdots \oplus g_n$ of cyclic subgroups $g_i$ such that the order of $g_i$ is a power $p_i^{\alpha_i}$ of a prime $p_i$ and $g_j$ is a subgroup of no cyclic subgroup of $3$ whose order is a higher power of $p_j$. Such a direct sum is

(1) When $3$ is the torsion subgroup of $A$, torsion coefficients and $ED$ of $3$ will be called torsion coefficients and $ED$ of $A$. 

---

[Image]
called a cyclic primary decomposition (abbreviated CPD) of \(\mathcal{Z}\). The prime powers
\[
p_{1}^{e_{1}}, \ldots, p_{r}^{e_{r}} \quad (e_{i} > 0; i = 1, \ldots, r)
\]
which are the orders of the respective summands in a CPD of \(\mathcal{Z}\) are called elementary divisors, ED of \(\mathcal{Z}\). The ED of \(\mathcal{Z}\) are said to be normally arranged if \(p_{1} \geq p_{2} \geq \ldots \geq p_{r}\) and if, when \(p_{j} = p_{j+1}\), then \(e_{i} > e_{i+1}\). \(\mathcal{Z}\) uniquely determines a set of normally ordered ED's.

We state a classical lemma.

**Lemma 8.1.** Cyclically ordered torsion coefficients of a FG non-trivial Abelian group \(\mathcal{Z}\), determine and are uniquely determined by normally ordered ED of \(\mathcal{Z}\). (See [8], p. 147.)

**Definition 8.1.** A basis of a FG Abelian group \(\mathcal{A}\) is a torsion subgroup with a CPD
\[
\mathcal{Z} = \langle x_{1} \rangle \oplus \ldots \oplus \langle x_{r} \rangle \quad (x_{i} \in \mathcal{Z})
\]
Let \(\mathcal{B}\) be a Betti subgroup of \(\mathcal{A}\) with a base \((u_{1}, \ldots, u_{g})\). The set of elements
\[
u_{1}, \ldots, u_{g}; x_{1}, \ldots, x_{r}
\]
of \(\mathcal{A}\) is called a basis for \(\mathcal{A}\).

An arbitrary element \(w \in \mathcal{A}\) has the form,
\[
0 = m_{1}x_{1} + \ldots + m_{g}x_{g}
\]
where \(m_{1}\) is an integer uniquely determined by \(w\) and the choice of the basis \((8.5)\), while each \(m_{j}\) is uniquely determined by \(w\) and the choice of the CPD \((8.4)\), provided \(m_{j}\) is restricted to integral values such that
\[
0 \leq m_{j} < \text{order} x_{j} \quad (j = 1, 2, \ldots, r)
\]
**Minimal data on \(\mathcal{A}\) and \(\mathcal{W}\).** In I, II, III, IV we present data adequate for meeting the objectives of §8 outlined above. These data follow.

I. A basis of \(\mathcal{A}\) (Definition 8.1) of form,
\[
u_{1}, \ldots, u_{g}; x_{1}, \ldots, x_{r}
\]
II. A normally ordered set
\[
i_{1}, \ldots, i_{g}
\]
of ED of \(\mathcal{A}\) of form \((8.3)\).

III. A generator \(w\) of the cyclic subgroup \(\langle w \rangle\) of \(\mathcal{A}\) and a profile of \(w\), that is, a set
\[
\mu_{1}, \ldots, \mu_{g}; \omega_{1}, \ldots, \omega_{g}
\]
We term \(\mathcal{W}\) the critical cyclic subgroup of \(\mathcal{A}\).

of coefficients in an admissible representation \((8.6)\) of \(w\), subject to \((8.7)\).

IV. An integer \(s \geq 0\), termed the free index of \(\mathcal{W}\), defined as the GCD of the integers \(\mu_{1}, \ldots, \mu_{g}\) of \((8.9)\), zero, if these integers all vanish.

We shall give another, but equivalent, definition of \(s\).

**Definition 8.2.** The free index \(s\) of \(\mathcal{W}\). We set \(s = 0\) if and only if \(\mathcal{W}\) has a finite order. If \(\mathcal{W}\) has an infinite order, \(s\) is finite and positive, with a value defined by the following lemma.

**Lemma 8.2.** With a "critical" cyclic subgroup \(\mathcal{W}\) of \(\mathcal{A}\) of infinite order there can be associated a positive integer \(s\) which is unique among positive integers with the following property.

