Spaces defined by topological games *

by

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Abstract. A pursuit-evasion game $G(\boldsymbol{K}, X)$ in which the pursuer and the evader choose certain subsets of a topological space X in a certain way is defined and studied here. Although this game resembles that of Banach–Mazur, it provides for completely different methods and problems to be introduced. Besides the thorough study of the game many interesting topological applications are investigated. Establishing the close relation between spaces of the class $\boldsymbol{K}$ and the space X in case of a winning strategy for one of the players makes it possible to prove many new theorems for different types of topological spaces (e.g., a very general product theorem for paracompact spaces is established). Many open questions and research problems are stated throughout the paper.

Introduction. Since its introduction the theory of games has found extensive applications in many scientific fields. D. Gale and F. M. Stewart [7] have introduced and studied the games of perfect information with an infinite number of strategies. The notion of a topological game with perfect information was introduced by C. Berge [2]. One specific game of this type is that of Banach–Mazur (see J. C. Oxtoby [17], Chapter 6). Since game-theoretical methods were implicitly used in the solution of some topological problems (e.g., [1] and [18]), it seems appropriate to conduct a more thorough investigation of the relation of game theory and general topology.

In the present paper we define and study the game $G(\boldsymbol{K}, X)$ and its applications to different problems in general topology. Although this game resembles that of Banach–Mazur, it provides for completely different methods and problems to be introduced. We focus our attention on topological and not on logical or set-theoretical aspects.

For a better orientation the paper is divided into 16 sections, each containing only the results closely related to the title. Section 1 contains the basic notation and the definition of the game $G(\boldsymbol{K}, X)$. Section 5 is

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concerned with s.c. $\boldsymbol{K}$-covers which are related to stationary strategies and to some cover properties of spaces. In Section 9 game-theoretical properties of $\boldsymbol{K}$-scattered spaces are studied. In Section 14 we consider games in product spaces which permit us to prove several new results on the paracompactness of product spaces. Section 15 is devoted to now-here locally $\boldsymbol{K}$ spaces which form a dual class to $\boldsymbol{K}$-scattered spaces. Finally, Section 16 treats of the determinacy of $G(\boldsymbol{K}, X)$.

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1. The games. Each space considered here is assumed to be completely regular and each map is assumed to be continuous. Spaces are denoted by the letters $X, Y, Z, \ldots$ By $\mathrm{Cl}_X E$ ($\mathrm{Int}_X E$) we denote the closure (resp. the interior) of the subset E of X. 2^X denotes the family of all closed subsets of X. The set of all natural numbers $0, 1, 2, \ldots$ is denoted by N and natural numbers are denoted by $m, n, k, \ldots$ The Greek letters $\alpha, \beta, \gamma, \ldots$ denote ordinal numbers.

The axiom of choice is used often and without special mention, but 16.16–16.18.

The topological terminology and some basic facts are taken from R. Engelking's book [5].

Let K denotes a non-void class of spaces for which $X \in \boldsymbol{K}$ implies $2^X \subseteq \boldsymbol{K}$.

We shall take for $\boldsymbol{K}$, in particular, the following classes of spaces: $\boldsymbol{1}$ — the class consisting of all one-point spaces and of the empty space; $\boldsymbol{F}$ — the class of all finite spaces; $\boldsymbol{C}$ — the class of all compact spaces; $\boldsymbol{\check{C}}$ — the class of all spaces which are complete in the sense of Čech; $\boldsymbol{D}$ — the class of all discrete spaces; and some other derived classes ($\boldsymbol{FK}$, $\sigma\boldsymbol{K}$, $\boldsymbol{DK}$, $\boldsymbol{LK}$ and $\boldsymbol{SK}$), which will be defined later.

For any class $\boldsymbol{K}$ and for any space X, $G(\boldsymbol{K}, X)$ denotes the following positional game with perfect information. There are two players I and II (the pursuer and the evader). They choose alternatively consecutive terms of a sequence $(E_n\colon n \in N)$ of subsets of X so that each player knows $\boldsymbol{K}, E_0, E_1, \ldots, E_n$ when he is choosing E_{n+1}.

A sequence $(E_n\colon n \in N)$ of subsets of X is a play of $G(\boldsymbol{K}, X)$ if $E_0 = X$ and if for each $n \in N$:

(1) E_{2n+1} is the choice of I;

(2) E_{2n+2} is the choice of II;

(3) $E_{2n+1} \in \boldsymbol{K}$;

(4) $E_n \in 2^X$;

(5) $E_{2n+1} \subseteq E_{2n}$;

(6) $E_{2n+2} \subseteq E_{2n}$;

(7) $E_{2n+1} \cap E_{2n+2} = 0$.

The player I wins if $\bigcap \{E_{2n}\colon n \in N\} = 0$. The player II wins if $\bigcap \{E_{2n}\colon n \in N\} \neq 0$.

1.1. Remark. The game $G(\boldsymbol{K}, X)$ is a special case of a game $G_\alpha(\boldsymbol{K}, X)$, where α is a given ordinal number. The players I and II choose consecutive terms of a transfinite sequence $(E_\xi\colon \xi < \alpha)$ of subsets of X. The game $G_{\omega_1}(\boldsymbol{K}, X)$ is implicitly used by A. V. Arhangel'skiĭ [1].

1.2. Remark. The game $G(\boldsymbol{1}, X)$ was discovered independently by F. Galvin. A. Ehrenfeucht, F. Galvin and J. Mycielski have some unpublished results on $G(\boldsymbol{1}, X)$.

A finite sequence $(E_m\colon m \leqslant n)$ of subsets of X is *admissible* for $G(\boldsymbol{K}, X)$ if the sequence $(E_0, E_1, \ldots, E_n, 0, 0, \ldots, 0, \ldots)$ is a play of $G(\boldsymbol{K}, X)$.

A function s is a *strategy* for I (II) in $G(\boldsymbol{K}, X)$ if the domain of s consists of admissible sequences $(E_0, \ldots, E_n)$ with n even (resp. odd); if $s(E_0, \ldots, E_n \subseteq 2^X$ and if $(E_0, \ldots, E_n, E_{n+1})$ is admissible for $E_{n+1} = s(E_0, \ldots, E_n)$.

A function S is a *many-valued strategy* for I (II) in $G(\boldsymbol{K}, X)$ if the domain of S consists of admissible sequences $(E_0, \ldots, E_n)$ with n even (resp. odd) if $S(E_0, \ldots, E_n) \subseteq 2^X$ and if $(E_0, \ldots, E_n, E_{n+1})$ is admissible for each $E_{n+1} \in S(E_0, \ldots, E_n)$.

A function $s\colon 2^X \to 2^X \cap \boldsymbol{K}$ is a *stationary strategy* for I in $G(\boldsymbol{K}, X)$ if $s(E) \subseteq E$ for each $E \in 2^X$. A play $(E_n\colon n \in N)$ of $G(\boldsymbol{K}, X)$ is played with the stationary strategy s if $E_{2n+1} = s(E_{2n})$ for each $n \in N$.

A function $s\colon 2^X \cap \boldsymbol{K} \to 2^X$ is a *stationary strategy* for II in $G(\boldsymbol{K}, X)$ if $s(E) \cap E = 0$ for each $E \in 2^X \cap \boldsymbol{K}$. A play $(E_n\colon n \in N)$ of $G(\boldsymbol{K}, X)$ is played with the stationary strategy s if $E_{2n+2} = s(E_{2n+1}) \cap E_{2n}$ for each $n \in N$.

A strategy s is said to be *winning* for I (for II) in $G(\boldsymbol{K}, X)$ if I (resp. II) using s wins each play of $G(\boldsymbol{K}, X)$.

$I(\boldsymbol{K}, X)$ ($II(\boldsymbol{K}, X)$) denotes the set of all winning strategies of I (resp. of II) in $G(\boldsymbol{K}, X)$.

$I_s(\boldsymbol{K}, X)$ ($II_s(\boldsymbol{K}, X)$) denotes the set of all stationary winning strategies of I (resp. of II) in $G(\boldsymbol{K}, X)$.

Clearly, $I_s(\boldsymbol{K}, X) \neq 0$ implies $I(\boldsymbol{K}, X) \neq 0$ and $II_s(\boldsymbol{K}, X) \neq 0$ implies $II(\boldsymbol{K}, X) \neq 0$.

A space X is said to be $\boldsymbol{K}$-*like* if $I(\boldsymbol{K}, X) \neq 0$. A space X is said to be *anti-$\boldsymbol{K}$-like* if $II(\boldsymbol{K}, X) \neq 0$.

If $I(\boldsymbol{K}, X) \neq 0$ and if $\boldsymbol{K} = \boldsymbol{F}$, $\boldsymbol{C}$, $\boldsymbol{\check{C}}$ or $\boldsymbol{D}$, we shall call X *finite-like*, *compact-like*, *Čech-like* or *discrete-like*, respectively.

2. Subclasses and closed subsets. The following theorem follows easily from the definition of the winning strategy:

2.1. THEOREM. *Let* $\boldsymbol{K}_1 \subseteq \boldsymbol{K}_2$. *Then*

2.1.1. $I(\boldsymbol{K}_1, X) \subseteq I(\boldsymbol{K}_2, X)$ *and therefore* $I(\boldsymbol{K}_1, X) \neq 0$ *implies* $I(\boldsymbol{K}_2, X) \neq 0$.

2.1.2. $II(\boldsymbol{K}_2, X) \subseteq II(\boldsymbol{K}_1, X)$ *and therefore* $II(\boldsymbol{K}_2, X) \neq 0$ *implies* $II(\boldsymbol{K}_1, X) \neq 0$.

By 2.1 it follows that

2.2. COROLLARY. *1-like implies finite-like, which implies compact-like, which implies Čech-like; finite-like implies discrete-like.*

2.3. COROLLARY. *Anti-Čech-like implies anti-compact-like, which implies anti-finite-like, which implies anti-1-like; anti-discrete-like implies anti-finite-like.*

2.4. THEOREM. *Let* $E \in 2^X$. *Then* $I(\boldsymbol{K}, X) \neq 0$ *implies* $I(\boldsymbol{K}, E) \neq 0$.

Proof. Let $s \in I(\boldsymbol{K}, X)$. We shall define $t \in I(\boldsymbol{K}, E)$. Let us set $E_0 = X$, $F_0 = E$, $E_1 = s(E_0)$, $F_1 = F_0 \cap E_1$ and $t(F_0) = F_1$. Then $F_1 \in 2^E \cap \boldsymbol{K}$. Let us take $F_2 \in 2^E$ with $F_1 \cap F_2 = 0$. We set $E_2 = F_2$, $E_3 = s(E_0, E_1, E_2)$, $F_3 = E_3$ and $t(F_0, F_1, F_2) = F_3$. Continuing in this manner, we get the play $(E_n: n \in N)$ of $G(\boldsymbol{K}, X)$ and the play $(F_n: n \in N)$ of $G(\boldsymbol{K}, E)$. Since $E_n = F_n$ for each $n > 1$, we have $\bigcap \{E_{2n}: n \in N\} = \bigcap \{F_{2n}: n \in N\}$. Since $s \in I(\boldsymbol{K}, X)$, we have $\bigcap \{E_{2n}: n \in N\} = 0$. Thus $\bigcap \{F_{2n}: n \in N\} = 0$. Hence $t \in I(\boldsymbol{K}, E)$.

2.5. THEOREM. *Let* $E \in 2^X$. *Then* $II(\boldsymbol{K}, E) \neq 0$ *implies* $II(\boldsymbol{K}, X) \neq 0$.

Proof. Let $s \in II(\boldsymbol{K}, E)$. We shall define $t \in II(\boldsymbol{K}, X)$. Let us set $E_0 = X$ and $F_0 = E$. Let us take $E_1 \in 2^X \cap \boldsymbol{K}$. We set $F_1 = F_0 \cap E_1$, $F_2 = s(F_0, F_1)$, $E_2 = F_2$ and $t(E_0, E_1) = E_2$. Let us take $E_3 \in 2^X \cap \boldsymbol{K}$ with $E_3 \subseteq E_2$. We set $F_3 = E_3$, $F_4 = s(F_0, F_1, F_2, F_3)$, $E_4 = F_4$ and

$$t(E_0, E_1, E_2, E_3) = E_4 .$$

Continuing in this manner, we get the play $(E_n: n \in N)$ of $G(\boldsymbol{K}, X)$ and the play $(F_n: n \in N)$ of $G(\boldsymbol{K}, E)$. Since $E_n = F_n$ for each $n > 1$, we have $\bigcap \{E_{2n}: n \in N\} = \bigcap \{F_{2n}: n \in N\}$. Since $s \in II(\boldsymbol{K}, E)$, we have $\bigcap \{F_{2n}: n \in N\} \neq 0$. Thus $\bigcap \{E_{2n}: n \in N\} \neq 0$. Hence $t \in II(\boldsymbol{K}, X)$.

3. Maps.

3.1. THEOREM. *Let us assume that there exists a map* f *from* X *onto* Y *for which* $f(E) \in 2^Y \cap \boldsymbol{K}_2$ *whenever* $E \in 2^X \cap \boldsymbol{K}_1$. *Then* $I(\boldsymbol{K}_1, X) \neq 0$ *implies* $I(\boldsymbol{K}_2, Y) \neq 0$.

Proof. Let $s \in I(\boldsymbol{K}_1, X)$. We shall define $t \in I(\boldsymbol{K}_2, Y)$. Let us set $E_0 = X$, $E_1 = s(E_0)$, $F_0 = Y$, $F_1 = f(E_1)$ and $t(F_0) = F_1$. Since $E_1 \in 2^X \cap \boldsymbol{K}_1$, we have $F_1 \in 2^Y \cap \boldsymbol{K}_2$. If F_2 is chosen in 2^Y so that $F_1 \cap F_2 = 0$, then letting $E_2 = f^{-1}(F_2)$ we get $E_1 \cap E_2 \subseteq f^{-1}f(E_1) \cap f^{-1}(F_2) = f^{-1}(F_1) \cap f^{-1}(F_2) = f^{-1}(F_1 \cap F_2) = 0$. We set $E_3 = s(E_0, E_1, E_2)$, $F_3 = f(E_3)$ and $t(F_0, F_1, F_2) = F_3$. Then $F_3 \subseteq F_2$, because $E_3 \subseteq E_2$. Moreover, $F_3 \in 2^Y \cap \boldsymbol{K}_2$. Continuing in this manner, we get the play $(E_n: n \in N)$ of $G(\boldsymbol{K}_1, X)$ and the play $(F_n: n \in N)$ of $G(\boldsymbol{K}_2, Y)$. Since $s \in I(\boldsymbol{K}_1, X)$, we have $\bigcap \{E_{2n}: n \in N\} = 0$. Since $\bigcap \{E_{2n}: n \in N\} = \bigcap \{f^{-1}(F_{2n}): n \in N\} = f^{-1}(\bigcap \{F_{2n}: n \in N\})$ and $f(X) = Y$, we have $\bigcap \{F_{2n}: n \in N\} = 0$. Hence $t \in I(\boldsymbol{K}_2, Y)$.

From 3.1 we get

3.2. COROLLARY. *Let us assume that there exists a map from* X *onto* Y. *Then*

3.2.1. *If* X *is finite-like, then* Y *is also finite-like.*

3.2.2. *If* X *is compact-like, then* Y *is also compact-like.*

3.3. THEOREM. *Let us assume that there exists a closed map from* X *onto* Y *for which* $f^{-1}(E) \in 2^X \cap \boldsymbol{K}_1$ *whenever* $E \in 2^Y \cap \boldsymbol{K}_2$. *Then* $I(\boldsymbol{K}_2, Y) \neq 0$ *implies* $I(\boldsymbol{K}_1, X) \neq 0$.

