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A theory of absolute proper retracts

by

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Abstract. We construct a theory of absolute *proper* retracts (APR's) for locally compact metric spaces analogous to the usual theory, only requiring that all maps be proper. The APR's are shown to be the non-compact ANR's having property SUV^{∞}. We obtain the standard extension theorems and a result characterizing the APR's by a property of their Freudenthal compactification.

1. Introduction. In this paper it is our aim to lay the foundation for a study of absolute *proper* retracts and absolute neighborhood *proper* retracts. The basic idea is to modify the definition of absolute retract and absolute neighborhood retract by requiring that all maps be proper.

Rather than concentrating at this time on the general properties of absolute proper retracts and absolute neighborhood proper retracts, we shall limit ourselves to the basic definitions and facts, and to the problem of identifying the absolute proper retracts and absolute neighborhood proper retracts among the ANR's. For absolute neighborhood proper retracts, the result is essentially trivial (and well-known). However, we include it here for completeness. It is that, for the class of spaces under consideration, X is an absolute neighborhood proper retract if and only if $X \in$ ANR. However, for absolute proper retracts, the situation is more complicated, and we show that X is an absolute proper retract if and only if X is non-compact, $X \in$ ANR, and X has a certain geometric property called *property* SUV^{∞}. As a tool, we obtain a result about the Freudenthal compactification of ANR's having property SUV^{∞} which is of interest in its own right.

2. Absolute proper retracts and absolute neighborhood proper retracts. A map $f\colon X \to Y$ is said to be *proper* if $f^{-1}(C)$ is compact for each compact set $C \subset Y$. Proper maps seem to make good geometric sense as a vehicle for the study of locally compact metric spaces (e.g., see the results of [2]), and throughout this paper we shall restrict our attention to this class of

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spaces. We therefore make the following standing hypothesis: *All spaces considered in this paper shall be locally compact metric spaces.* By ANR (AR) we mean absolute neighborhood retract (absolute retract) for metric spaces, and if X is such we write $X \in$ ANR ($X \in$ AR).

Suppose $X \subset Y$. Then X is a *neighborhood proper retract* in Y provided there exist a neighborhood U of X in Y and a proper retraction of U onto X. The space X is an *absolute neighborhood proper retract* if for every space Y and every embedding $h\colon X \to Y$, $h(X)$ is a neighborhood proper retract in Y. (Remark. All embeddings shall be *closed*.) Obviously, if X is an absolute neighborhood proper retract, then $X \in$ ANR. Conversely, if $X \in$ ANR and h is an embedding of X into the space Y, then there exist an open neighborhood V of $h(X)$ in Y and a retraction $r\colon V \to h(X)$. By [2; Lemma 3.2], there exists a closed neighborhood U of X lying in V so that $r|U$ is proper. Thus we have the following result.

THEOREM 2.1. *Suppose X is a locally compact metric space. Then X is an absolute neighborhood proper retract if and only if $X \in$ ANR.*

Suppose now that the space X is separable and that QX, the quasi-component space of X, is compact. Then the Freudenthal compactification of X, denoted by FX, is a metric space ([4]). We shall denote $(FX - X)$ by EX. EX is the space of *ends* of X, and has played an important role in the study of geometric properties of non-compact spaces and proper maps (e.g., see [2], [8]). An important fact for us here is that if $f\colon X \to Y$ is a proper map between separable spaces X and Y, where QX and QY are compact, then f has a unique extension to a map of pairs $Ff\colon (FX, EX) \to (FY, EY)$ ([2; Lemma 4.2] or [7; Theorem 3]).

Consider, for example, the following problem: Does there exist a proper retraction from the plane E^2 onto the x-axis? The answer is no, for if there were a proper map f from E^2 *onto* the real numbers R, then f would induce the map Ff from FE^2 into FR. Since E^2 has one end, while R has two, this would yield a map from the 2-sphere FE^2 *onto* a half-open interval, an impossibility. In fact, what this argument shows is that if f is a proper map from X *onto* Y, then Ff maps EX *onto* EY, so that $\operatorname{card} EX \geqslant \operatorname{card} EY$.

Suppose $X \subset Y$. Then X is a *proper retract* of Y if there exists a proper retraction from Y onto X. The above example suggests that any theory of *absolute* proper retracts should take into account the geometry inherent in the space of ends. (For absolute *neighborhood* proper retracts this was not necessary. Essentially this is because if $X \subset Y$, then X has a closed neighborhood V in Y so that the inclusion of X into V is end-preserving in the sense described in the next paragraph.) We shall be interested only in the case in which the Freudenthal compactification is a metric space, so we shall restrict ourselves to the class Σ of locally compact separable metric spaces X for which QX is compact.