If \(w\) is an arbitrary generator of \(\mathcal{W}\) and \(\mathcal{B}\) an arbitrary Betti subgroup of \(\mathcal{A}\), there exists a basis of \(\mathcal{B}\) with a first element \(w_{0}\) such that
\[
w = s \mu_{0} \mod 2 \quad (s)
\]
Lemma 8.2 follows from Lemma 3.1 of [16] and its proof.

Theorem 3.3 of [16] gives a first indication of the meaning of the free index \(s\) of \(\mathcal{W}\). It may be restated as follows.

**Theorem 8.0.** Suppose that \(\mathcal{A}\) is torsion free and that the free index of \(\mathcal{W}\) is \(s\). Then \(\mathcal{A}/\mathcal{W}\) is torsion free unless \(s > 1\), and when \(s > 1\), the first and only torsion coefficients of \(\mathcal{A}/\mathcal{W}\) is \(s\).

It will be noted that the minimal data contained in I and II depend upon \(\mathcal{A}\) alone, while the data contained in I, II, III, and IV depend upon both \(\mathcal{A}\) and \(\mathcal{W}\). In our application of this section in §9, \(\mathcal{A}\) and \(\mathcal{W}\) will be represented by \(\mathcal{W}_{\mathcal{A}}\) and \(\mathcal{W}_{\mathcal{A}}\), respectively, where \(\mathcal{W}_{\mathcal{A}}\) is the critical cyclic subgroup of \(\mathcal{W}\), to be introduced in Definition 9.1. The analogue of the integers \(\mu_{j}\) in III and \(s\) in IV, thereby appearing in §9, will be called "critical invariants" of the \(T\)-critical point \(p_{\mathcal{A}}\).

In the first of two principal theorems of this section we relate the Betti number of \(\mathcal{A}\) to that of \(\mathcal{A}/\mathcal{W}\). In the second of our two principal theorems we show that the above minimal data on \(\mathcal{A}\) and \(\mathcal{W}\) enable us to evaluate the torsion coefficients of \(\mathcal{A}/\mathcal{W}\). The minimal data on \(\mathcal{A}\) include the normally ordered ED of \(\mathcal{A}\), or equivalently by Lemma 8.1, the torsion coefficients of \(\mathcal{A}\).

**Theorem 8.1.** (i) If the free index \(s\) of \(\mathcal{W}\) is positive, \(\mathcal{A}/\mathcal{W}\) has a Betti number one less than that of \(\mathcal{A}\).

(ii) If the free index \(s = 0\), \(\mathcal{A}/\mathcal{W}\) has a Betti number equal to that of \(\mathcal{A}\).

Theorem 8.1 (i) is included in Theorem 3.2 of [16]. Theorem 8.1 (ii) is included in Lemma 4.1 of [16].

(1) Given \(x\) and \(y\) in \(\mathcal{A}\) we write \(x = y + n\mathcal{Z}\) if \(x - y \in \mathcal{Z}\), where \(\mathcal{Z}\) is the torsion subgroup of \(\mathcal{A}\).
To formulate Theorem 8.2 we introduce a $q + 1$ square matrix

$$
||a_{ij}|| = \begin{pmatrix}
    a_1 & a_2 & \cdots & a_q \\
    a_q & a_{q+1} & \cdots & a_{q+2} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{q+2} & a_{q+3} & \cdots & a_{2q+2}
\end{pmatrix}
$$

in which the entries in the diagonal are the $a_i$ (8.8) of $\overline{J}$ followed by the free index $s$ of $W$. The elements $a_1, \ldots, a_q$ in the last row are taken from the profile of a generator $w$ of $W$, as presented in (8.9). Elements in the matrix $||a_{ij}||$, other than those in the diagonal and last row, are zero. The integers $m_i$ are subject to the condition (8.7). The rank of this matrix is $q + 1$, or $q$, according as $s > 0$ or $s = 0$.

The second principal theorem of this section follows.

**Theorem 8.2.** The invariant factors exceeding 1 of the above $(q + 1)$-square matrix $||a_{ij}||$, if properly ordered, give the torsion coefficients of $A/W$.

Theorem 8.2 is proved in [16]. It is formulated separately in [16] as Corollary 3.1, when $s > 0$, and Corollary 4.1 when $s = 0$.

We add Theorem 3.1 of [16].

**Theorem 8.3.** If the free index $s$ of $W$ is 1 the torsion subgroup of $A/W$ is isomorphic to the torsion subgroup of $A$ and the Betti number of $A/W$ is one less than that of $A$.