Proof. Let $s \in I(\boldsymbol{K}_2, Y)$. We shall define $t \in I(\boldsymbol{K}_1, X)$. Let us set $E_0 = X$, $F_0 = Y$, $F_1 = s(F_0)$, $E_1 = f^{-1}(F_1)$ and $t(E_0) = E_1$. Since $F_1 \in 2^Y \cap \boldsymbol{K}_2$, we have $E_1 \in 2^X \cap \boldsymbol{K}_1$. If E_2 is chosen in 2^X so that $E_1 \cap E_2 = 0$, then, letting $F_2 = f(E_2)$, we have $F_1 \cap F_2 = 0$. For if $y \in F_1 \cap F_2$, then there exists an $x \in E_2$ with $f(x) = y$. Since $E_1 = f^{-1}(F_1)$ and $y \in F_1$, we have $x \in E_1$. Thus $x \in E_1 \cap E_2$ and we have obtained a contradiction. We set $F_3 = s(F_0, F_1, F_2)$, $E_3 = E_2 \cap f^{-1}(F_3)$ and $t(E_0, E_1, E_2) = E_3$. Then $E_3 \subseteq E_2$ and $E_3 \in 2^X \cap \boldsymbol{K}_1$. If $E_4 \in 2^X$, $E_3 \cap E_4 = 0$ and $E_4 \subseteq E_2$, then we set $F_4 = f(E_4)$. We claim that $F_3 \cap F_4 = 0$. If $y \in F_3 \cap F_4$, then there exists an $x \in E_4$ with $f(x) = y$. Since $E_4 \subseteq E_2$, we have $x \in E_2$. Since $y \in F_3$, we have $x \in f^{-1}(F_3)$. Hence $x \in E_3 \cap E_4$ and we have obtained a contradiction. We set $F_5 = s(F_0, \ldots, F_4)$, $E_5 = E_4 \cap f^{-1}(F_5)$ and $t(E_0, \ldots, E_4) = E_5$. Continuing in this manner, we get the play $(E_n: n \in N)$ of $G(\boldsymbol{K}_1, X)$ and the play $(F_n: n \in N)$ of $G(\boldsymbol{K}_2, Y)$. Since $s \in I(\boldsymbol{K}_2, Y)$, we have $\bigcap \{F_{2n}: n \in N\} = 0$. Since

$$\bigcap \{F_{2n}: n \in N\} = \bigcap \{f(E_{2n}): n \in N\} \supseteq f(\bigcap \{E_{2n}: n \in N\}) ,$$

we have $\bigcap \{E_{2n}: n \in N\} = 0$. Hence $t \in I(\boldsymbol{K}_1, X)$.

3.4. THEOREM. *Let us assume that there exists a closed map* f *from* X *onto* Y *for which* $f^{-1}(E) \in 2^X \cap \boldsymbol{K}_1$ *whenever* $E \in 2^Y \cap \boldsymbol{K}_2$. *Then* $II(\boldsymbol{K}_1, X) \neq 0$ *implies* $II(\boldsymbol{K}_2, Y) \neq 0$.

The proof of 3.4 is similar to that of 3.3 and thus it is omitted.

3.5. THEOREM. *Let us assume that there exists a map* f *from* X *onto* Y *for which* $f(E) \in 2^Y \cap \boldsymbol{K}_2$ *whenever* $E \in 2^X \cap \boldsymbol{K}_1$. *Then* $II(\boldsymbol{K}_2, Y) \neq 0$ *implies* $II(\boldsymbol{K}_1, X) \neq 0$.

The proof of 3.5 is similar to that of 3.1 and thus it is omitted.

From 3.5 we have

3.6. COROLLARY. *Let us assume that there exists a map from X onto Y. Then*

3.6.1. *If Y is anti-finite-like, then X is also anti-finite-like* (cf. 3.2.1).

3.6.2. *If Y is anti-compact-like, then X is also anti-compact-like* (cf. 3.2.2).

Recall that a map f from X onto Y is called *perfect* if $f(E) \in 2^Y$ for each $E \in 2^X$ and if $f^{-1}(y) \in \boldsymbol{C}$ for each $y \in Y$. A class $\boldsymbol{K}$ is said to be *perfect* if the condition $X \in \boldsymbol{K}$ is equivalent to $Y \in \boldsymbol{K}$, provided that there exists a perfect map from X onto Y.

Combining 3.1, 3.3, 3.4 and 3.5 we get

3.7. THEOREM. *Let $\boldsymbol{K}$ be a perfect class and let there exist a perfect map from X onto Y. Then*

3.7.1. $I(\boldsymbol{K}, X) \neq 0$ *iff* $I(\boldsymbol{K}, Y) \neq 0$, *and*

3.7.2. $II(\boldsymbol{K}, X) \neq 0$ *iff* $II(\boldsymbol{K}, Y) \neq 0$.

It is well known that the classes $\boldsymbol{C}$ and $\check{\boldsymbol{C}}$ are perfect. Hence, by 3.7, we have

3.8. COROLLARY. *Let us assume that there exists a perfect map from X onto Y. Then*

3.8.1. *X is compact-like iff Y is compact-like;*

3.8.2. *X is Čech-like iff Y is Čech-like;*

3.8.3. *X is anti-compact-like iff Y is anti-compact-like;*

3.8.4. *X is anti-Čech-like iff Y is anti-Čech-like.*

4. Finite and countable unions. We denote by $\boldsymbol{FK}$ the class of all $X = \bigcup\{X_m\colon m \leqslant n\}$, where $\{X_m\colon m \leqslant n\} \subset 2^X \cap \boldsymbol{K}$ and $n \in N$.

Clearly, $\boldsymbol{K} \subseteq \boldsymbol{FK}$ and $X \in \boldsymbol{FK}$ implies $2^X \subseteq \boldsymbol{FK}$.

4.1. THEOREM. $I(\boldsymbol{K}, X) \neq 0$ *iff* $I(\boldsymbol{FK}, X) \neq 0$.

Proof. ($\Rightarrow$) Let $I(\boldsymbol{K}, X) \neq 0$. Then $I(\boldsymbol{FK}, X) \neq 0$ by 2.1.1.

($\Leftarrow$) Let $s \in I(\boldsymbol{FK}, X)$. We shall define $t \in I(\boldsymbol{K}, X)$. Let us set $E_0 = X$, $E_1 = s(E_0)$ and $F_0 = E_0$. Since $E_1 \in 2^X \cap \boldsymbol{FK}$, we have

$$E_1 = \bigcup\{H_{1,m}\colon m \leqslant k_1\},$$

where $\{H_{1,m}\colon m \leqslant k_1\} \subseteq 2^X \cap \boldsymbol{K}$. We set $F_1 = H_{1,0}$ and $t(F_0) = F_1$. Let us take $F_2 \in 2^X$ with $F_1 \cap F_2 = 0$. We set $F_3 = F_2 \cap H_{1,1}$ and $t(F_0, F_1, F_2) = F_3$. Continuing in this manner, we get the admissible sequence $(F_0, \ldots, F_{2k_1})$ for $G(\boldsymbol{K}, X)$. We set $F_{2k_1+1} = F_{2k_1} \cap H_{1,k_1}$ and $t(F_0, \ldots, F_{2k_1}) = F_{2k_1+1}$. Let us take $F_{2k_1+2} \in 2^X$ with $F_{2k_1+2} \subseteq F_{2k_1}$ and $F_{2k_1+1} \cap F_{2k_1+2} = 0$. We set $E_2 = F_{2k_1+2}$. Clearly, $E_1 \cap E_2 = 0$. We set $E_3 = s(E_0, E_1, E_2)$. Since $E_3 \in 2^X \cap \boldsymbol{FK}$, we have $E_3 = \bigcup\{H_{3,m}\colon m \leqslant k_3\}$, where

$$\{H_{3,m}\colon m \leqslant k_3\} \subseteq 2^X \cap \boldsymbol{K}.$$

Continuing in this manner, we get the play $(E_n\colon n \in N)$ of $G(\boldsymbol{FK}, X)$ and the play $(F_n\colon n \in N)$ of $G(\boldsymbol{K}, X)$. Since $s \in I(\boldsymbol{FK}, X)$, we have $\bigcap\{E_{2n}\colon n \in N\} = 0$. Since $\{E_{2n}\colon n \in N\} \subseteq \{F_{2n}\colon n \in N\}$, it follows that $\bigcap\{E_{2n}\colon n \in N\} = \bigcap\{F_{2n}\colon n \in N\}$. Thus $\bigcap\{F_{2n}\colon n \in N\} = 0$. Hence $t \in I(\boldsymbol{K}, X)$.

4.2. THEOREM. $II(\boldsymbol{K}, X) \neq 0$ *iff* $II(\boldsymbol{FK}, X) \neq 0$.

The proof of 4.2 is similar to that of 4.1 and thus it is omitted.

From 4.1 and 4.2 immediately follow

4.3. COROLLARY. *X is 1-like iff X is finite-like;*

4.4. COROLLARY. *X is anti-1-like iff X is anti-finite-like.*

We denote by $\boldsymbol{\sigma K}$ the class of all $X = \bigcup\{X_n\colon n \in N\}$, where $\{X_n\colon n \in N\} \subseteq 2^X \cap \boldsymbol{K}$.

In the case of $\boldsymbol{K} = \boldsymbol{C}$, $\check{\boldsymbol{C}}$ or $\boldsymbol{D}$, the space $X \in \boldsymbol{\sigma K}$ is called σ-compact, σ-Čech or σ-discrete, respectively.

Clearly, $\boldsymbol{K} \subseteq \boldsymbol{FK} \subseteq \boldsymbol{\sigma K}$, and $X \in \boldsymbol{\sigma K}$ implies $2^X \subseteq \boldsymbol{\sigma K}$.

4.5. THEOREM. *If $X \in \boldsymbol{\sigma K}$, then* $I_s(\boldsymbol{K}, X) \neq 0$.

Proof. Let $X \in \boldsymbol{\sigma K}$. Then $X = \bigcup\{X_n\colon n \in N\}$, where $\{X_n\colon n \in N\} \subset 2^X \cap \boldsymbol{K}$. We shall define $s \in I_s(\boldsymbol{K}, X)$. If $E = 0$, then we set $s(E) = 0$. If $E \in 2^X$ and $E \neq 0$, then we set $s(E) = E \cap X_n$, where

$$n = \min\{m \in N\colon E \cap X_m \neq 0\}.$$

Let $r\colon 2^X - \{0\} \to N$ be the function defined by setting $r(E) = n$. Let $\{E_n\colon n \in N\}$ be a play of $G(\boldsymbol{K}, X)$ with $E_{2n+1} = s(E_{2n})$ for each $n \in N$. We set $E = \bigcap\{E_{2n}\colon n \in N\}$. Suppose that $E \neq 0$. Then $r(E) > r(E_{2n+2}) > r(E_{2n})$ for each $n \in N$, because $E_{2n+2} \cap s(E_{2n}) = 0$ for each $n \in N$. Hence we have $r(E) = \infty$ and we have obtained a contradiction. Thus $E = 0$. Hence $s \in I_s(\boldsymbol{K}, X)$.

4.6. QUESTION. Is $I_s(\boldsymbol{K}, X) \neq 0$ if $X = \bigcup\{X_n\colon n \in N\}$, where $X_n \in 2^X$ and $I_s(\boldsymbol{K}, X_n) \neq 0$ for each $n \in N$?

4.7. THEOREM. *If $X = \bigcup\{X_n\colon n \in N\}$, where $X_n \in 2^X$ and $I(\boldsymbol{K}, X_n) \neq 0$ for each $n \in N$, then $I(\boldsymbol{K}, X) \neq 0$.*

Proof. Let $X = \bigcup\{X_n\colon n \in N\}$, where $\{X_n\colon n \in N\} \subseteq 2^X$ and $I(\boldsymbol{K}, X_n) \neq 0$ for each $n \in N$. We decompose N into a family $\{N_k\colon n \in N\}$ of infinite subsets. Let $s_0 \in I(\boldsymbol{K}, X_0)$ and let $\{k_n\colon n \in N\}$ be the set N_0 ordered as a strictly increasing sequence. If $(E_n\colon n \in N)$ is a play of $G(\boldsymbol{K}, X)$, then $(X_0 \cap E_{2k_0}, X_0 \cap E_{2k_0+1}, X_0 \cap E_{2k_1}, X_0 \cap E_{2k_1+1}, \ldots)$ is a play of $G(K, X_0 \cap E_{2k_0})$. By 2.4 we have $I(\boldsymbol{K}, X_0 \cap E_{2k_0}) \neq 0$. Let

$$s_0' \in I(\boldsymbol{K}, X_0 \cap E_{2k_0}).$$

Then we have $\bigcap\{X_0 \cap E_{2k_n}: n \in N\} = 0$ provided that the player I uses s_0'. Hence $X_0 \cap \bigcap\{E_{2k_n}: n \in N\} = 0$. Since N_0 is infinite, it follows that $\bigcap\{E_{2k_n}: n \in N\} = \bigcap\{E_{2n}: n \in N\}$. Thus $X_0 \cap \bigcap\{E_{2n}: n \in N\} = 0$. It is clear now that $s \in I(\boldsymbol{K}, X)$ can be defined by the alternate application of strategies $s_n \in I(\boldsymbol{K}, X_n)$ which is prescribed by the decomposition $\{N_k: k \in N\}$ of N. Thus $I(\boldsymbol{K}, X) \neq 0$.

From 4.7 and 2.4 immediately follows

4.8. Corollary. *If E is an F_σ-set in X and if $I(\boldsymbol{K}, X) \neq 0$, then $I(\boldsymbol{K}, E) \neq 0$.*

From 4.7 we also get

4.9. Corollary. *If X is countable (σ-compact, σ-Čech, σ-discrete), then X is finite-like (compact-like, Čech-like, discrete-like, resp.).*

4.10. Question. Does $I(\sigma\boldsymbol{K}, X) \neq 0$ imply $I(\boldsymbol{K}, X) \neq 0$?

4.11. Theorem. *Let us assume that $II(\boldsymbol{K}, X) \neq 0$, $Y_n \in 2^X \cap \boldsymbol{K}$ for each $n \in N$, $Y = \bigcup\{Y_n: n \in N\}$ and $2^{X-Y} \cap K \subseteq 2^X \cap \boldsymbol{K}$. Then*

$$II(\boldsymbol{K}, X-Y) \neq 0 .$$

Proof. Let $s \in II(\boldsymbol{K}, X)$ and $Y = \bigcup\{Y_n: n \in N\}$, where $Y_n \in 2^X \cap \boldsymbol{K}$ for each $n \in N$ and $2^{X-Y} \cap \boldsymbol{K} \subseteq 2^X \cap \boldsymbol{K}$. We shall define $t \in II(\boldsymbol{K}, X-Y)$. Let us set $E_0 = X$, $E_1 = Y_0$, $E_2 = s(E_0, E_1)$ and $F_0 = X-Y$. Let us take $F_1 \in 2^{X-Y} \cap \boldsymbol{K}$. Then $F_1 \in 2^X \cap \boldsymbol{K}$. We set $E_3 = E_2 \cap F_1$, $E_4 = s(E_0, E_1, E_2, E_3)$ and $F_2 = E_4 - Y$. Then $F_2 \in 2^{X-Y}$. We set $t(F_0, F_1) = F_2$. Then (F_0, F_1, F_2) is admissible for $G(\boldsymbol{K}, X-Y)$ and $(E_0, \ldots, E_4)$ is admissible for $G(\boldsymbol{K}, X)$. We set $E_5 = E_4 \cap Y_1$ and $E_6 = s(E_0, \ldots, E_5)$. Let us take $F_3 \in 2^{X-Y} \cap \boldsymbol{K}$ with $F_3 \subseteq F_2$. We set $E_7 = E_6 \cap F_3$. Continuing in this manner, we get the play $(E_n: n \in N)$ of $G(\boldsymbol{K}, X)$ and the play $(F_n: n \in N)$ of $G(\boldsymbol{K}, X-Y)$. Since $s \in II(\boldsymbol{K}, X)$, we have $\bigcap\{E_{2n}: n \in N\} \neq 0$. It is clear that $\bigcap\{E_{2n}: n \in N\} \subseteq X-Y$. Since $\{F_{2n}: n \in N\} \subseteq \{E_{2n} - Y: n \in N\}$, it follows that $\bigcap\{F_{2n}: n \in N\} \neq 0$. Hence $t \in II(\boldsymbol{K}, X-Y)$.