Suppose $f\colon X \to Y$ is a proper map. Then f is *end-preserving* if $Ff|EX$ is injective. Now, $X \in \Sigma$ is said to be an *absolute proper retract* if for each space $Y \in \Sigma$ and end-preserving embedding $h\colon X \to Y$, $h(X)$ is a proper retract of Y. For such an X, we shall write $X \in$ APR.

Suppose now that $X \in$ APR. Then FX is a compact metric space and, as a consequence, we can suppose that FX lies in Q, the Hilbert cube. It follows from well-known characterizations of the Freudenthal compactification that $F(Q - EX)$ is Q. Therefore the inclusion of X into $(Q - EX)$ is end-preserving, and hence X is a proper retract of $(Q - EX)$. Since $(Q - EX) \in$ ANR, we have $X \in$ ANR. As a matter of fact, we can suppose that FX is embedded in Q as a Z-set, from which it follows that $(Q - EX)$ is contractible, and hence an absolute retract. (The information on Z-sets required for our purposes can be found in [1].) Also, since no compact space is a proper image of a non-compact space, X must be non-compact. Thus we have the following result.

THEOREM 2.2. *If $X \in$ APR, then X is non-compact and $X \in$ AR.*

The converse of Theorem 2.2 is false. For example, let B denote a 3-cell and let $p \in \operatorname{Bd} B$. Let $E^2 = \operatorname{Bd} B - \{p\}$ and $H = B - \{p\}$. Then the inclusion of E^2 into H is end-preserving. If there were a proper retraction $r\colon H \to E^2$, then Fr would yield a retraction of B onto $\operatorname{Bd} B$, an impossibility. Hence $E^2 \notin$ APR.

We wish now to classify the APR's among the ANR's. The results of the next section shall provide the necessary tools for this.

3. The Freudenthal compactification of an SUV$^\infty$-ANR. Suppose $X \in$ ANR. There are apparently few known results which connect some geometric property of X with the property of having a certain compactification $\hat{X}$ such that $\hat{X} \in$ AR. In this section we provide one such result where $\hat{X}$ is FX. This result will then be applied in the next section to obtain the classification of the APR's.

The geometric property we use is called *property* SUV$^\infty$, and if X has this property we write $X \in$ SUV$^\infty$. For information on property SUV$^\infty$ and its geometric significance, see [5], [8], and [9]. For our purposes we need not even state the definition of property SUV$^\infty$ (given in [5] and [8]), but shall require only statements (*) and (**) below. In these statements, by *tree* we mean a locally finite, connected and simply connected simplicial 1-complex.

(*) If $X \in$ ANR, then $X \in$ SUV$^\infty$ if and only if there exists a tree T such that X is property homotopically dominated by T. (This follows from [8; Corollary 3.5] and [2; Theorem 3.12].)

(**) If $X \in$ SUV$^\infty$ and X is embedded as a Z-set in the Q-manifold M, then there exists a tree T such that X has arbitrarily close closed neighborhoods homeomorphic to $T \times Q$. (See [9; Corollary 2].)

If X is a space and $X' \subset X$, then X' is an *unstable* subset of X if there exists a homotopy $H\colon X\times I \to X$ such that $H(x,0) = x$ for all $x \in X$ and $H(x,t) \notin X'$ for all $x \in X$ and $0 < t \leqslant 1$. (See [6].)

Now we are in a position to state the main result of this section; it was inspired by, and answers, a question raised by T. A. Chapman during a coffee-break conversation with the author.

THEOREM 3.1. *Suppose X is a locally compact connected metric* ANR. *Then $X \in$ SUV$^\infty$ if and only if*

(1) *$FX \in AR$, and*

(2) *EX is an unstable subset of FX.*

We precede the proof of Theorem 3.1 by three lemmas. The first of these has an easy proof and was stated as Lemma 2.2 of [8]. We restate it here for convenience.

LEMMA 3.2. *If T is a tree, then $FT \in$* AR *and ET is an unstable subset of FT.*

LEMMA 3.3. *Suppose $X' \subset X \subset Y$ and $r\colon Y\to X$ is a retraction. Then X' is an unstable subset of X if $r^{-1}(X')$ is an unstable subset of Y.*

Proof. Let $H\colon Y\times I \to Y$ be a homotopy such that $H(y,0) = y$ for all $y \in Y$ and $H(y,t) \notin r^{-1}(X')$ for all $y \in Y$ and $0 < t \leqslant 1$. Define $G\colon X\times I\to X$ by $G(x,t) = r\big(H(x,t)\big)$. Then $G(x,0) = r\big(H(x,0)\big) = r(x) = x$ for all $x \in X$. If $x \in X$ and $0 < t \leqslant 1$, $G(x,t) = r\big(H(x,t)\big) \notin X'$ since $H(x,t) \notin r^{-1}(X')$. Hence X' is an unstable subset of X.