§ 9. From $\overline{J}^{n-1}_{q-1}$ to $\overline{J}^{n-1}_{q-1}; \ k = k_a > 0$. We shall apply the theorems of § 8 on $A/W$, setting

$$\begin{align*}
A &= \overline{J}^{n-1}_{q-1}; \ W = \overline{J}^{n-1}_{q-2} \quad (k = k_a > 0) \\
W_{q-2}^n &= \text{a cyclic subgroup of } \overline{J}^{n-1}_{q-1}, \text{ now to be defined.}
\end{align*}$$

**Definition 9.1.** The critical cyclic subgroup $W_{q-1}^n$ of $\overline{J}^{n-1}_{q-1}$. Let $k_a$ be a saddle $k$-cell of $p_a$ (Definition 4.3). Then $k_a$ is a $(k-1)$-cycle on $F_a$ whose carrier is a topological $(k-1)$-sphere. We shall set

$$W_{q-1}^n = \left\{ \left[([a_k], F_a) \right] \right\} \quad (k = k_a > 0)$$

and term $W_{q-1}^n$ the critical cyclic subgroup of $\overline{J}^{n-1}_{q-1}$. According to (4.8) of Corollary 4.1, the pair of homology classes $\pm ([a_k], F_a)$ in $\overline{J}^k_a$ is independent of the choice of $k_a$ as a saddle $k$-cell of $p_a$.

A principal property of a saddle $k$-cell $k_a$ on $F_a$ is that, taken mod $F_a$, it is a rel. $k$-cycle and, as such, a prebase for the rel. homology group $H_k^F(F_{q-1}, \overline{J}^{n-1}_{q-1}; F_a, Z)$. Cf. Theorem 4.3 (ii).

We shall prove the following theorem.

**Theorem 9.0.** The critical cyclic subgroup $W_{q-1}^n$ of $\overline{J}^{n-1}_{q-1}$ is the kernel of the inclusion induced $\overline{J}^k_a$-homomorphism,

$$\overline{J}^{n-1}_{q-1}; \overline{J}^{n-1}_{q-1} \rightarrow \overline{J}^{n-1}_{q-1},$$

of Definition 3.2.

**Proof of Theorem 9.0.** We must show that both of the inclusions

(a) $W_{q-1}^n \subset \ker J_{q-1}^n$; \qquad (b) $\ker J_{q-1}^n \subset W_{q-1}^n$,

are valid.

**Proof of (a).** It suffices to show that a generator

$$w = ([a_k], F_a) \text{ of } W_{q-1}^n \quad (\text{see } 9.3)$$

annihilates $J_{q-1}^n$. Now $w$ and $J_{q-1}^n(w)$, by definition are the homology classes of $[a_k]$ on $F_a$ and on $\overline{J}^{n-1}_{q-1}$, respectively. Since $[a_k]$ is $0$ on $F_a, w \in \ker J_{q-1}^n$.

The inclusion (a) is thus valid.

**Proof of (b).** It suffices to show that if a $(k-1)$-cycle $a_{k-1}$ on $F_a$ bounds on $F_a$, then

$$((a_{k-1}, F_a) \in W_{q-1}^n.$$  \hfill (9.3)

**Proof of (9.3).** By hypothesis on $a_{k-1}$, there exists a $k$-chain $c_k$ on $F_a$ such that

$$\partial c_k = a_{k-1}.$$  \hfill (9.4)

The $k$-chain $c_k$ is thus a $k$-cycle on $F_a \text{mod} F_a$. As a prebase of $H_k(F_a, Z, F_a)$, $c_k$ is such that there exists an integer $\mu$, a chain $c_{k+1}$ on $F_a$ and a chain $c_k$ on $F_a$ such that

$$c_k = \mu a_k + \partial c_{k+1} + \partial c_k. \quad \hfill (9.5)$$

The application of $\partial$ to both members of (9.5) shows (with the aid of (9.4)) that

$$\partial c_k = a_{k-1} + \partial c_{k+1} + \partial c_k, \quad \hfill (9.6)$$

from which (9.3) follows.

Thus the inclusion (b), as well as the inclusion (a) holds, and Theorem 9.0 follows.

Theorem 9.0 has the following corollary.