If $E \subseteq Y \subseteq X$ and $E \in \boldsymbol{C}$, then $E \in 2^X \cap 2^Y$. Hence, by 4.11, we get

4.12. Corollary. *If X is anti-compact-like and Y is a σ-compact subset of X, $X-Y$ is anti-compact-like.*

5. $\boldsymbol{K}$-covers. A family $\mathcal{A}$ of open subsets of X is said to be a *$\boldsymbol{K}$-cover* if for each $E \in 2^X \cap \boldsymbol{K}$ there exists an $A \in \mathcal{A}$ for which $E \subseteq A$; i.e. if $2^X \cap \boldsymbol{K}$ refines $\mathcal{A}$.

Clearly, 1-covers and open covers coincide.

5.1. Theorem. *Let $I(\boldsymbol{K}, X) \neq 0$. Then for each sequence $(\mathcal{A}_n : n \in N)$ of $\boldsymbol{K}$-covers there exists a sequence $(A_n : n \in N)$ such that $A_n \in \mathcal{A}_n$ for each $n \in N$ and $\bigcup\{A_n : n \in N\} = X$.*

Proof. Let $s \in I(\boldsymbol{K}, X)$ and let $(\mathcal{A}_n : n \in N)$ be a sequence of $\boldsymbol{K}$-covers of X. For each $n \in N$ and $E \in 2^X \cap \boldsymbol{K}$ there exists an $A_n(E) \in \mathcal{A}_n$ with $E \subseteq A_n(E)$. We define a strategy t for the player II as follows. We set $t(E_0, \ldots, E_{2n+1}) = \bigcap\{X - A_k(E_{2k+1}) : k \leqslant n\}$ for each admissible sequence $(E_0, \ldots, E_{2n+1})$ for $G(\boldsymbol{K}, X)$. Let $(E_n : n \in N)$ be a play of $G(\boldsymbol{K}, X)$, where $E_{2n+1} = s(E_0, \ldots, E_{2n})$ and $E_{2n+2} = t(E_0, \ldots, E_{2n+1})$ for each $n \in N$. Since $s \in I(\boldsymbol{K}, X)$, it follows that $\bigcap\{E_{2n} : n \in N\} = 0$. Thus $\bigcap\{X - A_n(E_{2n+1}) : n \in N\} = 0$. Hence $\bigcup\{A_n(E_{2n+1}) : n \in N\} = X$.

5.2. Remark. The previous theorem is provable under weaker conditions, e.g. if $I(\boldsymbol{K}', X) \neq 0$, where $\boldsymbol{K}' = \{Y : I(\boldsymbol{K}, Y) \neq 0\}$.

From 5.1 we immediately have

5.3. Corollary. *Let $I(\boldsymbol{K}, X) \neq 0$. Then each $\boldsymbol{K}$-cover of X contains a countable cover of X.*

From 5.1 we also get:

5.4. Corollary. *If X is finite-like, then X has property C''* (see [8], p. 527).

5.5. Corollary. *If X is compact-like, then X is a Hurewicz space* (see [10], p. 209).

Let $\mathfrak{m}$ be an infinite cardinal. We say that a space X has the *$\mathfrak{m}$-Lindelöf property* (or, that X is an *$\mathfrak{m}$-Lindelöf space*) if each open cover of X contains a subcover of cardinality $\leqslant \mathfrak{m}$.

By the definition, $\aleph_0$-Lindelöf spaces and Lindelöf spaces coincide.

5.6. Theorem. *If $I(\boldsymbol{K}, X) \neq 0$ and if each $E \in 2^X \cap \boldsymbol{K}$ has the $\mathfrak{m}$-Lindelöf property, then X also has the $\mathfrak{m}$-Lindelöf property.*

Proof. Let $\mathcal{A}$ be an open cover of X. Let $\mathcal{B}$ be the family of all $B \subseteq X$ for which there exists a family $\{A_i : i \in I\} \subseteq \mathcal{A}$ with $\operatorname{card} I \leqslant \mathfrak{m}$ and $\bigcup\{A_i : i \in I\} = B$. Assume that each $E \in 2^X \cap \boldsymbol{K}$ has the $\mathfrak{m}$-Lindelöf property. Then $\mathcal{B}$ is a $\boldsymbol{K}$-cover of X. Assume that $I(\boldsymbol{K}, X) \neq 0$. Then, by 5.3, $\mathcal{B}$ contains a countable cover $\{B_n : n \in N\}$ of X. However, $B_n = \bigcup\{A_i : i \in I_n\}$ for some subfamily $\{A_i : i \in I_n\}$ of $\mathcal{A}$ with $\operatorname{card} I_n \leqslant \mathfrak{m}$. Hence $\{A_i : i \in I_n$ and $n \in N\}$ is a subcover of $\mathcal{A}$ and its cardinality is $\leqslant \mathfrak{m}$.

Since each σ-compact space has the Lindelöf property, we have from 5.6 the following

5.7. Corollary. *If X is $\sigma\boldsymbol{C}$-like, then X has the Lindelöf property.*

5.8. Theorem. *If there exists a $\boldsymbol{K}$-cover $\mathcal{A}$ of X so that no countable subfamily of $\mathcal{A}$ covers X, then $II_s(\boldsymbol{K}, X) \neq 0$.*

Proof. We shall define $s \in II_s(\boldsymbol{K}, X)$. Let $\mathcal{A}$ be a $\boldsymbol{K}$-cover of X for which no countable subfamily covers X. Then for each $E \in 2^X \cap \boldsymbol{K}$ there exists an $A(E) \in \mathcal{A}$ for which $E \subseteq A(E)$. We set $s(E) = X - A(E)$ for each $E \in 2^X \cap \boldsymbol{K}$. Let $(E_n : n \in N)$ be a play of $G(\boldsymbol{K}, X)$, where $E_{2n+2} = s(E_{2n+1}) \cap E_{2n}$ for each $n \in N$. Then $\bigcap\{E_{2n} : n \in N\} = X - \bigcup\{A(E_{2n+1}) : n \in N\} \neq 0$. Hence $s \in II_s(\boldsymbol{K}, X)$.

From 5.8 we have

5.9. Corollary. *If X does not have the Lindelöf property, then X is anti-σC-like* (cf. 5.7).

5.10. Theorem. *Assume that there exists a sequence $(\mathcal{A}_n : n \in N)$ of K-covers of X for which the following condition holds: if $(A_n : n \in N)$ is a sequence of sets, where $A_n \in \mathcal{A}_n$ for each $n \in N$, then $\bigcup\{A_n : n \in N\} \neq X$. Then $II(K, X) \neq 0$* (cf. 5.1).

Proof. We shall define $s \in II(K, X)$. Let $(\mathcal{A}_n : n \in N)$ be a sequence of K-covers of X such that if $A_n \in \mathcal{A}_n$ for each $n \in N$, then $\bigcup\{A_n : n \in N\} \neq X$. For each $n \in N$ and $E \in 2^X \cap K$ there exists an $A_n(E) \in \mathcal{A}_n$ for which $E \subseteq A_n(E)$. Let $(E_0, \ldots, E_{2n+1})$ be an admissible sequence for $G(K, X)$. Then we set

$$s(E_0, \ldots, E_{2n+1}) = \bigcap\{X - A_k(E_{2k+1}) : k \leqslant n\}.$$

If $(E_n : n \in N)$ is a play of $G(K, X)$, where $E_{2n+2} = s(E_0, \ldots, E_{2n+1})$ for each $n \in N$, then $\bigcap\{E_{2n} : n \in N\} = X - \bigcup\{A_n(E_{2n+1}) : n \in N\} \neq 0$. Hence $s \in II(K, X)$.

Recall that a Baire measure on X is a non-negative, real-valued σ-additive set function m defined on the family of all Baire sets (i.e. on the σ-algebra generated by cozero subsets of X). A Baire measure m on X is said to be *regular* if $m(E) = \inf\{m(U) : E \subseteq U$ and U is a cozero set$\}$ for each Baire set $E \subseteq X$. If m is a Baire measure on X, then we set $m^*(A) = \inf\{m(E) : A \subseteq E$ and E is a Baire set$\}$ for each $A \subseteq X$.

5.11. Theorem. *If there exists a regular Baire measure m on X such that $m(X) > 0$ and $m^*(E) = 0$ for each $E \in 2^X \cap K$, then $II(K, X) \neq 0$.*

Proof. Let m be a regular Baire measure on X such that $m(X) > 0$ and $m^*(E) = 0$ for each $E \in 2^X \cap K$. Let $\mathcal{A}_n$ be the family of all cozero sets U with $m(U) \leqslant r \cdot 2^{-n-1}$ for each $n \in N$, where $0 < r < m(X)$. From the regularity of m it follows that $\mathcal{A}_n$ is a K-cover of X for each $n \in N$. If $A_n \in \mathcal{A}_n$ for each $n \in N$, then $m(\bigcup\{A_n : n \in N\}) \leqslant \Sigma\{m(A_n) : n \in N\} \leqslant \Sigma\{r \cdot 2^{-n-1} : n \in N\} = r < m(X)$. Hence $\bigcup\{A_n : n \in N\} \neq X$. Thus, by 5.10, we have $II(K, X) \neq 0$.

Recall that the Sorgenfrey line R_S is the real line R retopologized as follows: the family $\{[x, y) : x < y\}$ is taken for a base of open sets. Clearly, each open set in R is open in R_S. It is easy to prove that each σ-compact subset of R_S is countable.

5.12. Theorem. *The Sorgenfrey line is anti-σC-like.*

Proof. It is easy to check that the σ-algebra generated by $\{[x, y) : x < y\}$ is the same as the σ-algebra generated by $\{(x, y) : x < y\}$. Let m be the restriction of the Lebesgue measure with respect to the σ-algebra. Then m is a regular Baire measure on R_S. If E is a countable subset of R_S, then $m(E) = 0$. Since each σ-compact subset of R_S is countable, we have $II(\sigma C, R_S) \neq 0$ by 5.11.

From 5.10 we have:

5.13. Corollary. *If X has not property C'', then X is anti-finite-like* (cf. 5.4).

5.14. Corollary. *If X is not a Hurewicz space, then X is anti-compact-like* (cf. 5.5).

6. Pseudo-characters. The pseudo-character $\psi(E, X)$ of a subset E of a space X is the least cardinal number $\mathfrak{m}$ for which E is the intersection of $\mathfrak{m}$ open sets.

6.1. Theorem. *Let $I(K, X) \neq 0$ and $\psi(E, X) \leqslant \mathfrak{m}$ for each $E \in 2^X \cap K$, where $\mathfrak{m}$ is an infinite cardinal. Then X has a cover $\mathcal{A}$ such that $\mathcal{A} \subseteq 2^X \cap K$ and* $\operatorname{card}\mathcal{A} \leqslant \mathfrak{m}$.

Proof. Let $I(K, X) \neq 0$ and $\psi(E, X) \leqslant \mathfrak{m}$ for each $E \in 2^X \cap K$, where $\mathfrak{m}$ is an infinite cardinal. Then for each $E \in 2^X$ and $F \in 2^X \cap K$ with $F \subseteq E$ there exists a family $S(E, F) \subseteq 2^X$ such that $\bigcup\{H : H \in S(E, F)\} = E - F$ and $\operatorname{card} S(E, F) \leqslant \mathfrak{m}$. Let $s \in I(K, X)$ and let T_n be the set of all admissible sequences $(E_0, \ldots, E_{2n})$ for $G(K, X)$, where $E_{2k+1} = s(E_0, \ldots, E_{2k})$ and $E_{2k+2} \in S(E_{2k}, E_{2k+1})$ for each $k < n$. We set $\mathcal{A}_n = \{s(E_0, \ldots, E_{2n}) : (E_0, \ldots, E_{2n}) \in T_n\}$ for each $n \in N$. It is easy to point out that $\operatorname{card}\mathcal{A}_n \leqslant \mathfrak{m}$ for each $n \in N$. Let us set $\mathcal{A} = \bigcup\{\mathcal{A}_n : n \in N\}$. Then $\mathcal{A} \subseteq 2^X \cap K$ and $\operatorname{card}\mathcal{A} \leqslant \mathfrak{m}$. It remains to prove that $\mathcal{A}$ covers X. Suppose that there exists a point x in $X - \bigcup\{A : A \in \mathcal{A}\}$. Then $x \notin s(E_0) = E_1$, where $E_0 = X$. Thus there exists an $E_2 \in S(E_0, E_1)$ for which $x \in E_2$. Since $x \notin s(E_0, E_1, E_2) = E_3$, there exists and $E_4 \in S(E_2, E_3)$ for which $x \in E_4$. Continuing in this manner, we infer the existence of a play $(E_n : n \in N)$ of $G(K, X)$, where $(E_0, \ldots, E_{2n}) \in T_n$ for each $n \in N$ and $\bigcap\{E_{2n} : n \in N\} \neq 0$. We have obtained a contradiction with $s \in I(K, X)$. Hence $\mathcal{A}$ covers X.

From 6.1 we have the following

6.2. Corollary. *Let $I(K, X) \neq 0$ and $\psi(E, X) \leqslant \mathfrak{m}$ and $\operatorname{card} E \leqslant \mathfrak{m}$ for each $E \in 2^X \cap K$, where $\mathfrak{m}$ is an infinite cardinal. Then* $\operatorname{card} X \leqslant \mathfrak{m}$.

6.3. Theorem. *If each $E \in 2^X \cap K$ is a G_δ-set in X, then the following conditions are equivalent:*

6.3.1. $I(K, X) \neq 0$;

6.3.2. $I_s(K, X) \neq 0$;

6.3.3. $X \in \sigma K$.

Proof. Assume that each $E \in 2^X \cap K$ is a G_δ-set in X. If $X \in \sigma K$, then $I_s(K, X) \neq 0$ by 4.5. If $I_s(K, X) \neq 0$, then obviously $I(K, X) \neq 0$. If $I(K, X) \neq 0$, then $X \in \sigma K$ by 6.1.

As a consequence of 6.3 and 4.9 we get

6.4. COROLLARY. *Let X be a space such that each $E \in 2^X$ is a G_δ-set in X. Then*

6.4.1. *X is finite-like iff X is countable.*

6.4.2. *X is compact-like iff X is σ-compact.*

6.4.3. *X is Čech-like iff X is σ-Čech.*

6.4.4. *X is discrete-like iff X is σ-discrete.*

Since compactness and Čech completeness are preserved by f and f^{-1}, where f is a perfect map, we have from 6.4 and 3.8 the following

6.5. THEOREM. *Assume that there exists a perfect map from X onto Y, where each $E \in 2^Y$ is a G_δ-set in Y. Then*

6.5.1. *X is compact-like iff X is σ-compact;*

6.5.2. *X is Čech-like iff X is σ-Čech.*

7. Discrete unions. We denote by $\boldsymbol{DK}$ the class of all spaces X for which there exists a discrete cover $\{X_i : i \in I\}$ with $\{X_i : i \in I\} \subseteq \boldsymbol{K}$.

