On movability and other similar shape properties

by

Juliusz Olgąski (Warszawa)

Abstract. The hereditary shape property called the $\mathcal{R}$-movability has been defined. Some relations between the $\mathcal{R}$-movability and other shape properties: the movability, the $\mathcal{R}$-movability and the $\mathcal{A}$-movability have been established. There are answers the following questions:

1° Is it true that if compacta $X$ and $Y$ are $\mathcal{R}$-movable, then $X \times Y$ is $\mathcal{R}$-movable?
2° Is it true that if a compactum $X$ is $\mathcal{A}$-movable and $\mathcal{B}$-movable then $X$ is $\mathcal{A}$-$\mathcal{B}$-movable?

where the binary operation $\times$ is the Cartesian product or the join or the one-point union or the topological sum.

1. Introduction. K. Borsuk introduced hereditary shape properties: $\mathcal{R}$-movability and $\mathcal{A}$-movability ([3] and [4]). Let $\mathcal{I}$ be a family of compacta. In this paper we define $\mathcal{R}$-movability, which is a generalization of those shape properties. The aim of this paper is to study the properties of $\mathcal{R}$-movability and to determine the relations between $\mathcal{R}$-movability, $\mathcal{A}$-movability, $\mathcal{R}$-movability and movability.

2. $\mathcal{R}$-movability. Let $A$ and $X$ be compacta and let $X \subset M \times \mathcal{A}B(3)$. $X$ is said to be $A$-movable if for every neighborhood $U$ of $X$ in $M$ there exists a neighborhood $U_0$ of $X$ in $M$ such that for every neighborhood $U$ of $X$ in $M$ every map $a: A \to U_0$ is homotopic in $U$ to a map with values in $U$. K. Borsuk showed ([4]) that $A$-movability does not depend upon the choice of a space $M$ and that $A$-movability is a hereditary shape property, i.e., if $Sh(X) \subseteq Sh(Y)$ and $Y$ is $A$-movable, then $X$ is $A$-movable. To generalize this property, consider a family $\mathcal{I}$ of compacta. A compactum $X \subset M \times \mathcal{A}B(3)$ is said to be $\mathcal{R}$-movable if for every neighborhood $U$ of $X$ in $M$ there exists a neighborhood $U_0$ of $X$ in $M$ such that for every neighborhood $U$ of $X$ in $M$ and for every $A \in \mathcal{I}$ every map $a: A \to U_0$ is homotopic in $U$ to a map with values in $U$. By a slight change in the proofs of an analogous theorem for $A$-movability ([4]) one proves that
The choice of the space $M$ and the embedding of $X$ into $M$ are not important for the definition of $\mathcal{K}$-movability and that

If $X$ is $\mathcal{K}$-movable and $\text{Sh}(X) \geq \text{Sh}(Y)$, then $Y$ is $\mathcal{K}$-movable.

It is easy to see that

If a compactum $X$ is movable, then $X$ is $\mathcal{K}$-movable for every family $\mathcal{K}$.

A compactum $X$ lying in the Hilbert cube $Q$ is said to be $n$-movable ([3], p. 859) if for every neighborhood $U$ of $X$ in $Q$ there exists a neighborhood $U_0$ of $X$ in $Q$ such that for every compactum $A \subset U_0$ with $\text{dim} A < n$ and for every neighborhood $\bar{U}$ of $X$ in $Q$ there exists a homeomorphism $\psi: A \times [0,1] \rightarrow U$ satisfying conditions: $\psi \circ \alpha = \alpha$ and $\psi(\alpha(1),0) \in U$ for every point $\alpha \in A$. Put $\mathcal{M} = Q$ in the definition of $\mathcal{K}$-movability. Since for every compactum $A \subset Q$ and its neighborhood $U_0$ in $Q$ every map $\alpha: A \rightarrow U_0$ is homotopic to an embedding of $A$ into $U_0$, we get

If $\mathcal{K}$ is a family of all compacta of dimension $n$, then $\mathcal{K}$-movability is equivalent to $n$-movability.

Let $\mathcal{K}$ and $\mathcal{K}'$ be families of compacta, $\mathcal{K}$ is said to be $\mathcal{M}$-dominated by $\mathcal{K}'$ ($\mathcal{K} \subseteq \mathcal{K}'$) if every $\mathcal{K}$-movable compactum $X$ is also $\mathcal{K}'$-movable. If $\mathcal{K} \subseteq \mathcal{K}'$, then $\mathcal{K}$ and $\mathcal{K}'$ are said to be $\mathcal{M}$-equivalent ($\mathcal{K} \cong \mathcal{K}'$).

If families $\mathcal{K}$ and $\mathcal{K}'$ consist of single elements $A$ and $A'$ respectively, then we write $A \cong A' \mathcal{M}$ ($A \cong A'$) instead of $\mathcal{K} \cong \mathcal{K}'$.