**Corollary 9.1.** The natural homomorphism

$$\overline{J}^{n-1}_{q-1}; \overline{J}^{n-1}_{q-1} \rightarrow \overline{J}^{n-1}_{q-1}, \quad (k = k_a > 0)$$

induced by $J_{q-1}^n$ is an isomorphism.

That the mapping (9.7) is biunique follows from Theorem 9.0. That it is surjective follows from (ii) of Corollary 4.3. Thus Corollary 9.1 is true.
Applications of the theorems on $W/\partial$ of § 8. The objective of § 9 is to relate the invariants $\{a_k\}$ to the corresponding invariants of $H^{\infty}_{\omega-1}$, understanding always that $k = k_{\omega0}$ is considered in § 7 but not in § 9. See scheme outlined in § 6. Here as elsewhere $a$ is a critical value of $F$. In § 9, $a > a_0$.

Suppose then that $k = k_0 > 0$. In this case the sequence of homology groups

$$H^\infty_{\omega-1}, H^\infty_{\omega-1}/W^\infty_{\omega-1}, H^\infty_{\omega-1}$$

will serve our purpose of relating $H^\infty_{\omega-1}$ to $H^\infty_{\omega-1}$, with $H^\infty_{\omega-1}/W^\infty_{\omega-1}$ serving as a mediator, $A/W$, between $H^\infty_{\omega-1}$ and $H^\infty_{\omega-1}$. According to Corollary 9.1, the last two groups in the sequence (9.8) are isomorphic. The data needed to relate the invariants of the first two groups in the sequence (9.8) include the following.

**Definition 9.2.** The free index $s^*$. When $k = k_0 > 0$ one identifies $H^\infty_{\omega-1}$ and $W^\infty_{\omega-1}$ with $A$ and $W$ of § 8. The free index of $W = W^\infty_{\omega-1}$, given by Definition 8.2, is denoted by $s^*$. There are two cases. In both cases $k = k_0 > 0$.

Case I. Order $W^\infty_{\omega-1}$ finite. In this case $s^* = 0$ and $s^*$ exists by Definitions 8.2 and 4.4 respectively.

Case II. Order $W^\infty_{\omega-1}$ infinite. In this case $s^*$ is the finite positive value $a$ associated with $W = W^\infty_{\omega-1}$ in Lemma 8.2. A torsion index fails to exist by Definition 4.4.

Theorem 8.0 gives the following first indication of the meaning of $s^*$ when $k = k_0 > 0$.

**Theorem 9.1.** Suppose that $k = k_0 > 0$ and that $H^\infty_{\omega-1}$ is torsion free. Then $H^\infty_{\omega-1}$ is torsion free unless $s^* > 1$ and when $s^* > 1$, has a unique torsion coefficient $s^*$.

Proof. Theorem 8.0 implies that when $k = k_0 > 0$ and $H^\infty_{\omega-1}$ is torsion free, then $H^\infty_{\omega-1}/W^\infty_{\omega-1}$ and hence its isomorph $H^\infty_{\omega-1}$ (Corollary 9.1) is torsion free unless $s^* > 1$, and when $s^* > 1$, has a unique torsion coefficient $s^*$.

With a similar use of Corollary 9.1 and of the three groups (9.8) we infer the following from Theorem 8.1. Here $k = k_0 > 0$.

**Theorem 9.2.** (i) If the free index $s^*$ of $W^\infty_{\omega-1}$ is positive, $H^\infty_{\omega-1}$ has a Betti number which is one less than that of $H^\infty_{\omega-1}$.

(ii) If the free index $s^* = 0$, $H^\infty_{\omega-1}$ has a Betti number equal to that of $H^\infty_{\omega-1}$.

Similarly if $a_k$ is a "saddle $k$-cell" of $p_a$ (Definition 4.3), Theorem 8.3 implies the following.

**Theorem 9.3.** If $n_1, ..., n_k$ are the elementary divisors of $H^\infty_{\omega-1}$, if the integers $m_1, ..., m_k$ are taken from the profile (8.9) of a generator $(\partial \nu, \xi')$ of $W^\infty_{\omega-1}$, and if $s = s^*$ is the free index of $W^\infty_{\omega-1}$, then the torsion coefficients of $H^\infty_{\omega-1}$ are such that the invariant factors of the matrix $[a_k]$ of § 8 exceed 1.

Theorem 8.3 similarly implies the following.

**Theorem 9.4.** If $k = k_0 > 0$ and if the free index $s^*$ of $W^\infty_{\omega-1}$ is 1, the torsion subgroup of $H^\infty_{\omega-1}$ is isomorphic to the torsion subgroup of $H^\infty_{\omega-1}$ and the $(k-1)$st Betti number of $F_a$ is one less than that of $F_a$.