Let us remark that $\boldsymbol{DF} = \boldsymbol{D}$, $\boldsymbol{DD} = \boldsymbol{D}$ and $\boldsymbol{D\check{C}} = \boldsymbol{\check{C}}$. Clearly, $X \in \boldsymbol{DK}$ implies $2^X \subseteq \boldsymbol{DK}$.

7.1. LEMMA. *If X has a discrete cover $\{X_i : i \in I\}$, where $I(\boldsymbol{K}, X_i) \neq 0$ for each $i \in I$, then $I(\boldsymbol{DK}, X) \neq 0$.*

Proof. Let us take $s_i \in I(\boldsymbol{K}, X_i)$ for each $i \in I$. We shall define $s \in I(\boldsymbol{DK}, X)$. Let us set $E_0 = X$, $E_1 = \bigcup \{s_i(X_i) : i \in I\}$ and $s(E_0) = E_1$. Clearly, $E_1 \in 2^X \cap \boldsymbol{DK}$. Let $(E_0, \ldots, E_{2n})$ be an admissible sequence for $G(\boldsymbol{DK}, X)$, where $E_{2k+1} = s(E_0, \ldots, E_{2k})$ for each $k < n$. We set $E_{2n+1} = \bigcup \{s_i(X_i \cap E_0, \ldots, X_i \cap E_{2n}) : i \in I\}$ and $s(E_0, \ldots, E_{2n}) = E_{2n+1}$. Clearly, $E_{2n+1} \in 2^X \cap \boldsymbol{DK}$ and $(X_i \cap E_0, \ldots, X_i \cap E_{2n})$ is admissible for $G(\boldsymbol{K}, X_i)$, where $X_i \cap E_{2k+1} = s_i(X_i \cap E_0, \ldots, X_i \cap E_{2k})$ for each $k < n$ and $i \in I$. Let $(E_n : n \in N)$ be a play of $G(\boldsymbol{DK}, X)$, where $E_{2n+1} = s(E_0, \ldots, E_{2n})$ for each $n \in N$. Then $\bigcap \{E_{2n} : n \in N\} = \bigcup \{\bigcap \{X_i \cap E_{2n} : n \in N\} : i \in I\}$. Since $s_i \in I(\boldsymbol{K}, X_i)$, we have $\bigcap \{X_i \cap E_{2n} : n \in N\} = 0$. Thus $\bigcap \{E_{2n} : n \in N\} = 0$. Hence $s \in I(\boldsymbol{DK}, X)$.

From 7.1 and 4.7 we get

7.2. THEOREM. *If X has a σ-discrete cover $\{X_i : i \in I\}$, where $X_i \in 2^X$ and $I(\boldsymbol{K}, X_i) \neq 0$ for each $i \in I$, then $I(\boldsymbol{DK}, X) \neq 0$.*

8. Locally $\boldsymbol{K}$ spaces. A space X is said to be *locally $\boldsymbol{K}$* if for each point $x \in X$ there exists an open nbhd U of x in X for which $\mathrm{Cl}_X U \in \boldsymbol{K}$. We denote by $\boldsymbol{LK}$ the class of all locally $\boldsymbol{K}$ spaces.

Let us remark that $\boldsymbol{DK} \subseteq \boldsymbol{LK}$. Clearly, $X \in \boldsymbol{LK}$ implies $2^X \subseteq \boldsymbol{LK}$.

Let us recall that a space X is said to be *subparacompact* if each open cover of X has a σ-discrete closed refinement. Subparacompact spaces were studied by D. K. Burke [3] and L. F. McAuley [11].

8.1. THEOREM. *If X is subparacompact and $X \in \boldsymbol{LK}$, then $X \in \boldsymbol{\sigma DK}$.*

Proof. Let X be a subparacompact space with $X \in \boldsymbol{LK}$. Then X has an open cover $\mathcal{A}$, where $\mathrm{Cl}_X A \in \boldsymbol{K}$ for each $A \in \mathcal{A}$. Since X is subparacompact, it follows that $\mathcal{A}$ has a σ-discrete refinement $\mathcal{B}$ with $\mathcal{B} \subseteq 2^X$. If $B \in \mathcal{B}$, then $B \in 2^X \cap \boldsymbol{K}$, because $B \subseteq A \subseteq \mathrm{Cl}_X A$ for some $A \in \mathcal{A}$. Hence $X \in \boldsymbol{\sigma DK}$.

From 8.1 and 4.7 we get

8.2. COROLLARY. *If X is subparacompact and $X \in \boldsymbol{LK}$, then $I(\boldsymbol{DK}, X) \neq 0$.*

9. $\boldsymbol{K}$-scattered spaces. A space X is said to be *$\boldsymbol{K}$-scattered* if for each non-void $E \in 2^X$ there exist an $x \in E$ and an open nbhd U of x in X for which $E \cap \mathrm{Cl}_X U \in \boldsymbol{K}$. We denote by $\boldsymbol{SK}$ the class of all $\boldsymbol{K}$-scattered spaces.

$\boldsymbol{K}$-scattered spaces were investigated by A. H. Stone [21]. We shall recall some basic facts about $\boldsymbol{K}$-scattered spaces. For any $E \in 2^X$, E^* denotes the set of all $x \in X$ such that $E \cap \mathrm{Cl}_X U \notin \boldsymbol{K}$ for each open nbhd U of x in X. We set $X^{(0)} = X$, $X^{(\alpha+1)} = (X^{(\alpha)})^*$ and $X^{(\alpha)} = \bigcap \{X^{(\beta)} : \beta < \alpha\}$ for the limit ordinal α. $X^{(\alpha)}$ is called a *$\boldsymbol{K}$-derivative* of X of order α. We set $\xi(X) = \inf\{\alpha : X^{(\alpha)} = 0\}$ and $\xi(X) = \infty$ otherwise. Hence it follows that X is $\boldsymbol{K}$-scattered iff $\xi(X) = \alpha$ for some α. Finally, we set $X^{\#} = X^{(\alpha)}$ if $\xi(X) = \alpha+1$ and $X^{\#} = 0$ otherwise. We denote by $\boldsymbol{S_\alpha K}$ the class of all $\boldsymbol{K}$-scattered spaces X with $\xi(X) < \alpha$.

It is easy to point out that $X^{(\alpha)} \in 2^X$ for each α, $X \in \boldsymbol{SK}$ implies $2^X \subseteq \boldsymbol{SK}$, $\boldsymbol{S_2K} = \boldsymbol{LK}$ and $\boldsymbol{SSK} = \boldsymbol{SLK} = \boldsymbol{SDK} = \boldsymbol{SK}$. In particular, $\boldsymbol{S1} = \boldsymbol{SF} = \boldsymbol{SD}$ and $\boldsymbol{SC} = \boldsymbol{SDC} = \boldsymbol{SLC} = \boldsymbol{SSC}$. Clearly, 1-scattered and scattered spaces coincide. Scattered spaces were studied by Z. Semadeni [19] and $\boldsymbol{C}$-scattered spaces were studied by R. Telgársky [22].

9.1. THEOREM. *Assume that there exists a perfect map from X onto Y. Then*

9.1.1. *X is $\boldsymbol{SC}$-like iff Y is $\boldsymbol{SC}$-like.*

9.1.2. *X is anti-$\boldsymbol{SC}$-like iff Y is anti-$\boldsymbol{SC}$-like.*

Proof. The class $\boldsymbol{SC}$ is perfect by Theorem 1.3 of [22]. Hence, by 3.7, the theorem follows.

9.2. LEMMA. *If there exists an $F \in 2^X \cap \boldsymbol{K}$ so that $I(\boldsymbol{K}, E) \neq 0$ for each $E \in 2^X$ with $E \cap F = 0$, then $I(\boldsymbol{K}, X) \neq 0$.*

Proof. Assume that there exists an $F \in 2^X \cap \boldsymbol{K}$ so that $I(\boldsymbol{K}, E) \neq 0$ for each $E \in 2^X$ with $E \cap F = 0$. Let us pick $s_E \in I(\boldsymbol{K}, E)$, where $E \in 2^X$ and $E \cap F = 0$. We shall define $s \in I(\boldsymbol{K}, X)$. Set $E_0 = X$, $E_1 = F$ and $s(E_0) = E_1$. Let us take $E_2 \in 2^X$ with $E_1 \cap E_2 = 0$. Then we set $F_0 = E_2$, $F_1 = s_{F_0}(F_0)$, $E_3 = F_1$ and $s(E_0, E_1, E_2) = E_3$. Let us take $E_4 \in 2^X$ with $E_4 \subseteq E_2$ and $E_3 \cap E_4 = 0$. We set $F_2 = E_4$, $F_3 = s_{F_0}(F_0, F_1, F_2)$, $E_5 = F_3$ and $s(E_0, \ldots, E_4) = E_5$. Continuing in this manner, we get the play $(E_n : n \in N)$ of $G(\boldsymbol{K}, X)$ and the play $(F_n : n \in N)$ of $G(\boldsymbol{K}, F_0)$, where

$F_n = E_{n+2}$ for each $n \in N$. Since $s_{F_0} \in I(\boldsymbol{K}, F_0)$, we have $\bigcap \{F_{2n} : n \in N\} = 0$. Thus $\bigcap \{E_{2n} : n \in N\} = 0$. Hence $s \in I(\boldsymbol{K}, X)$.

9.3. THEOREM. *If X is a Lindelöf space and $X \in \boldsymbol{SK}$, then $I(\boldsymbol{K}, X) \neq 0$.*

Proof. We shall prove the statement by induction with respect to $\xi(X)$. If $\xi(X) = 0$, then $X = 0$. Thus $I(\boldsymbol{K}, X) \neq 0$. If $\xi(X) = \alpha+1$, then $X^{(\alpha)} = X^{\#} \neq 0$ and $X^{\#} \in 2^X \cap \boldsymbol{LK}$. Hence for each $x \in X$ there exists an open nbhd U_x of x in X with $X^{\#} \cap \mathrm{Cl}_X U_x \in \boldsymbol{K}$. If X has the Lindelöf property, then $\{U_x : x \in X\}$ contains a countable subcover $\{U_{x_n} : n \in N\}$ of X. Hence we have $X = \bigcup \{\mathrm{Cl}_X U_{x_n} : n \in N\}$. According to 4.7, it suffices to prove that $I(\boldsymbol{K}, \mathrm{Cl}_X U_{x_n}) \neq 0$ for each $n \in N$. Let $H \in \{\mathrm{Cl}_X U_{x_n} : n \in N\}$. Then $H \cap X^{\#} \in 2^H \cap \boldsymbol{K}$. If $E \in 2^H$ and $E \cap X^{\#} = 0$ then $\xi(E) \leqslant \alpha$. Thus, by the inductive assumption, we have $I(\boldsymbol{K}, E) \neq 0$. Applying 9.2 we get $I(\boldsymbol{K}, H) \neq 0$. Hence $I(\boldsymbol{K}, X) \neq 0$. If $\xi(X)$ is a limit ordinal, then $\{X - X^{(\alpha)} : \alpha < \xi(X)\}$ is an open cover of X. Hence for each $x \in X$ there exist an open nbhd U_x of x in X and an ordinal $\alpha < \xi(X)$ for which $\mathrm{Cl}_X U_x \subseteq X - X^{(\alpha)}$. Thus $\xi(\mathrm{Cl}_X U_x) < \xi(X)$ for each $x \in X$. Hence, by the inductive assumption, we have $I(\boldsymbol{K}, \mathrm{Cl}_X U_x) \neq 0$ for each $x \in X$. If X has the Lindelöf property, then $\{U_x : x \in X\}$ contains a countable subcover $\{U_{x_n} : n \in N\}$. Since $X = \bigcup \{\mathrm{Cl}_X U_{x_n} : n \in N\}$, we infer from 4.7 that $I(\boldsymbol{K}, X) \neq 0$.

From 9.3 and 4.7 immediately follows

9.4. COROLLARY. *If X is a Lindelöf space and $X \in \boldsymbol{\sigma SK}$, then $I(\boldsymbol{K}, X) \neq 0$.*

From 9.4 we get

9.5. COROLLARY. *Let X be a Lindelöf space. Then*

9.5.1. *If $X \in \boldsymbol{\sigma SF}$, then X is finite-like.*

9.5.2. *If $X \in \boldsymbol{\sigma SC}$, then X is compact-like.*

9.5.3. *If $X \in \boldsymbol{\sigma S\check{C}}$, then X is Čech-like.*

9.6. QUESTION. Does there exist a scattered Lindelöf space X for which $I_s(\boldsymbol{F}, X) = 0$?

9.7. THEOREM. *If X is subparacompact and $X \in \boldsymbol{SK}$, then $I(\boldsymbol{DK}, X) \neq 0$.*

Proof. We shall prove the statement by induction with respect to $\xi(X)$. If $\xi(X) = 0$, then $X = 0$. Thus $I(\boldsymbol{DK}, X) \neq 0$. If $\xi(X) = \alpha+1$, then $X^{(\alpha)} = X^{\#} \neq 0$ and $X^{\#} \in 2^X \cap \boldsymbol{LK}$. Hence for each $x \in X$ there exists an open nbhd U_x of x in X with $X^{\#} \cap \mathrm{Cl}_X U_x \in \boldsymbol{K}$. If X is subparacompact, then the open cover $\{U_x : x \in X\}$ of X has a σ-discrete refinement $\{X_i : i \in I\}$, where $X_i \in 2^X$ for each $i \in I$. According to 7.2, it suffices to prove that $I(\boldsymbol{DK}, X_i) \neq 0$ for each $i \in I$. Let $i \in I$. Then $X^{\#} \cap X_i \in \boldsymbol{K}$. If E is a closed subset of X_i with $E \cap X^{\#} = 0$, then $\xi(E) \leqslant \alpha$, because $E^{(\alpha)} = 0$. Thus, by the inductive assumption, we have $I(\boldsymbol{DK}, E) \neq 0$. Applying 9.2 (where $\boldsymbol{K}$ is replaced by $\boldsymbol{DK}$), we get $I(\boldsymbol{DK}, X_i) \neq 0$. Hence $I(\boldsymbol{DK}, X) \neq 0$. If $\xi(X)$ is a limit ordinal, then $\{X - X^{(\alpha)} : \alpha < \xi(X)\}$ is an open cover of X. If X is subparacompact, then $\{X - X^{(\alpha)} : \alpha < \xi(X)\}$ has a σ-discrete refinement $\{X_i : i \in I\}$ with $X_i \in 2^X$ for each $i \in I$. Let $i \in I$. Then there exists an ordinal $\alpha < \xi(X)$ for which $X_i \subseteq X - X^{(\alpha)}$. Hence $\xi(X_i) < \xi(X)$. Thus, by the inductive assumption, $I(\boldsymbol{DK}, X_i) \neq 0$. Applying 7.2, we get $I(\boldsymbol{DK}, X) \neq 0$.