Hence for the neighborhood $U_n$ of $A'$ in $Q$ there exists a neighborhood $V_n$ of $B$ in $N$ and an integer $n$ such that

$$g_{n, [n]} \approx g_{n+1, [n]} \text{ in } U_n, \quad \text{for } n \geq n_0.$$

Let $U$ be a neighborhood of $X$ in $Q$. Since $g_{n, [n]}$ carries $B$ into $U_n$ and $X$ is $\mathcal{K}$-movable, there exists a homotopy $\psi: B \times [0,1] \rightarrow U$ such that $\psi(p,0) = g_{n, [n]}(p)$ and $\psi(p,1) \in U$ for every point $p \in B$. Define a map $F: B \times [0,1] \cup (N \times \{0\}) \rightarrow U$ by

$$F(p,t) = \begin{cases} \psi(p,t) & \text{for } (p,t) \in B \times [0,1], \\ g_{n, [n]}(p) & \text{for } (p,t) \in N \times \{0\}. \end{cases}$$

Since $B \times [0,1] \cup N \times \{0\}$ is a compactum and $U$ is open in $Q$, there exists a neighborhood $W$ of $B \times [0,1] \cup N \times \{0\}$ in $N \times \{0\}$ and a map $F: W \rightarrow U$ extending $F$. There exists a neighborhood $V$ of $B$ in $N$ satisfying three conditions: $V \subset V_n$, $V \times [0,1] \subset W$ and $F(V \times \{1\}) \subset U$. Since $I$ is a fundamental sequence and $g \cong g_{1, [1]}$, there exists an $n_0 \geq n$ such that

$$f_{n_0}(A') \subset V$$

and

$$g_{n_0} \circ f_{n_0, [1]} \cong g_{n_0, [2]} \text{ in } U.$$

By (2.7) and (2.8) it follows that

$$g_{n_0} \circ f_{n_0, [1]} \cong g_{n_0, [2]} \text{ in } U'. $$

$$F|_{f_{n_0}(A' \times [0,1])} = f_{n_0}(A') \times [0,1] \rightarrow U$$

is a homotopy satisfying conditions $F(p,0) = g_{n_0, [n]}(p)$ and $F(p,1) \in U$ for every $p \in f_{n_0}(A')$. Let $F_1: f_{n_0}(A') \rightarrow U$ be defined by the formula $F_1(p) = F(p,1)$. Therefore

$$g_{n_0} \circ f_{n_0, [1]} \approx F_1 \circ f_{n_0, [1]} \text{ in } U.$$ 

By (2.6), (2.7), (2.10) and (2.11), $g$ is homotopic in $U$ to a map with values in $U_0$, and thus $X$ is $\mathcal{K}$-movable.

**Corollary.** If $\text{Sh}(A) \cong \text{Sh}(B)$, then $A \cong B$.

**Example.** Let $S^j_i$ be $i$-dimensional spheres for $i = 1, 2, j = 1, 2$ and let $S^j_i \cap S^j_{i'} = \emptyset$ for $(i,j) \neq (i',j')$. Let $A = S^j_i \cup S^j_{i'} \cup S^j_i$ and $B = S^j_i \cup S^j_{i'} \cup S^j_{i'}$.

Shapes of $A$ and $B$ are not comparable, but one can easily see that

$$A \cong B.$$
Example. Let \( R \) be a family of solenoids and let a family \( \{G_i\} \) consist of a single circle \( G \). Then \( \mathcal{H} \subset \{G_i\} \).

Assume that a compactum \( X \subset N \in \operatorname{AR}(3) \) is \( \mathcal{H} \)-movable. By Corollary (2.12), \( X \) is \( T \)-movable, where \( T \) is a solid torus. Let \( U \) be a neighborhood of \( X \) in \( N \). There exists a neighborhood \( \tilde{U} \) of \( X \) in \( N \) such that for every neighborhood \( V \) of \( X \) in \( N \) every map \( \alpha : \tilde{U} \to U \) is homotopic in \( U \) to a map with values in \( \tilde{U} \). Let \( \tilde{U} \) be a neighborhood of \( X \) in \( N \) and take \( \beta \in \mathcal{H} \) and \( \beta : \tilde{U} \to U \). The solenoid \( S \) can be described as an intersection of a decreasing sequence of solid tori \( T_i, i = 1, 2, ... \).

There exists an integer \( n_i \) and a map \( \beta_i : T_{n_i} \to U \) extending \( \beta \). Since \( \beta_i \) is homotopic in \( U \) to a map with values in \( \tilde{U} \), \( \beta \) is homotopic in \( U \) to such a map. Thus \( X \) is \( \mathcal{H} \)-movable.

Problem. Is the family of all solenoids \( M \)-equivalent to a circle?

3. Relations between movability, \( n \)-movability, \( A \)-movability, and \( \mathcal{H} \)-movability. S. Mardešić and J. Segal ([8], p. 651) proved the following.

Lemma. If \( X = \lim \{X_n, \rho_m\} \), where \( X_n \in \operatorname{ANR} \) for \( n = 1, 2, ... \), then \( X \) is movable if and only if for every integer \( n \) there exists an integer \( n \geq n \) such that for every \( n > n \) there exists a map \( r : X_{n-1} \to X_n \) satisfying the condition \( p_m \circ r = \rho_m \).

By a slight modification of the proof of this lemma one can easily show the following.