**§ 10. The existence of $T$-critical points.** We are supposing that $M_a$ is a compact, connected topological $n$-dimensional manifold upon which a TND function $F$ is defined with critical values

$$a_0 < a_1 < ... < a_s,$$

of singleton type. If $a$ is any one of these values, $p_a$ denotes the corresponding $T$-critical point. As we have seen, the $T$-index $k = k_0$ of $p_a$ is on the range $0, 1, ..., n$. Let $m_a$ be the number of $T$-critical points with $T$-index $k$ (Definition 1.2). For $q = 0, 1, ..., \beta_k(M_a)$ is the $q$th Betti number of $M_a$.

We shall prove the following. Cf. Theorem 30.1 of [15].

**Theorem 10.1.** i) The Betti numbers $\beta_k(M_a)$ are finite and vanish for $q > n$.

(ii) The following relations are valid:

$$m_n = \beta_n(M_a),$$

$$m_n - m_{n-1} = \beta_n(M_a) - \beta_{n-1}(M_a),$$

$$m_n = m_{n-1} + m_{n-2} = \beta_n(M_a) - \beta_{n-1}(M_a) + \beta_{n-2}(M_a),$$

$$m_n = m_{n-1} + m_{n-2} + ... + (-1)\beta_n(M_a),$$

(iii) The relations (10.1) imply the inequalities,

$$m_k > \beta_k(M_a) (k = 0, ..., n).$$

Proof of (i). $\beta_k(M_a)$ is finite, since $H^\infty_{\omega-1}$ is PD (Theorem 5.1). Moreover $\beta_k(M_a) = 0$ when $q > n$, by Theorem 5.3.

The following two lemmas aid in proving the relations (10.1) of (ii).

**Lemma 10.1.** Let $a$ be a $T$-critical value of a $T$-critical point $p_a$ of positive index $k = k_0$. Then

$$\beta_{k-1} (F_a) = 0 = \beta_{k-1} (\hat{F}_a),$$

$$\beta_k (F_a) = \beta_k (\hat{F}_a) = 1 = 0,$$

according as } s^* = 0 \text{ or } s^* > 0. \text{ Moreover }

\tag{10.5}
\beta_q \left( F_{n+1} \right) = \beta_q \left( F_n \right) \quad (k \neq q \text{ or } q+1).

Theorem 9.2 implies (10.3).

According to Definition 9.2 when } k = n > 0, s^* = 0 \text{ or } s^* > 0, \text{ according as the torsion index } s^* \text{ exists or fails to exist. With this understood Theorem 7.2, (i) and (ii), imply (10.4). Moreover (10.5) follows from Corollary 4.2 (iii). Thus Lemma 10.1 is true.}

A second lemma is needed to prove the relation (10.1). It makes use of integers } a_q \text{ and } \gamma_q \text{ defined for } q = 0, 1, 2, \ldots \text{ as follows.}

I. \text{ When } q > n, a_q = \gamma_q = 0.

II. \text{ When } q = 1, 2, \ldots, n, a_q \text{ and } \gamma_q \text{ equal the number of } T \text{-critical points } p_{q_i} \text{ of } T \text{-index } q \text{ with } s^* = 0 \text{ and } s^* > 0, \text{ respectively.}

III. \text{ When } q = 0, a_q \text{ is the number of } T \text{-critical points with } T \text{-index } 0, \text{ and } \gamma_q = 0.

\text{Lemma 10.2. The Betti number }

\tag{10.6}
\beta_q \left( M_{n+1} \right) = a_q - \gamma_{q+1} \quad (q = 0, 1, 2, \ldots).

\text{Proof of (10.6). We refer to the } T \text{-critical values } a_{a_1} < a_2 < \ldots < a_{a} \text{ of } F \text{ and, for } i = 1, \ldots, n \text{ and } q = 0, 1, \ldots \text{ set}

\tag{10.7'}
A^q_i = \beta_q \left( F_{a_{q_i}} \right) - \beta_q \left( F_{a_{q_i+1}} \right),

\tag{10.7''}
D^q_i = \beta_q \left( F_{a_{q_i}} \right) - \beta_q \left( F_{a_{q_{i+1}}} \right)

\text{so that}

\tag{10.8}
A^q_i + D^q_i = \beta_q \left( F_{a_{q_i}} \right) - \beta_q \left( F_{a_{q_{i+1}}} \right).