As a consequence of 9.7 and 4.7 we have

9.8. COROLLARY. *If X is subparacompact and $X \in \boldsymbol{\sigma SK}$, then $I(\boldsymbol{DK}, X) \neq 0$.*

From 9.8 we get

9.9. COROLLARY. *Let X be a subparacompact space. Then*

9.9.1. *If $X \in \boldsymbol{\sigma SD}$, then X is $\boldsymbol{D}$-like.*

9.9.2. *If $X \in \boldsymbol{\sigma SC}$, then X is $\boldsymbol{DC}$-like.*

9.9.3. *If $X \in \boldsymbol{\sigma S\check{C}}$, then X is $\boldsymbol{\check{C}}$-like.*

9.10. QUESTION. Is $X \in \boldsymbol{\sigma SK}$ if X is a paracompact space with $I(\boldsymbol{K}, X) \neq 0$?

9.11. QUESTION. Is $I(\boldsymbol{K}, X) \neq 0$ if X is a Lindelöf space for which $I(\boldsymbol{DK}, X) \neq 0$?

9.12. THEOREM. *If $X \in \boldsymbol{\sigma S_{\omega_0} K}$, then $I_s(\boldsymbol{LK}, X) \neq 0$.*

Proof. Let $X \in \boldsymbol{\sigma S_{\omega_0} K}$. Then $X = \bigcup \{X_n : n \in N\}$, where $X_n \in 2^X \cap \boldsymbol{S_{\omega_0} K}$ for each $n \in N$. Let $E \in 2^X$ and $n \in N$. If $E \cap X_n \neq 0$, then $(E \cap X_n)^{\#} \neq 0$, because $\xi(X_n) < \omega_0$. We shall define $s \in I_s(\boldsymbol{LK}, X)$ as follows. If $E = 0$, then we set $s(E) = 0$. Let $E \neq 0$. Then there exists a least $n \in N$ for which $E \cap X_n \neq 0$. We set $s(E) = (E \cap X_n)^{\#}$. It is clear that $s(E) \in 2^X \cap \boldsymbol{LK}$ and $s(E) \subseteq E$ for each $E \in 2^X$. Let $\{E_n : n \in N\}$ be a play of $G(\boldsymbol{LK}, X)$, where $E_{2n+1} = s(E_{2n})$ for each $n \in N$. Suppose that there exists an $x \in \bigcap \{E_{2n} : n \in N\}$. Then there exists the least $n \in N$ for which $x \in X_n$. Hence we conclude that $\xi(E_0 \cap X_n) > \xi(E_2 \cap X_n) > \ldots$ and this is a contradiction. Thus $s \in I_s(\boldsymbol{LK}, X)$.

9.13. QUESTION. Is $X \in \boldsymbol{\sigma S_{\omega_0} K}$ if $I_s(\boldsymbol{LK}, X) \neq 0$?

From 9.12 we get

9.14. COROLLARY. *If $X \in \boldsymbol{\sigma S_{\omega_0} D}$ ($X \in \boldsymbol{\sigma S_{\omega_0} C}$, $X \in \boldsymbol{\sigma S_{\omega_0} \check{C}}$), then X is $\boldsymbol{D}$-like ($\boldsymbol{LC}$-like, $\boldsymbol{L\check{C}}$-like, resp.).*

10. Closure-preserving covers.

10.1. (H. B. Potoczny [18], Lemma 4 and Lemma 6). *If X has a closure-preserving cover by compact sets, then there exists a function $s: 2^X \to 2^X \cap \boldsymbol{DC}$ for which the following holds:*

10.1.1. *$s(E) \subseteq E$ for each $E \in 2^X$, and*

10.1.2. *if $(U_n : n \in N)$ is a sequence of open sets, where $s(X) \subseteq U_0$ and $s(X - \bigcup \{U_k : k \leqslant n\}) \subseteq U_{n+1}$ for each $n \in N$, then $\bigcup \{U_n : n \in N\} = X$.*

It is clear from 10.1.1 and 10.1.2 that $s \in I_s(\boldsymbol{DC}, X)$.

Thus from 10.1 and 4.7 we get

10.2. COROLLARY. *If X has a σ-closure-preserving cover by compact sets (i.e. a cover $\mathcal{A} = \bigcup \{\mathcal{A}_n : n \in N\}$, where $\mathcal{A}_n$ is a closure-preserving family of compact sets for each $n \in N$), then X is* ***DC**-like.*

From 10.2 and 6.3 we have

10.3. COROLLARY. *Assume that X has a σ-closure-preserving cover by compact sets and that each $E \in 2^X$ is a G_δ-set in X. Then $X \in \boldsymbol{\sigma DC}$.*

In the sequel we shall need this result:

10.4. (R. Telgársky [25], Theorem 6). *If X has a closure-preserving cover by finite sets, then $X \in \boldsymbol{\sigma S_{\omega_0} D}$.*

10.5. THEOREM. *If X has a σ-closure-preserving cover by finite sets, then X is discrete-like.*

Proof. Let $\{X_i : i \in I_n$ and $n \in N\}$ be a cover of X, where the family $\{X_i : i \in I_n\}$ is closure-preserving for each $n \in N$ and X_i is finite for each $i \in \bigcup \{I_n : n \in N\}$. We set $X_n = \bigcup \{X_i : i \in I_n\}$ for each $n \in N$. Then $X_n \in 2^X$ and, by 10.4, $X_n \in \boldsymbol{\sigma S_{\omega_0} D}$ for each $n \in N$. Thus $X \in \boldsymbol{\sigma\sigma S_{\omega_0} D} = \boldsymbol{\sigma S_{\omega_0} D}$. Hence, by 9.14, X is discrete-like.

10.6. THEOREM. *If X is collectionwise normal and* ***σDC**-like, then X is paracompact.*

Proof. Let X be a collectionwise normal space and let $I(\boldsymbol{\sigma DC}, X) \neq 0$. Let $\mathcal{A}$ be an open cover of X. If $E \in 2^X \cap \boldsymbol{\sigma DC}$, then $E = \bigcup \{E_i : i \in I_n$ and $n \in N\}$, where each family $\{E_i : i \in I_n\}$ is discrete and each E_i is compact. For each $i \in \bigcup \{I_n : n \in N\}$ there exists a finite subfamily $\mathcal{A}_i$ of $\mathcal{A}$ for which $E_i \subseteq \bigcup \{A : A \in \mathcal{A}_i\}$. For each $n \in N$ there exists a discrete family $\{U_i : i \in I_n\}$ of open sets such that $E_i \subseteq U_i$ for each $i \in I_n$. It is easy to point out that $\{A \cap U_i : A \in \mathcal{A}_i, i \in I_n$ and $n \in N\}$ is a σ-locally finite family of open sets, and that it refines $\mathcal{A}$ and covers E. Let us set $E' = \bigcup \{A \cap U_i : A \in \mathcal{A}_i, i \in I_n$ and $n \in N\}$. Then $\{E' : E \in 2^X \cap \boldsymbol{\sigma DC}\}$ is a $\boldsymbol{\sigma DC}$-cover of X. Hence, by 5.3, $\{E' : E \in 2^X \cap \boldsymbol{\sigma DC}\}$ contains a countable cover $\{E'_m : m \in N\}$ of X. It is clear that $\{E'_m : m \in N\}$ determines a σ-locally finite open refinement of $\mathcal{A}$. Hence X is paracompact.

From 10.6 and 10.2 we have

10.7. COROLLARY. *If a collectionwise normal space X has a σ-closure-preserving cover by compact sets, then X is paracompact.*

As a corollary of 10.7 we get

10.8. (H. B. Potoczny [18], Theorem). *If a collectionwise normal space X has a closure-preserving cover by compact sets, then X is paracompact.*

10.9. THEOREM. *If X is countably compact and* ***DC**-like, then X i[s] compact.*

Proof. Let X be a countably compact space. Then each $E \in 2^X \cap \boldsymbol{DC}$ is compact. Thus $I(\boldsymbol{DC}, X) = I(\boldsymbol{C}, X)$. Assume that $I(\boldsymbol{DC}, X) \neq 0$. Then $I(\boldsymbol{C}, X) \neq 0$ and according to 5.7 we conclude that X is a Lindelöf space. Thus X is compact, because X is a countably compact Lindelöf space.

As a corollary of 10.9 and 10.2 we have

10.10. THEOREM. *If a countably compact space X has a σ-closure-preserving cover by compact sets, then X is compact.*

Let us note that if a space X has a closure-preserving cover by compact sets, then X need not be subparacompact (see [25], Added in proof).

The following result is, in a certain sense, a conversion of 10.3:

10.11. (R. Telgársky [25], Theorem 2 and Theorem 5). *Let X be a paracompact space. If $X \in \boldsymbol{\sigma D}$ ($X \in \boldsymbol{\sigma LC}$), then X has a closure-preserving cover by finite sets (by compact sets, resp.).*

11. Hereditarily paracompact spaces.

11.1. THEOREM. *If X is hereditarily paracompact and $I(\boldsymbol{K}, X) \neq 0$, then X has a cover $\{X_n : n \in N\}$, where each X_n is a* ***FK**-scattered closed subset of X with $\xi(X_n) \leq n+1$; hence $X \in \boldsymbol{\sigma S_{\omega_0} FK}$.*

Proof. Let E and F be closed subsets of X with $F \subseteq E$. If X is hereditarily paracompact, then there exists a family $S(E, F)$ of closed subsets of X such that

11.1.1. $\bigcup \{H : H \in S(E, F)\} = E - F$, and

11.1.2. $S(E, F)$ is locally finite at each point of $E - F$.

Let $s \in I(\boldsymbol{K}, X)$ and let T_n be the set of all admissible sequences $(E_0, \ldots, E_{2n})$ for $G(\boldsymbol{K}, X)$, where for each $k < n$:

11.1.3. $E_{2k+1} = s(E_0, \ldots, E_{2k})$, and

11.1.4. $E_{2k+2} \in S(E_{2k}, E_{2k+1})$.

We set for each $n \in N$:

$$P_n = \{E_{2n} : (E_0, \ldots, E_{2n}) \in T_n\},$$

$$Y_n = \bigcup \{s(E_0, \ldots, E_{2n}) : (E_0, \ldots, E_{2n}) \in T_n\},$$

and

$$X_n = \bigcup \{Y_k : k \leq n\}.$$

It is easy to point out that for each $n \in N$:

11.1.5. P_{n+1} is a refinement of P_n,

11.1.6. $\bigcup \{E : E \in P_{n+1}\} = X - X_n$,

11.1.7. P_{n+1} is locally finite at each point of $X - X_n$,

11.1.8. $Y_{n+1} \subseteq X - X_n$,

11.1.9. Y_{n+1} is locally closed in X (see [8], p. 65), and

11.1.10. $Y_{n+1} \in \boldsymbol{LFK}$.

The proofs of 11.1.5–11.1.10 are omitted.

Clearly, $P_0 = \{X\}$ and $X_0 \in 2^X \cap \boldsymbol{K}$. Therefore, X_0 is an $\boldsymbol{FK}$-scattered closed subset of X and $\xi(X_0) \leqslant 1$. Let us assume that X_n is a $\boldsymbol{FK}$-scattered closed subset of X with $\xi(X_n) \leqslant n+1$ for some $n \in N$. We shall prove that X_{n+1} is a $\boldsymbol{FK}$-scattered closed subset of X with $\xi(X_{n+1}) \leqslant n+2$. We claim that X_{n+1} is $\boldsymbol{FK}$-scattered. Let H be a closed non-void subset of X_{n+1}. If $H \subseteq X_n$, then H is $\boldsymbol{FK}$-scattered, because H is a closed subset of X_n. Thus there exist an $x \in H$ and an open nbhd U of x in X for which $H \cap \mathrm{Cl}_X U \in \boldsymbol{FK}$. If $H \cap Y_{n+1} \neq 0$ and $x \in H \cap Y_{n+1}$, then there exists an open nbhd U of x in X for which $Y_{n+1} \cap \mathrm{Cl}_X U \in \boldsymbol{FK}$, because $Y_{n+1} \in \boldsymbol{LFK}$ by 11.1.10. Since X is a regular space and $X_n \in 2^X$, we may assume, without loss of generality, that $X_n \cap \mathrm{Cl}_X U = 0$. Hence $H \cap \mathrm{Cl}_X U \subseteq Y_{n+1}$ and therefore $H \cap \mathrm{Cl}_X U \subseteq Y_{n+1} \cap \mathrm{Cl}_X U$. Thus $H \cap \mathrm{Cl}_X U \in \boldsymbol{FK}$. Hence it follows that X_{n+1} is $\boldsymbol{FK}$-scattered. Since $Y_{n+1} \in \boldsymbol{LFK}$ and X_n is closed, we have $\xi(X_{n+1}) \leqslant n+2$, because the $\boldsymbol{FK}$-derivative of X_{n+1} will certainly remove the set Y_{n+1}. We claim that X_{n+1} is closed in X. Let $x \in X - X_{n+1}$. Since $x \notin X_n$, there exists an open nbhd U of x in X for which $X_n \cap \mathrm{Cl}_X U = 0$. Since P_{n+1} covers $X - X_n$ and is locally finite at each point of $X - X_n$ (see 11.1.6 and 11.1.7), there exists an open nbhd V of x in X for which the family $R = \{E \in P_{n+1}\colon E \cap \mathrm{Cl}_X V \neq 0\}$ is finite and $V \subseteq U$. Hence $\mathrm{Cl}_X V \subseteq \subseteq \bigcup \{E\colon E \in R\}$. From the definition of Y_{n+1}, 11.1.7 and 11.1.8 it follows that $Y_{n+1} \cap \bigcup \{E\colon E \in R\}$ is closed in X. Thus $W = V - Y_{n+1}$ is an open nbhd of x in X and $W \cap X_{n+1} = 0$. Hence $X_{n+1} \in 2^X$. It remains to prove that $\{X_n\colon n \in N\}$ covers X. Let us suppose that $x \notin X_n$ for each $n \in N$. Then $x \notin s(E_0) = E_1$, where $E_0 = X$. Thus there exists an $E_2 \in S(E_0, E_1)$ for which $x \in E_2$. Since $x \notin s(E_0, E_1, E_2) = E_3$, there exists an $E_4 \in S(E_2, E_3)$ for which $x \in E_4$. Continuing in this manner, we infer the existence of a play $(E_n\colon n \in N)$ of $G(\boldsymbol{K}, X)$, where $(E_0, \ldots, E_{2n}) \in T_n$ for each $n \in N$ and $\bigcap \{E_{2n}\colon n \in N\} \neq 0$. We have obtained a contradiction with $s \in I(\boldsymbol{K}, X)$. Thus $\{X_n\colon n \in N\}$ covers X.

In the sequel we shall need the following:

11.2. (R. Telgársky [24], 2.9 and 2.10). *Let X be a hereditarily paracompact space. If X is scattered ($\boldsymbol{C}$-scattered), then X has a closure-preserving cover by finite sets (by compact sets, resp.).*

11.3. Theorem. *Let X be a hereditarily paracompact space. Then the following conditions are equivalent:*

11.3.1. *X is discrete-like.*

11.3.2. *X is σ-scattered (i.e., $X \in \boldsymbol{\sigma SD}$).*

11.3.3. *X has a σ-closure-preserving cover by finite sets.*

Proof. Let X be hereditarily paracompact. If X is discrete-like, then $X \in \boldsymbol{\sigma S_{\omega_0} FD}$ by 11.1. Since $\boldsymbol{FD} = \boldsymbol{D}$ and $\boldsymbol{S_{\omega_0} D} \subseteq \boldsymbol{SD}$, it follows that X is σ-scattered. If X is σ-scattered, then X has a σ-closure-preserving cover by finite sets by 11.2. If X has a σ-closure-preserving cover by finite sets, then X is discrete-like by 10.5.