LEMMA 3.4. *If T is a tree, then $F(T\times Q) \in$* AR *and $E(T\times Q)$ is an unstable subset of $F(T\times Q)$.*

Proof. If i is a positive integer and $0 \leqslant t \leqslant 1$, let $I_i^t = [-t, t]$. We let $Q_t = \prod_{i=1}^{\infty} I_i^t$, and we let Q_1 be our model for Q. Let $\varphi\colon FT\to[0,1]$ be a map such that $\varphi(x) = 0$ if and only if $x \in ET$. Let $Y = \{(x,y) \in FT\times Q \mid y \in Q_{\varphi(x)}\}$ and $Y_0 = \{(x,y) \in Y \mid x \in ET\}$. It is easy to construct a retraction $r\colon FT\times Q\to Y$ such that $r^{-1}(Y_0) = ET\times Q$. It is also easy to show that the pair (Y, Y_0) is homeomorphic to the pair $\big(F(T\times Q), E(T\times Q)\big)$. Now, by Lemma 3.2, $FT \in$ AR, and hence $FT\times Q \in$ AR. Then, since Y is a retract of $FT\times Q$, $Y \in$ AR and, since $Y \cong F(T\times Q)$, $F(T\times Q) \in$ AR.

By Lemma 3.2, ET is an unstable subset of FT. It follows easily that $ET\times Q$ is an unstable subset of $FT\times Q$. Since $r^{-1}(Y_0) = ET\times Q$, it follows from Lemma 3.3 that Y_0 is an unstable subset of Y. Since $(Y, Y_0) \cong \big(F(T\times Q), E(T\times Q)\big)$, $E(T\times Q)$ is an unstable subset of $F(T\times Q)$.

Proof of Theorem 3.1. Suppose first of all that $X \in$ SUV$^\infty$. By [1; Theorem 3.1], we may suppose that X is a Z-set in $K = Q - \{pt\}$. Since $X \in$ ANR, it follows from Theorem 2.1 that there exist a neighborhood U of X in K and a *proper* retraction $s\colon U\to X$. By (**), there exist a tree T and a closed neighborhood W of X in Q such that $W \subset U$ and $W \cong T\times Q$. We denote $s|W$ by s^*. Since s^* is proper, s^* extends to $Fs^*\colon FW\to FX$. Hence, regarding FX as a subspace of FW, FX is a retract of FW via a retraction $r = Fs^*$ so that $r^{-1}(EX) = EW$. Since $W \cong T\times Q$, it follows from Lemma 3.4 and Lemma 3.3 that $FX \in$ AR and that EX is an unstable subset of FX.

Now suppose that $FX \in$ AR and that EX is an unstable subset of FX. We may regard FX as being embedded in Q so that EX is a Z-set. It is easy to construct a tree $T \subset (Q - EX)$ so that T is a strong proper deformation retract of $(Q - EX)$. From this it follows that T and $(Q-EX)$ are of the same proper homotopy type and hence, by (*), that $(Q - EX) \in$ SUV$^\infty$.

By [8; Lemma 2.1] there exists a retraction $s\colon Q\to FX$ such that $s(Q-EX) \subset X$. Then $r = s|(Q-EX)$ is a proper retraction of $(Q-EX)$ onto X. It follows that X is properly homotopically dominated by $(Q-EX)$. But $(Q-EX) \in$ SUV$^\infty$ and so, by (*), $X \in$ SUV$^\infty$.

In the second half of the above proof, the general hypothesis that $X \in$ ANR follows from the fact that X is an open subset of FX, assumed to be an AR. In the first half of the proof, the hypothesis that $X \in$ ANR is necessary, and cannot be replaced by the weaker condition that X be locally contractible. A compact example is the space Y constructed in [3; Corollary 11.2, p. 126], and non-compact examples can be obtained by attaching a tree to Y at its vertex. (My thanks to B. J. Ball for pointing this out.) One can also construct in E^3 a 2-dimensional locally connected (although, of course, not locally contractible) space X such $X \in$ SUV$^\infty$ but $FX \notin$ AR.

4. The classification of absolute proper retracts. We are now equipped to prove the main classification theorem for APR's.

THEOREM 4.1. *Suppose X is a locally compact metric space. Then $X \in$* APR *if and only if X is non-compact, $X \in$* ANR, *and $X \in$* SUV$^\infty$.