\text{Hence for } q = 0, 1, 2, \ldots

\tag{10.8}
\beta_q \left( M_{n+1} \right) - \beta_q \left( F_{a_{q+1}} \right) = (A^q_1 + D^q_1) + \cdots + (A^q_r + D^q_r).

\text{Now each value } A^q = 0 \text{ in (10.8), since there exists an } F \text{-traction of } \tilde{F}_n \text{ into } F_{a_{q+1}} \text{ by Theorem 2.2, and hence an isomorphic mapping of } B_q(F_{a_{q+1}}, Z) \text{ onto } B_q(F_{a_{q+1}}, Z), \text{ by Theorem 3.1. We infer from (10.8) that}

\tag{10.9}
\beta_q \left( M_{n+1} \right) - \beta_q \left( F_{a_{q+1}} \right) = D^q_1 + \cdots + D^q_r.

\text{The difference } D^q_i \text{ has the value}

\tag{10.10}
\beta_q \left( F_{a_{q_i}} \right) - \beta_q \left( F_{a_{q_{i+1}}} \right) \quad (i = 1, 2, \ldots, r)

\text{and by Lemma 10.1, equals 1 when } q = k_{a_i} \text{ and } s^* = 0. \text{ It equals } -1 \text{ when } q+1 = k_{a_i} \text{ and } s^* > 0. \text{ Otherwise the difference } (10.10) \text{ is zero.} \text{ Relation (10.6) follows.}

\text{References}

P-ideals and F-ideals in rings of continuous functions

by

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Abstract. A ring of continuous functions is a ring of the form \( C(X) \), the ring of all continuous real-valued functions on a completely-regular Hausdorff space \( X \).

The author defines two classes of ideals in \( C(X) \), P-ideals and P-ideals, which are analogs of F-spaces and F-spaces. He then discusses properties of these ideals, such as their structure spaces and zero-sets of their members, and characterizes those spaces \( X \) for which there exist P-ideals (or F-ideals) in \( C(X) \).

Introduction. If \( X \) is a space so that every prime ideal in \( C(X) \) is maximal, then \( X \) is said to be a P-space. We extend this concept to ideals in rings of continuous functions by defining a non-zero ideal \( I \) to be a P-ideal if every proper prime ideal in \( I \) is a maximal ideal in \( I \). It is known [2, 14,29] that \( C(X) \) is a P-ideal, i.e. \( X \) is a P-space, if and only if its real structure space \( (eX) \) is a P-space. We show that a modified version of this theorem holds for P-ideals. We also characterize those spaces whose rings of continuous functions possess a P-ideal.

If \( X \) is a space so that \( mM := \{ f \in C(X) : \text{sup} f = 0 \} \) is prime for every maximal ideal \( M \) in \( C(X) \), then \( X \) is said to be an F-space. We extend this concept also to ideals, by defining a non-zero ideal \( I \) to be an F-ideal if \( mM \) is prime whenever \( M \not\subseteq I \) and \( M \) is a maximal ideal in \( C(X) \). We are then able to show that \( I \) is an F-ideal if and only if its structure space is an F-space, an analog to the theorem that \( X \) is an F-space if and only if \( \beta X \) is an F-space. We are also able to characterize those spaces whose rings of continuous functions possess an F-ideal.

Preliminaries and notations. The reader is referred to section 2 in [4] for most of the preliminaries. Familiarity with [2] is also assumed.

If \( f \in C(X) \), then \( Z(f) = \{ x \in X : f(x) = 0 \} \), \( \text{pos} f = \{ x \in X : f(x) > 0 \} \), and \( \text{neg} f = \{ x \in X : f(x) < 0 \} \). If \( f \in C(X) \) (i.e. \( f \) is bounded), then \( \text{int} \) denotes the extension of \( f \) to \( \beta X \). In general, \( Z(f) \subseteq Z(f) \supseteq Z(f) \supseteq Z(f) \supseteq Z(f) \supseteq Z(f) \).

We shall use the letter \( M \) for maximal ideals of \( C(X) \), and \( M = \{ f \in C(X) : f(x) = 0 \} \).

We regard \( \beta X \) as the structure space of \( C(X) \). Thus if \( U \) is open in \( \beta X \), \( U = \{ M : M \not\subseteq I \} \) for some ideal \( I \) in \( C(X) \).