11.4. Theorem. *Let X be a hereditarily paracompact space. Then the following conditions are equivalent:*

11.4.1. *X is $\boldsymbol{DC}$-like.*

11.4.2. *X is $\boldsymbol{LC}$-like.*

11.4.3. *X is $\boldsymbol{SC}$-like.*

11.4.4. *$X \in \boldsymbol{\sigma SC}$.*

11.4.5. *X has a σ-closure-preserving cover by compact sets.*

Proof. Let X be hereditarily paracompact. Since $\boldsymbol{DC} \subseteq \boldsymbol{LC} \subseteq \boldsymbol{SC}$, we infer from 2.1 that $\boldsymbol{DC}$-like implies $\boldsymbol{LC}$-like, which implies $\boldsymbol{SC}$-like. By 11.1 we have $X \in \boldsymbol{\sigma SC}$ provided that X is $\boldsymbol{SC}$-like, because $\boldsymbol{FSC} = \boldsymbol{SC}$ (see Theorem 1.1 of [22]) and $\boldsymbol{\sigma S_{\omega_0} SC} = \boldsymbol{\sigma SC}$. If $X \in \boldsymbol{\sigma SC}$, then X has a σ-closure-preserving cover by compact sets, because of 11.2. Finally, 11.4.5 implies 11.4.1 by 10.2.

12. Metrizable spaces.

12.1. Theorem. *If X is metrizable, then the following conditions are equivalent:*

12.1.1. *X is discrete-like.*

12.1.2. *X is σ-scattered.*

12.1.3. *X is σ-discrete.*

12.1.4. *X has a closure-preserving cover by finite sets.*

12.1.5. *X has a σ-closure-preserving cover by finite sets.*

Proof. Let X be a metrizable space. Then X is hereditarily paracompact and each closed subset of X is a G_δ-set in X. Thus 12.1.1, 12.1.2 and 12.1.5 are equivalent by 11.3. From 6.4 we obtain the equivalence of 12.1.1 and 12.1.3. From 10.11 it follows that 12.1.3 implies 12.1.4. Clearly, 12.1.4 implies 12.1.5.

12.2. Theorem. *If X is metrizable, then the following conditions are equivalent:*

12.2.1. *X is $\boldsymbol{DC}$-like.*

12.2.2. *X is $\boldsymbol{LC}$-like.*

12.2.3. *X is $\boldsymbol{SC}$-like.*

12.2.4. *$X \in \boldsymbol{\sigma DC}$.*

12.2.5. *$X \in \boldsymbol{\sigma LC}$.*

12.2.6. *$X \in \boldsymbol{\sigma SC}$.*

12.2.7. *X has a closure-preserving cover by compact sets.*

12.2.8. *X has a σ-closure-preserving cover by compact sets.*

Proof. Let X be a metrizable space. Then X is hereditarily paracompact and each closed subset of X is a G_δ-set in X. Thus 12.2.1, 12.2.2, 12.2.3, 12.2.6 and 12.2.8 are equivalent by 11.4. It follows from 6.3 that 12.2.1 is equivalent to 12.2.4. Clearly, 12.2.4 implies 12.2.5, and 12.2.5 implies 12.2.6. By 10.11 we infer that 12.2.5 implies 12.2.7. Finally, it is clear that 12.2.7 implies 12.2.8.

13. M-spaces. Recall that X is said to be an *M-space* if there exists a closed map f from X onto a metric space Y such that $f^{-1}(y)$ is countably compact for each $y \in Y$.

It is immediate that each metrizable and each countably compact space is an M-space. M-spaces were introduced by K. Morita (see [14]) and studied by many authors.

13.1. THEOREM. *If X is an M-space, then the following conditions are equivalent*:

13.1.1. *X is **DC**-like.*

13.1.2. *$X \in \sigma\mathbf{DC}$.*

13.1.3. *X has a closure-preserving cover by compact sets.*

13.1.4. *X has a σ-closure-preserving cover by compact sets.*

Proof. Let X be an M-space. Then there exists a closed map f from X onto a metric space Y such that $f^{-1}(y)$ is countably compact for each $y \in Y$. If X is *DC-like*, then, by 10.9, $f^{-1}(y)$ is compact for each $y \in Y$. Since 13.1.2 implies 13.1.1 by 4.5, 13.1.3 implies 13.1.4, and 13.1.4 implies 13.1.1 by 10.2, we may assume that f is a perfect map. Hence, if X is *DC-like*, then Y is **SC**-like by 9.1.1. Since f^{-1} preserves properties 13.1.1–13.1.4, the theorem follows from 12.2.

14. Product spaces.

14.1. QUESTION. Is $X \times X$ finite-like if X is finite-like?

14.2. THEOREM. *If X is compact-like and Y is compact, then $X \times Y$ is compact-like.*

Proof. Let p be the projection from $X \times Y$ onto X. If Y is compact, then p is perfect. Thus, by 3.8.1, if X is compact-like, then $X \times Y$ is also compact-like.

14.3. EXAMPLE. The product space $X \times X$ of a $\sigma\mathbf{C}$-like space X by itself need not be $\sigma\mathbf{C}$-like. To show that, we refer to Example 1.4 (with $n = 1$) of E. A. Michael [13]. The space X is obviously $\sigma\mathbf{C}$-like (even $\sigma\mathbf{F}$-like) and $X \times X$ does not have the Lindelöf property. Thus, by 5.7, $X \times X$ is not $\sigma\mathbf{C}$-like.

14.4. EXAMPLE. The product space $X \times Y$ of two Lindelöf spaces $X \in \mathbf{S}\sigma\mathbf{C}$ and $Y \in \mathbf{S}\check{\mathbf{C}}$ need not be normal. To show that, we refer to paper [12] of E. A. Michael. The space X is a modification of an uncountable subset X_0 of the real line, where each nowhere dense subset of X_0 is countable. The space Y is the space of irrational numbers.

14.5. THEOREM. *If X is **SC**-like and Y is compact, then $X \times Y$ is **SC**-like.*

Proof. Let p be the projection from $X \times Y$ onto X. If Y is compact, then p is perfect. Hence, by 9.2.1, if X is **SC**-like, then $X \times Y$ is also **SC**-like.

14.6. THEOREM. *If X is paracompact and **SC**-like and Y is paracompact, then the product space $X \times Y$ is paracompact.*

Proof. For each paracompact space Z there exist a paracompact space Z_0 with $\dim Z_0 = 0$ and a perfect map p_Z from Z_0 onto Z (see [16]). By 9.2, if Z is **SC**-like, then Z_0 is also **SC**-like.

Let X be a paracompact, **SC**-like space and let Y be a paracompact space. If $X_0 \times Y_0$ is paracompact, then $X \times Y$ is also paracompact, because the map $p_X \times p_Y$ is perfect and paracompactness is preserved by perfect maps (see [16]). Hence, without loss of generality, we may assume that $\dim X = \dim Y = 0$. Let $\mathcal{A}$ be an open cover of $X \times Y$. We shall make use of the following property of paracompact spaces with $\dim = 0$: Each open cover has a discrete (closed-open) refinement. We shall construct a discrete refinement of $\mathcal{A}$. We consider a game $G(\mathbf{K}_0, X \times Y)$, where $\mathbf{K}_0 = \{E \in 2^{X \times Y}: \mathrm{proj}_X E \in 2^X \cap \mathbf{SC}\}$. We assume that all moves of the players are restricted to closed rectangular subsets of $X \times Y$ (i.e., sets of the form $E \times F$, where $E \in 2^X$ and $F \in 2^Y$). Let $E \in 2^{X \times Y}$. We set $E' = \mathrm{proj}_X E$ and $E'' = \mathrm{proj}_Y E$. It is now clear that E is rectangular iff $E = E' \times E''$. Let $s \in I(\mathbf{SC}, X)$. We define a strategy t for the player I in $G(\mathbf{K}_0, X \times Y)$ as follows: If $(E_0, \ldots, E_{2n})$ is an admissible sequence for $G(\mathbf{K}_0, X \times Y)$, then we set $t(E_0, \ldots, E_{2n}) = s(E'_0, \ldots, E'_{2n}) \times E''_{2n}$. Next, we shall define a many-valued strategy T for the player II in $G(\mathbf{K}_0, X \times X)$; simultaneously we shall define an auxiliary function S. Let E be a closed-open rectangular subset of $X \times Y$ and let F be a closed rectangular subset of $X \times Y$, where $F' \subseteq E'$, $F' \in \mathbf{SC}$ and $F'' = E''$. We assert that there exist two families $S(E, F)$ and $T(E, F)$ of closed-open rectangular subsets of $X \times Y$ for which the following conditions are satisfied:

14.6.1. $S(E, F) \cup T(E, F)$ is a discrete family in $X \times Y$.

14.6.2. $\bigcup \{H: H \in S(E, F) \cup T(E, F)\} = E$.

14.6.3. $\bigcup \{H: H \in S(E, F)\} \supseteq F$.

14.6.4. For each $H \in S(E, F)$ there exists a finite subfamily $\mathcal{A}(H)$ of $\mathcal{A}$ for which $H \subseteq \bigcup \{A: A \in \mathcal{A}(H)\}$.

The existence of S and T can be proved (by induction with respect to $\alpha = \xi(F')$) in the same manner as Theorem 2.3 of [22]; the situation here is much simpler because of $\dim = 0$. There are two steps in the proof. The basic step: the assertion holds if F' is compact. The inductive step: if there exists a discrete family $\mathcal{H}$ of closed-open rectangular subsets of E such that $\bigcup \{H: H \in \mathcal{H}\} = E$ and the assertion is true for each $(H, H \cap F)$, where $H \in \mathcal{H}$, then the assertion is true for (E, F).

It can easily be seen that t is a winning strategy for the player I if the player II is using the strategy T. Let P_n be the set of all admissible sequences $(E_0, \ldots, E_{2n})$ for $G(\boldsymbol{K}_0, X \times Y)$, where for each $k < n$ we have $E_{2k+1} = t(E_0, \ldots, E_{2k})$ and $E_{2k+2} \in T(E_{2k}, E_{2k+1})$. Let us set $\mathcal{B}_0 = 0$ and $\mathcal{B}_{n+1} = \bigcup \{S(E_{2n}, E_{2n+1}): (E_0, \ldots, E_{2n+2}) \in P_{n+1}\}$. It is easy to point out (by induction with respect to n) that the family $\mathcal{B}_n$ is discrete in $X \times Y$. Moreover, we have

14.6.5. If $m < n$, $H_1 \in \mathcal{B}_m$ and $H_2 \in \mathcal{B}_n$, then $H_1 \cap H_2 = 0$.

Let us set $\mathcal{B} = \bigcup \{\mathcal{B}_n: n \in N\}$. We claim that $\mathcal{B}$ covers $X \times Y$. Suppose that there exists a point (x, y) in $(X \times Y) - \bigcup \{B: B \in \mathcal{B}\}$. Since (x, y) is not covered by $\mathcal{B}_1$, there exists a sequence (E_0, E_1, E_2) in P_1 for which $(x, y) \in E_2 \in T(E_0, E_1)$. Since (x, y) is not covered by $\mathcal{B}_2$, there exist a set E_3 and a set E_4 for which $(E_0, \ldots, E_4) \in P_2$ and $(x, y) \in E_4 \in T(E_2, E_3)$. Continuing in this manner, we get the play $(E_n: n \in N)$ of $G(\boldsymbol{K}_0, X \times Y)$ for which $(x, y) \in \bigcap \{E_{2n}: n \in N\}$. However, $(E'_n: n \in N)$ is a play of $G(\boldsymbol{SC}, X)$ and $E'_{2n+1} = s(E'_0, \ldots, E'_{2n})$ for each $n \in N$. Thus $\bigcap \{E'_{2n}: n \in N\} = 0$ and this is a contradiction. Since $\mathcal{B}$ covers $X \times Y$, it follows from 14.6.5 that $\mathcal{B}$ is a discrete cover of $X \times Y$. If $B \in \mathcal{B}$, then there exists a finite subfamily $\mathcal{A}(B)$ of $\mathcal{A}$ for which $B \subseteq \bigcup \{A: A \in \mathcal{A}(B)\}$. Thus the family $\{A \cap B: A \in \mathcal{A}(B)$ and $B \in \mathcal{B}\}$ is a locally finite open refinement of $\mathcal{A}$. Hence $X \times Y$ is paracompact.

As a corollary of 14.6 and 10.2 we have

14.7. THEOREM. *If X is a paracompact space with a σ-closure-preserving cover by compact sets and Y is a paracompact space, then $X \times Y$ is paracompact.*

As a corollary of 14.6 and 4.5 we have

14.8. THEOREM. *If X is a paracompact space with $X \in \sigma\boldsymbol{SC}$ and Y is paracompact, then $X \times Y$ is paracompact.*

We shall need the following result:

14.9. (R. Telgársky [25], Theorem 3). *Let X be a paracompact space. If X has two order locally finite covers $\{E_\xi: \xi < \alpha\}$* and *$\{U_\xi: \xi < \alpha\}$, where $E_\xi \in 2^X \cap \boldsymbol{SC}$ and U_ξ is an open* nbhd *of E_ξ* for *each $\xi < \alpha$, then $X \in \sigma\boldsymbol{SC}$.*

As a corollary of 14.8 and 14.9 we have

14.10. (R. Telgársky [22], Theorem 2.5). *Let X and Y be paracompact spaces. If X has two order locally finite covers $\{E_\xi: \xi < \alpha\}$ and $\{U_\xi: \xi < \alpha\}$, where $E_\xi \in 2^X \cap \boldsymbol{SC}$ and U_ξ is an open* nbhd *of E_ξ for each $\xi < \alpha$, then $X \times Y$ is paracompact.*

In the same way as in 14.6 the following results can be proved.

14.11. THEOREM. *If a Lindelöf space X is $\boldsymbol{SC}$-like and Y is a Lindelöf space, then $X \times Y$ is a Lindelöf space.*

14.12. THEOREM. *If X is a compact-like space and Y is a Hurewicz space, then $X \times Y$ is a Hurewicz space.*

As a corollary of 14.12 and 9.4 we get

14.13. (R. Telgársky [23], Theorem 3.5). *If X is a $\boldsymbol{C}$-scattered Lindelöf space and Y is a Hurewicz space, then $X \times Y$ is a Hurewicz space.*

From 14.11 and 2.1 we get

14.14. COROLLARY. *If X is a compact-like space and Y it a Lindelöf space, then $X \times Y$ is a Lindelöf space.*

From 14.14 immediately follows

14.15. COROLLARY. *If X is compact-like, then $X \times X$ has the Lindelöf property.*

14.16. QUESTION. Is $X \times X$ paracompact if X is paracompact and $X \in \boldsymbol{S\check{C}}$? The question is very natural, because if we replace "$X \in \boldsymbol{S\check{C}}$" by "$X \in \boldsymbol{SC}$" or by "$X \in \boldsymbol{\check{C}}$", then the answer is positive (see R. Telgársky [22] and Z. Frolík [6], resp.).

15. Nowhere locally $\boldsymbol{K}$ spaces. A space X is said to be *nowhere locally $\boldsymbol{K}$* if each $E \in 2^X \cap \boldsymbol{K}$ is nowhere dense in X. We denote by $\boldsymbol{NLK}$ the class of all nowhere locally $\boldsymbol{K}$ spaces.

It is important to note that $X \in \boldsymbol{NLK}$ does not imply $2^X \subseteq \boldsymbol{NLK}$. However, if E is a regularly closed subset of X (i.e., $\mathrm{Cl}_X \mathrm{Int}_X E = E$) and $X \in \boldsymbol{NLK}$, then $E \in \boldsymbol{NLK}$. Similarly, if U is an open subset of X and $X \in \boldsymbol{NLK}$, then $U \in \boldsymbol{NLK}$.