Proof. Suppose first of all that $X \in$ APR. We have already seen, in Theorem 2.2, that X is non-compact and $X \in$ ANR. We may assume that FX is embedded as a Z-set in Q, from which it follows, as in the proof of Theorem 3.1, that $(Q-EX) \in$ SUV$^\infty$. The inclusion of X into $(Q-EX)$ is end-preserving, so that X is a proper retract of $(Q-EX)$. Hence, X is properly homotopically dominated by $(Q-EX)$ and, by (*), $X \in$ SUV$^\infty$.

Now suppose that X is non-compact, $X \in$ ANR, and $X \in$ SUV$^\infty$. Since $X \in$ SUV$^\infty$, X is separable and connected, hence $X \in \Sigma$. By Theorem 3.1, $FX \in$ AR and EX is an unstable subset of FX. Also, since X is non-compact, EX is non-empty. Now suppose $Y \in \Sigma$ and that $h\colon X\to Y$ is an end-preserving embedding. Then Fh is an embedding of FX into FY.

Let $X^* = (Fh)(FX)$, and let $f\colon (X^*) \cup (EY) \to X^*$ be a map such that $f|X^* = \mathrm{id}_{X^*}$ and $f(EY) \subset (Fh)(EX)$. (The existence of f follows from the fact that every non-empty closed subset of a compact, 0-dimensional space is a retract of that space.) By [8; Lemma 2.1], f extends to a map $f^*\colon FY \to X^*$ such that $f^*(Y - EY) \subset h(X)$. Define $r\colon Y \to h(X)$ by $r(y) = f^*(y)$ for all $y \in Y$. Then r is a proper retraction of Y onto $h(X)$, and thus $X \in \mathrm{APR}$.

Combining Theorems 3.1 and 4.1, we have the following

THEOREM 4.2. *Suppose X is a locally compact metric space. Then $X \in \mathrm{APR}$ if and only if X is non-compact, $FX \in \mathrm{AR}$, and EX is an unstable subset of FX.*

5. Some extension theorems for proper maps. While not going deeply into the theory of APR's at this time, we shall at least give the obvious analogs of the well-known extension theorems of [3; Chapter IV, Section 4].

To simplify the statements of these results, we shall introduce the following notation: $(X, X_0) \in \Sigma'$ shall mean that X_0 is a closed subspace of X, each of X and X_0 are in the class Σ, and the inclusion of X_0 into X is end-preserving.

The proof of the following theorem is evident and shall be omitted.

THEOREM 5.1. *Suppose Y is a locally compact metric space and X is a closed subspace of Y.*

(a) *If $X \in \mathrm{ANR}$, then there exists a neighborhood U of X in Y such that if Z is a space and $f\colon X \to Z$ is a proper map, then f extends to a proper map $f^*\colon U \to Z$.*

(b) *If $(X, Y) \in \Sigma'$, $X \in \mathrm{APR}$, Z is a space, and $f\colon X \to Z$ is a proper map, then f extends to a proper map $f^*\colon Y \to Z$.*

THEOREM 5.2. *Suppose $X \in \Sigma$. Then $X \in \mathrm{APR}$ if and only if for each $(Z, Z_0) \in \Sigma'$, every proper map from Z_0 into X extends to a proper map from Z into X.*

Proof. Suppose first of all that $X \in \mathrm{APR}$. Suppose $(Z, Z_0) \in \Sigma'$ and $f\colon Z_0 \to X$ is a proper map. Then Ff maps the pair (FZ_0, EZ_0) into (FX, EX). Since the inclusion of Z_0 into Z is end-preserving, Ff extends to a map G of $(FZ_0) \cup (EZ)$ into FX such that $G(EZ) \subset EX$. Now, $FX \in \mathrm{AR}$ and EX is an unstable subset of FX so, by [8; Lemma 2.1], there exists a map $k\colon FZ \to FX$ such that $k|(FZ_0) \cup (EZ) = G$ and $k\big(FZ - (FZ_0) \cup (EZ)\big) \subset \subset FX - EX = X$. Define $f^*\colon Z \to X$ by $f^*(z) = k(z)$ for all $z \in Z$. Then f^* is proper and $f^*|Z_0 = f$.

To prove the sufficiency of the given condition, suppose $Y \in \Sigma$ and that $h\colon X \to Y$ is an end-preserving embedding. Then $\big(Y, h(X)\big) \in \Sigma'$ and $h^{-1}\colon h(X) \to X$ is proper, so there exists a proper map $g\colon Y \to X$ such that for all $y \in h(X)$, $g(y) = h^{-1}(y)$. Let $r = h \circ g\colon Y \to h(X)$. Then r is a proper retraction of Y onto $h(X)$, and hence $X \in \mathrm{APR}$.

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