The definitions of $\boldsymbol{SK}$ and $\boldsymbol{NLK}$ easily imply

15.1. THEOREM. *X is $\boldsymbol{K}$-scattered iff $2^X \cap \boldsymbol{NLK} = \{0\}$; i.e. $\boldsymbol{SK} \cap \cap \boldsymbol{NLK} = \{0\}$.*

15.2. THEOREM. *If $X \in \boldsymbol{NLK}$ and $I(\boldsymbol{K}, X) \neq 0$, then X is the union of a countable family of its nowhere dense subsets.*

Proof. Let $s \in I(\boldsymbol{K}, X)$, where X is nowhere locally $\boldsymbol{K}$. For each pair (E, F), where E is regularly closed in X and $F \in 2^E \cap \boldsymbol{K}$ there exists a family $S(E, F)$ of pairwise disjoint regularly closed subsets of X for which the following conditions hold:

15.2.1. If $H \in S(E, F)$, then $H \cap F = 0$, and

15.2.2. $\bigcup \{H: H \in S(E, F)\}$ is a dense subset of E.

It is easy to check that $\bigcup \{\mathrm{Int}_X H: H \in S(E, F)\}$ is also dense in E. Let P_n be the set of all admissible sequences $(E_0, \ldots, E_{2n})$ for $G(\boldsymbol{K}, X)$, where for each $k < n$ we have $E_{2k+1} = s(E_0, \ldots, E_{2k})$ and $E_{2k+2} \in S(E_{2k}, E_{2k+1})$. Let us set $B_n = \bigcup \{\mathrm{Int}_X E_{2n}: (E_0, \ldots, E_{2n}) \in P_n\}$ for each $n \in N$. It is easy to point out (by induction with respect to n) that each B_n is an open and dense subset of X. We claim that $\bigcap \{B_n: n \in N\} = 0$. Let us suppose that there exists a point x in $\bigcap \{B_n: n \in N\}$. Then $x \in B_0 = X$. Let us set $E_0 = X$ and $E_1 = s(E_0)$. Since $x \in B_1$, there exists a set $E_2 \in 2^X$ with $x \in \mathrm{Int}_X E_2$ and $(E_0, E_1, E_2) \in P_1$. Continuing in this manner, we get the play $(E_n: n \in N)$ of $G(\boldsymbol{K}, X)$, where $E_{2n+1} = s(E_0, \ldots, E_{2n})$ and $x \in E_{2n}$ for each $n \in N$. But this is a contradiction. Thus $\bigcap \{B_n: n \in N\} = 0$ and hence $\bigcup \{X - B_n: n \in N\} = X$. Each $X - B_n$ is a nowhere dense closed subset of X, because B_n is dense and open in X.

15.3. THEOREM. *If $I(\boldsymbol{K}, X) \neq 0$, then there exists a set $Y \in 2^X$ such that Y is a set of the first category in X and $X - Y \in \boldsymbol{LK}$.*

Proof. Let Y be the $\boldsymbol{K}$-derivative X^* of X. Then $Y \in 2^X$ and $X - Y \in \boldsymbol{LK}$. Clearly, $\mathrm{Int}_X Y \in \boldsymbol{NLK}$. Let us set $Z = \mathrm{Cl}_X \mathrm{Int}_X Y$. Then also $Z \in \boldsymbol{NLK}$. Let us assume that $I(\boldsymbol{K}, X) \neq 0$. Then $I(\boldsymbol{K}, Z) \neq 0$ by 2.4. Thus, by 15.2, Z is a set of the first category in itself. But then Z is a set of the first category in X. Clearly, $Y = (Y - \mathrm{Int}_X Y) \cup Z$. Since $Y - \mathrm{Int}_X Y$ is nowhere dense in X, it follows that Y is a set of the first category in X.

A space X is said to be *countably basiscompact* (cf. [20], p. 24) if there is an open basis $\mathcal{B}$ for open sets in X so that $\bigcap \{\mathrm{Cl}_X B_n: n \in N\} \neq 0$ for each decreasing sequence $\{B_n: n \in N\}$ with $B_n \in \mathcal{B}$ for each $n \in N$.

It is obvious that each countably compact space is countably basiscompact.

15.4. THEOREM. *If $X \in \boldsymbol{NLK}$, $X \subseteq Y$, Y is countably basiscompact and $Y - X$ is a set of the first category in Y, then $II(\boldsymbol{K}, X) \neq 0$.*

Proof. Let $X \in \boldsymbol{NLK}$, $X \subseteq Y$, $Y - X = \bigcup \{F_n: n \in N\}$, where each F_n is nowhere dense in Y, and let $\mathcal{B}$ be an open basis of Y for which Y is countably basiscompact. We shall define $s \in II(\boldsymbol{K}, X)$. Let us set $E_0 = X$. Let us take $E_1 \in 2^X \cap \boldsymbol{K}$. Then E_1 is nowhere dense in X. Thus E_1 is nowhere dense in Y as well. Thus there exist a $y_0 \in Y$ and a $B_0 \in \mathcal{B}$ with $y_0 \in B_0$ and $(E_1 \cup F_0) \cap \mathrm{Cl}_Y B_0 = 0$. We set $E_2 = X \cap \mathrm{Cl}_Y B_0$ and $s(E_0, E_1) = E_2$. Let us take $E_3 \in 2^X \cap \boldsymbol{K}$ with $E_3 \subseteq E_2$. Since $X \cap B_0$ is open in X and $X \cap B_0$ is dense in E_2, it follows that $E_2 \in \boldsymbol{NLK}$. Thus $E_3 \cup (F_1 \cap E_2)$ is nowhere dense in E_2. Hence there exist a $y_1 \in Y$ and a $B_1 \in \mathcal{B}$ with $y_1 \in B_1$, $\mathrm{Cl}_Y B_1 \subseteq B_0$ and $(E_3 \cup F_1) \cap \mathrm{Cl}_Y B_1 = 0$. We set $E_4 = X \cap \mathrm{Cl}_Y B_1$ and $s(E_0, E_1, E_2, E_3) = E_4$. Continuing in this manner, we get the play $(E_n: n \in N)$ of $G(\boldsymbol{K}, X)$, where $E_{2n+2} = X \cap \mathrm{Cl}_Y B_n$ for each $n \in N$. Clearly, $\bigcap \{E_{2n}: n \in N\} = X \cap \bigcap \{\mathrm{Cl}_Y B_n: n \in N\}$. Let us set $E = \bigcap \{\mathrm{Cl}_Y B_n: n \in N\}$. Since the sequence $\{B_n: n \in N\}$ is decreasing, we have $E \neq 0$. However, $E \cap F_n = 0$ for each $n \in N$. Hence $E \subseteq X$ and thus $\bigcap \{E_{2n}: n \in N\} \neq 0$. This proves that $s \in II(\boldsymbol{K}, X)$.

15.5. THEOREM. *If $X \in \boldsymbol{NLK}$ and $X \subseteq Y$, where $Y \in \check{\boldsymbol{C}}$ and $Y - X$ is a set of the first category in Y, then $II(\boldsymbol{K}, X) \neq 0$.*

Proof. Let $X \in \boldsymbol{NLK}$ and $X \subseteq Y$, where $Y \in \check{\boldsymbol{C}}$ and $Y - X$ is a set of the first category in Y. Let Z be a compactification of Y. Then $Z - Y$ is σ-compact. Thus $Z - Y$ is a set of the first category in Z. Since $Y - X$ is a set of the first category in Y, it follows that $Y - X$ is a set of the first category in Z. Hence $Z - X$ is a set of the first category in Z. Finally, by 15.4, we get $II(\boldsymbol{K}, X) \neq 0$.

15.6. THEOREM. *If $X \in \check{\boldsymbol{C}}$ and $X \notin \boldsymbol{SK}$, then $II(\boldsymbol{K}, X) \neq 0$.*

Proof. Let $X \notin \boldsymbol{SK}$. Then there exists an $E \in 2^X \cap \boldsymbol{NLK}$ such that $E \neq 0$. Let $X \in \check{\boldsymbol{C}}$. Then $E \in \check{\boldsymbol{C}}$. Hence, by 15.5, we have $II(\boldsymbol{K}, E) \neq 0$. Thus $II(\boldsymbol{K}, X) \neq 0$ by 2.5.

15.7. THEOREM. *If $X \in \check{\boldsymbol{C}}$ and $I(\boldsymbol{K}, X) \neq 0$, then $X \in \boldsymbol{SK}$* (cf. 15.6).

Proof. Let $X \in \check{\boldsymbol{C}}$. If $X \notin \boldsymbol{SK}$, then $II(\boldsymbol{K}, X) \neq 0$ by 15.6. Hence, if $I(\boldsymbol{K}, X) \neq 0$, then $X \in \boldsymbol{SK}$.

From 15.7 we have the following

15.8. COROLLARY. *Let $X \in \check{\boldsymbol{C}}$. Then*

15.8.1. *If X is finite-like, then X is scattered.*

15.8.2. *If X is discrete-like, then X is scattered.*

15.8.3. *If X is compact-like, then X is $\boldsymbol{C}$-scattered.*

15.8.4. *If X is $\boldsymbol{SC}$-like, then X is $\boldsymbol{C}$-scattered.*

For each $Z \subseteq X$ we set $Z' = \bigcap \{Z^{(\alpha)}: \alpha$ is an ordinal$\}$, where $Z^{(\alpha)}$ is the $\boldsymbol{\sigma LC}$-derivative of Z of order α (see Section 9).

We shall need the following result:

15.9. (A. H. Stone [21], Theorem 4′, p. 63, $\boldsymbol{P} = \boldsymbol{\sigma LC}$). *Let X be a metric space. Then $X - X' \in \boldsymbol{\sigma D \sigma LC} = \boldsymbol{\sigma LC}$. Hence, in particular, if $X \notin \boldsymbol{\sigma LC}$, then $X' \neq 0$.*

A subset X of a space Y is said to be an *A-set* if there exists a family $\{F_{i_0 \ldots i_n}: (i_0, \ldots, i_n) \in N^{n+1}$ and $n \in N\} \subseteq 2^Y$ such that $X = \bigcup \{\bigcap \{F_{i_0 \ldots i_n}: n \in N\}: (i_0, i_1, \ldots) \in N^N\}$.

15.10. THEOREM. *If X is an A-set in a complete metric space Y and X is not σ-locally compact, then there exists a set $E \in 2^X \cap 2^Y$ which is homeomorphic to the space of irrational numbers.*

Proof. We shall modify a construction of A. G. El'kin [4]. Let X be an A-set in a complete metric space Y. Assume that X is not σ-locally compact. Then $X' \neq 0$ by 15.9. Moreover, $X' \in \boldsymbol{NL\sigma LC}$. Since $X' \in 2^X$, it follows that X' is also an A-set in Y. We shall prove that X' contains a set $E \in 2^Y$ which is homeomorphic to the space of irrational numbers. Let $X' = \bigcup \{\bigcap \{F_{i_0 \ldots i_n}: n \in N\}: (i_0, i_1, \ldots) \in N^N\}$, where $\{F_{i_0 \ldots i_n}: (i_0, \ldots, i_n) \in N^{n+1}$ and $n \in N\} \subseteq 2^Y$. We may assume, without loss of generality, that $F_{i_0 \ldots i_n i_{n+1}} \subseteq F_{i_0 \ldots i_n}$ for each $(i_0, \ldots, i_n, i_{n+1}) \in N^{n+2}$ and $n \in N$.

Set $X_{j_0\ldots j_m} = \bigcup\{\bigcap\{F_{i_0\ldots i_n}\colon n \in N\}\colon (i_0, i_1, \ldots) \in N^N$ and $i_0 = j_0, \ldots, i_m = j_m\}$ for each $(j_0, \ldots, j_m) \in N^{m+1}$ and $m \in N$. It is easy to see that

15.10.1. $X' = \bigcup\{X_j\colon j \in N\}$,

15.10.2. $X_{j_0\ldots j_m} = \bigcup\{X_{j_0\ldots j_m j}\colon j \in N\}$ and

15.10.3. $X_{j_0\ldots j_m} \subseteq F_{j_0\ldots j_m}$

for each $(j_0, \ldots, j_m) \in N^{m+1}$ and $m \in N$. Now we shall construct a family $\{G_{i_0\ldots i_n}\colon (i_0, \ldots, i_n) \in N^{n+1}$ and $n \in N\}$ of non-void open sets in Y such that

15.10.4. $G_{i_0\ldots i_n i_{n+1}} \subseteq G_{i_0\ldots i_n}$ for each $(i_0, \ldots, i_n, i_{n+1}) \in N^{n+2}$ and $n \in N$,

15.10.5. the family $\{G_{i_0\ldots i_n}\colon (i_0, \ldots, i_n) \in N^{n+1}\}$ is discrete in Y for each $n \in N$,

15.10.6. $\operatorname{diam} G_{i_0\ldots i_n} \leqslant \dfrac{1}{n+1}$ for each $(i_0, \ldots, i_n) \in N^{n+1}$ and $n \in N$,

and

15.10.7. for each a sequence $(i_0, i_1, \ldots) \in N^N$ there exists a sequence $(j_0, j_1, \ldots) \in N^N$ such that $G_{i_0\ldots i_n} \cap X'_{j_0\ldots j_n} \neq 0$ for each $n \in N$.

We proceed by induction with respect to $n \in N$. Since $X' \in \boldsymbol{NL\sigma LC}$, it follows that $X' \notin \boldsymbol{C}$. Thus there exists a discrete family $\{G_i\colon i \in N\}$ of open sets in Y such that $G_i \cap X' \neq 0$ and $\operatorname{diam} G_i \leqslant 1$ for each $i \in N$. We claim that

15.10.8. for each $i \in N$ there exists a $j \in N$ such that $G_i \cap X'_j \neq 0$.

Suppose that there exists an $i \in N$ such that $G_i \cap X'_j \neq 0$ for each $j \in N$. Hence $G_i \cap X_j \subseteq X_j - X'_j$ for each $j \in N$. By 15.10.1 we have $G_i \cap X' = \bigcup\{G_i \cap X_j\colon j \in N\}$. Thus $G_i \cap X' \subseteq \bigcup\{X_j - X'_j\colon j \in N\}$. However, we have $X_j - X'_j \in \boldsymbol{\sigma LC}$ by 15.9. Hence $\bigcup\{X_j - X'_j\colon j \in N\} \in \boldsymbol{\sigma LC}$. Since $G_i \cap X' \neq 0$, there exists an open set U in Y for which $\operatorname{Cl}_Y U \subseteq G_i$ and $U \cap X' \neq 0$. Clearly, this implies that $X' \cap \operatorname{Cl}_Y U \in \boldsymbol{\sigma LC}$ and thus $X' \notin \boldsymbol{NL\sigma LC}$. This is a contradiction.

Assume that for some $n \in N$ the family $\{G_{i_0\ldots i_n}\colon (i_0, \ldots, i_n) \in N^{n+1}\}$ with the required properties is constructed. By the inductive assumption, for each $(i_0, \ldots, i_n) \in N^{n+1}$ we have $(j_0, \ldots, j_n) \in N^{n+1}$ such that $G_{i_0\ldots i_n} \cap X'_{j_0\ldots j_n} \neq 0$. Since $X'_{j_0\ldots j_n} \in \boldsymbol{NL\sigma LC}$, $X'_{j_0\ldots j_n} \notin \boldsymbol{C}$. Hence there exists a discrete family $\{G_{i_0\ldots i_n i}\colon i \in N\}$ of open sets in Y such that $G_{i_0\ldots i_n i} \subseteq G_{i_0\ldots i_n}$, $G_{i_0\ldots i_n i} \cap X'_{j_0\ldots j_n} \neq 0$ and $\operatorname{diam} G_{i_0\ldots i_n i} \leqslant \dfrac{1}{n+2}$ for each $i \in N$. We claim that

15.10.9. for each $i \in N$ there exists a $j \in N$ such that $G_{i_0\ldots i_n i} \cap X'_{j_0\ldots j_n j} \neq 0$.

Suppose that there exists an $i \in N$ such that $G_{i_0\ldots i_n i} \cap X'_{j_0\ldots j_n j} = 0$ for each $j \in N$. Then $G_{i_0\ldots i_n i} \cap X_{j_0\ldots j_n j} \subseteq X_{j_0\ldots j_n j} - X'_{j_0\ldots j_n j}$ for each $j \in N$.

Using 15.10.2, we get

$$G_{i_0\ldots i_n i} \cap X_{j_0\ldots j_n} = \bigcup\{G_{i_0\ldots i_n i} \cap X_{j_0\ldots j_n j}\colon j \in N\}$$
$$\subseteq \bigcup\{X_{j_0\ldots j_n j} - X'_{j_0\ldots j_n j}\colon j \in N\}.$$

However, by 15.9, we have $X_{j_0\ldots j_n j} - X'_{j_0\ldots j_n j} \in \boldsymbol{\sigma LC}$. Hence $\bigcup\{X_{j_0\ldots j_n j} - X'_{j_0\ldots j_n j}\colon j \in N\} \in \boldsymbol{\sigma LC}$. Since $G_{i_0\ldots i_n i} \cap X'_{j_0\ldots j_n} \neq 0$, there exists an open set U in Y such that $\operatorname{Cl}_Y U \subseteq G_{i_0\ldots i_n i}$ and $U \cap X'_{j_0\ldots j_n} \neq 0$. Clearly, this implies that $X'_{j_0\ldots j_n} \cap \operatorname{Cl}_Y U \in \boldsymbol{\sigma LC}$ and thus $X'_{j_0\ldots j_n} \notin \boldsymbol{NL\sigma LC}$. This is a contradiction. Set

$$E = \bigcup\{\bigcap\{\operatorname{Cl}_Y G_{i_0\ldots i_n}\colon n \in N\}\colon (i_0, i_1, \ldots) \in N^N\}.$$

By 15.10.5 we may write

$$E = \bigcap\{\bigcup\{\operatorname{Cl}_Y G_{i_0\ldots i_n}\colon (i_0, \ldots, i_n) \in N^{n+1}\}\colon n \in N\} \qquad \text{(cf. [8], p. 32)}.$$

Hence it follows that E is closed in Y.

Let $(i_0, i_1, \ldots) \in N^N$. Then $\bigcap\{\operatorname{Cl}_Y G_{i_0\ldots i_n}\colon n \in N\}$ is just a singleton set by 15.10.6 and by the completeness of Y. Let $f\colon N^N \to E$ be the function defined by setting

$$f((i_0, i_1, \ldots)) \in \bigcap\{\operatorname{Cl}_Y G_{i_0\ldots i_n}\colon n \in N\}.$$

Then $f(N^N) = E$ and, by 15.10.4 and 15.10.5, f is a homeomorphism, where N^N is considered as a product of N copies of the discrete space N (cf. [8], p. 438). However, N^N is homeomorphic to the space of irrational numbers (cf. [8], p. 442, Corollary 3a). Hence it remains to prove that $E \subseteq X'$. By 15.10.7, for each $(i_0, i_1, \ldots) \in N^N$ there exists a $(j_0, j_1, \ldots) \in N^N$ such that $G_{i_0\ldots i_n} \cap X'_{j_0\ldots j_n} \neq 0$ for each $n \in N$. Since $X'_{j_0\ldots j_n} \subseteq X_{j_0\ldots j_n} \subseteq F_{j_0\ldots j_n}$, it follows that $F_{j_0\ldots j_n} \cap \operatorname{Cl}_Y G_{i_0\ldots i_n} \neq 0$ for each $n \in N$. Thus

$$\bigcap\{\operatorname{Cl}_Y G_{i_0\ldots i_n}\colon n \in N\} = \bigcap\{F_{j_0\ldots j_n} \cap \operatorname{Cl}_Y G_{i_0\ldots i_n}\colon n \in N\}$$
$$\subseteq \bigcap\{F_{j_0\ldots j_n}\colon n \in N\} \subseteq X'.$$

Hence $E \subseteq X'$.

15.11. Theorem. *Let X be an A-set in a complete metric space. Then X is anti-$\boldsymbol{LC}$-like iff X is not σ-locally compact.*

Proof. Let X be an A-set in a complete metric space. If X is σ-locally compact, then $I(\boldsymbol{LC}, X) \neq 0$ by 4.5 and therefore X is not anti-$\boldsymbol{LC}$-like. If X is not σ-locally compact, then, by 15.10, X contains a closed subset E which is homeomorphic to the space of irrational numbers. Since $E \in \boldsymbol{\check{C}} \cap \boldsymbol{NLC}$, E is anti-$\boldsymbol{LC}$-like by 15.5. Hence, by 2.5, X is anti-$\boldsymbol{LC}$-like.

16. Determinacy of $G(\boldsymbol{K}, X)$. The game $G(\boldsymbol{K}, X)$ is said to be *determined* if $I(\boldsymbol{K}, X) \neq 0$ or $II(\boldsymbol{K}, X) \neq 0$.

16.1. THEOREM. *If X has a cover $\{X_n: n \in N\}$, where $X_n \in 2^X$ and $G(\boldsymbol{K}, X_n)$ is determined for each $n \in N$, then $G(\boldsymbol{K}, X)$ is determined.*

Proof. If $I(\boldsymbol{K}, X_n) \neq 0$ for each $n \in N$, then $I(\boldsymbol{K}, X) \neq 0$ by 4.7. Let $I(\boldsymbol{K}, X_m) = 0$ for some $m \in N$. If $G(\boldsymbol{K}, X_n)$ is determined for each $n \in N$, then $II(\boldsymbol{K}, X_m) \neq 0$. Hence $II(\boldsymbol{K}, X) \neq 0$ by 2.5.

As a corollary of 3.7 we have

16.2. THEOREM. *Let $\boldsymbol{K}$ be a perfect class and assume that there exists a perfect map from X onto Y. Then $G(\boldsymbol{K}, X)$ is determined iff $G(\boldsymbol{K}, Y)$ is determined.*

16.3. THEOREM. *If $X \in \sigma\check{\boldsymbol{C}}$ and X has the Lindelöf property, or if X is σ-discrete, then $G(\boldsymbol{K}, X)$ is determined.*

Hence, in particular, if X is σ-compact, or if X is an F_σ-set in a Polish space, then $G(\boldsymbol{K}, X)$ is determined.

Proof. Let $X \in \check{\boldsymbol{C}}$. If $X \in \boldsymbol{SK}$ and X has the Lindelöf property, then $I(\boldsymbol{K}, X) \neq 0$ by 9.3. If $X \notin \boldsymbol{SK}$, then $II(\boldsymbol{K}, Y) \neq 0$ by 15.6. Now let $X \in \boldsymbol{D}$. If $X \in \sigma\boldsymbol{K}$, then $I(\boldsymbol{K}, X) \neq 0$ by 4.7. If $X \notin \sigma\boldsymbol{K}$, then $2^X \cap \boldsymbol{K}$ is such a $\boldsymbol{K}$-cover of X that $2^X \cap \boldsymbol{K}$ does not contain a countable cover of X. Thus $II(\boldsymbol{K}, X) \neq 0$ by 5.8. By 16.1 the theorem follows.

16.4. THEOREM. *If X is subparacompact and $X \in \sigma\check{\boldsymbol{C}}$, then $G(\boldsymbol{DK}, X)$ and $G(\boldsymbol{LK}, X)$ are determined.*

Hence, in particular, if X is an F_σ-set in a complete metric space, then $G(\boldsymbol{DK}, X)$ and $G(\boldsymbol{LK}, X)$ are determined.

Proof. Let X be a subparacompact space with $X \in \check{\boldsymbol{C}}$. If $X \in \boldsymbol{SK}$, then $I(\boldsymbol{DK}, X) \neq 0$ by 9.7. Since $\boldsymbol{DK} \subseteq \boldsymbol{LK}$, we also have $I(\boldsymbol{LK}, X) \neq 0$ by 2.1.1. If $X \notin \boldsymbol{SK}$, then $II(\boldsymbol{LK}, X) \neq 0$ by 15.6. Since $\boldsymbol{DK} \subseteq \boldsymbol{LK}$, we also have $II(\boldsymbol{DK}, X) \neq 0$ by 2.1.2. Hence $G(\boldsymbol{DK}, X)$ and $G(\boldsymbol{LK}, X)$ are determined. By 16.1, the theorem follows.

16.5. THEOREM. *If $X \in \sigma\check{\boldsymbol{C}}$ or if $X \in \sigma\boldsymbol{SC}$, then $G(\boldsymbol{F}, X)$ and $G(\boldsymbol{C}, X)$ are determined.*

Hence, in particular, if $X \in \sigma\boldsymbol{C}$ or $X \in \sigma\boldsymbol{LC}$ or $X \in \sigma\boldsymbol{SF}$ or $X \in \sigma\boldsymbol{D}$, then $G(\boldsymbol{F}, X)$ and $G(\boldsymbol{C}, X)$ are determined.

Proof. Let $X \in \sigma\check{\boldsymbol{C}}$. If X has the Lindelöf property, then $G(\boldsymbol{F}, X)$ and $G(\boldsymbol{C}, X)$ are determined by 16.3. If X does not have the Lindelöf property, then $II(\boldsymbol{C}, X) \neq 0$ and $II(\boldsymbol{F}, X) \neq 0$ by 5.9 and 2.1.2. Now let $X \in \boldsymbol{SC}$. If $2^X \cap \boldsymbol{C} \subseteq \boldsymbol{SF}$, then $X \in \boldsymbol{SF}$. If $X \in \boldsymbol{SF}$ and X has the Lindelöf property, then $I(\boldsymbol{F}, X) \neq 0$ by 9.3. If $X \in \boldsymbol{SF}$ and X does not have the Lindelöf property, then $II(\boldsymbol{F}, X) \neq 0$ by 5.9 and 2.1.2. If $2^X \cap \boldsymbol{C} \not\subseteq \boldsymbol{SF}$, then there exists a compact dense-in-itself subset E of X i.e. $E \in 2^X \cap \boldsymbol{C} \cap \boldsymbol{NLF}$. Thus $II(\boldsymbol{F}, E) \neq 0$ by 15.5 and therefore $II(\boldsymbol{F}, X) \neq 0$ by 2.1.2. Hence $G(\boldsymbol{F}, X)$ is determined. If X has the Lindelöf property, then $I(\boldsymbol{C}, X) \neq 0$ by 9.3. If X does not have the Lindelöf property, then $II(\boldsymbol{C}, X) \neq 0$ by 5.9 and 2.1.2. Hence $G(\boldsymbol{C}, X)$ is determined. If $X \in \sigma\boldsymbol{SC}$, then the determinacy of $G(\boldsymbol{C}, X)$ follows from 16.1.

16.6. THEOREM. *If X is countably compact, then $G(\boldsymbol{F}, X)$, $G(\boldsymbol{C}, X)$, $G(\sigma\boldsymbol{C}, X)$, $G(\boldsymbol{D}, X)$ and $G(\boldsymbol{DC}, X)$ are determined.*

Proof. If X is countably compact, then $2^X \cap \sigma\boldsymbol{C} = 2^X \cap \boldsymbol{C}$, $2^X \cap \boldsymbol{D} = 2^X \cap \boldsymbol{F}$ and $2^X \cap \boldsymbol{DC} = 2^X \cap \boldsymbol{C}$. Thus it suffices to prove the determinacy of $G(\boldsymbol{F}, X)$ and $G(\boldsymbol{C}, X)$. Let X be countably compact. If X has the Lindelöf property, then X is compact and therefore $G(\boldsymbol{F}, X)$ and $G(\boldsymbol{C}, X)$ are determined by 16.5. If X does not have the Lindelöf property, then $II(\boldsymbol{F}, X) \neq 0$ and $II(\boldsymbol{C}, X) \neq 0$ by 5.9 and 2.1.2.

16.7. THEOREM. *If X is an A-set in a complete metric space, then $G(\boldsymbol{F}, X)$, $G(\boldsymbol{C}, X)$, $G(\boldsymbol{D}, X)$, $G(\boldsymbol{DC}, X)$ and $G(\boldsymbol{LC}, X)$ are determined.*

Proof. Let X be an A-set in a complete metric space. Assume that $X \in \sigma\boldsymbol{LC}$. Then $G(\boldsymbol{F}, X)$ and $G(\boldsymbol{C}, X)$ are determined by 16.5, because $\sigma\boldsymbol{LC} \subseteq \sigma\check{\boldsymbol{C}}$. The games $G(\boldsymbol{D}, X)$, $G(\boldsymbol{DC}, X)$ and $G(\boldsymbol{LC}, X)$ are determined by 16.4. Now assume that $X \notin \sigma\boldsymbol{LC}$. Then, by 15.10, X has a closed subset E which is homeomorphic to the space of irrational numbers. By 15.5 we have $II(\boldsymbol{LC}, X) \neq 0$. Thus, by 2.1.2, it follows that $II(\boldsymbol{F}, X) \neq 0$, $II(\boldsymbol{C}, X) \neq 0$, $II(\boldsymbol{D}, X) \neq 0$ and $II(\boldsymbol{DC}, X) \neq 0$.

16.8. THEOREM. *If X is a subset of a Polish space Y for which $Y - X$ is totally metacompact* (see [10]), *then $G(\boldsymbol{F}, X)$ is determined.*

Proof. The theorem follows immediately from the following result of A. Lelek [9]: If X is a subset of a Polish space Y for which $Y - X$ is totally metacompact, then X is countable or X contains a copy of the Cantor discontinuum. In the first case $I(\boldsymbol{F}, X) \neq 0$ by 4.7 and in the second case we have $II(\boldsymbol{F}, X) \neq 0$ by 15.6.

16.9. QUESTION. Does there exist a scattered space X for which $G(\boldsymbol{D}, X)$ is not determined?

J. Mycielski [15] has investigated the consequences of a game-theoretical axiom (A) which postulates the determinacy of a game on each subset of the Cantor discontinuum. (A) implies that each uncountable separable metric space contains a copy of the Cantor discontinuum ([15], p. 207). (A) also implies a weak form of the axiom of choice ([15], p. 207) which enables us to construct winning strategies in $G(\boldsymbol{F}, X)$, where X is any metric separable space. Hence (A) implies the determinacy of $G(\boldsymbol{F}, X)$ for any metric separable space X.

F. Galvin has communicated to me that he can prove the following: If the continuum hypothesis, or if the Martin axiom, is assumed, then there exists a subset X of the real line for which $G(\boldsymbol{F}, X)$ is not determined.

Let us state the main problem of this paper which remains unsolved: What is the topological characterization of the space X for which $I(\boldsymbol{K}, X) \neq 0$ (or $I_s(\boldsymbol{K}, X) \neq 0$, or $II(\boldsymbol{K}, X) \neq 0$, or $II_s(\boldsymbol{K}, X) \neq 0$)?

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