Lemma. If \( X = \lim \{X_n, \rho_m\} \), where \( X_n \in \operatorname{ANR} \) for \( n = 1, 2, ... \), and \( \mathcal{H} \) is a family of compacta, then \( X \) is \( \mathcal{H} \)-movable if and only if for every integer \( n \) there exists an integer \( n \geq n \) such that for every \( n > n \) and for every \( A \in \mathcal{H} \) and for every map \( f : A \to X_n \) there exists a map \( A \to X_n \) satisfying the condition:

\[
p_m \circ f = \rho_m \circ A.
\]

Corollary. If \( X = \lim \{X_n, \rho_m\} \), where \( X_n \in \operatorname{ANR} \) for \( n = 1, 2, ... \), and \( \mathcal{H} \) is a family of compacta and for all \( n \) and for every \( A \in \mathcal{H} \), every map \( f : A \to X_n \) is homotopic to a constant map, then \( X \) is \( \mathcal{H} \)-movable.

Lemma. Let \( X = \lim \{X_n, \rho_m\} \), where \( X_n \in \operatorname{ANR} \) for \( n = 1, 2, ... \) and let \( \mathcal{H} \). If \( X \) is \( \mathcal{H} \)-movable, then \( X \) is movable.

Proof. Let \( n \) be an integer. It follows by Lemma (3.2) that there exists an integer \( n \geq n \) such that for every \( n > n \) and for every \( A \in \mathcal{H} \) and for every map \( f : A \to X_n \) there exists a map \( A \to X_n \) satisfying condition (3.3). Let \( \mathcal{H} \). Take \( A = X_n \) and \( A = \rho_m \). Hence there exists a map \( A \to X_n \) satisfying condition (3.3). Thus \( p_m = \rho_m \circ A \). Put \( r = A \). By Lemma (3.1), \( X \) is movable.

Theorem. There exists a countable family \( \mathcal{W} \) of polyhedra such that \( \mathcal{W} \)-movability is equivalent to \( \mathcal{H} \)-movability.

Proof. One knows that there are only countably many homotopy types of polyhedra. Let \( \mathcal{W} \) consist of elements taken singly from all homotopy types of polyhedra. Let a compactum \( X \in \mathcal{W} \)-movable. \( X \) can be described as an inverse sequence of a decreasing sequence of solid tori \( T \).

Conversely, if \( X \) is movable, then by (2.3) \( X \) is \( \mathcal{W} \)-movable for every family \( \mathcal{W} \) of compacta, in particular for \( \mathcal{W} \).

Corollary. There exists a compactum \( W \) such that \( W \)-movability is equivalent to \( \mathcal{H} \)-movability.

Proof. Let \( W \) be a one-point compactification of a disjoint union of elements of \( \mathcal{W} \). It is clear that the family consisting of the single element \( W \) is \( \mathcal{H} \)-equivalent to \( \mathcal{W} \).

Corollary. There exists a maximal element (the family \( \mathcal{W} \) of compacta or the compactum \( W \)) in the partial ordering “\( \leq \)”.

Problem. Does there exist, for every family \( \mathcal{W} \) of compacta, a compactum \( \mathcal{A} \) such that \( \mathcal{A} \mathcal{W} \mathcal{A} \mathcal{W} \)?

Theorem. For \( n = 1, 2, ... \), there exists a countable family \( \mathcal{W} \) of polyhedra of dimension \( \leq n \) such that \( \mathcal{W} \)-movability is equivalent to \( \mathcal{W} \)-movability.

Proof. There is only a countable number of homotopy types of polyhedra of dimension \( \leq n \). Let \( \mathcal{W} \) consist of polyhedra taken singly from all these types. Let \( \mathcal{H} \) be a family of all compacta of dimension \( \leq n \).

By (2.4), \( \mathcal{W} \)-movability is equivalent to \( \mathcal{H} \)-movability. Since \( \mathcal{H} \subset \mathcal{W} \), \( \mathcal{W} \)-movability implies \( \mathcal{W} \)-movability. Assume now that \( X \) is \( \mathcal{W} \)-movable. Let \( X \subset N \in \operatorname{AR}(3) \) and let \( U \) be a neighborhood of \( X \) in \( N \). Since \( X \) is \( \mathcal{W} \)-movable, there exists a neighborhood \( U \) of \( X \) in \( N \) such that for every neighborhood \( \tilde{U} \) of \( X \) in \( N \) and for every \( K \subset \tilde{U} \) there exists a map \( \varphi : K \to U \) satisfying the conditions:

\[
\varphi \approx \varphi' \quad \text{in} \quad U \quad \text{and} \quad \varphi(K) \subset \tilde{U}.
\]

Take \( A \in \mathcal{H} \) and a map \( \alpha : A \to U \) and let \( \tilde{U} \) be a neighborhood of \( X \) in \( N \). Since \( \dim A < n \), there exist polyhedra \( K_i \) for \( i = 1, 2, ... \), and maps \( p_i : K_i \to K_i \) for \( i > n \) such that \( \dim K_i = n \) and \( A = \lim (K_i, p_i) \).

Let the maps \( p_i : K_i \to K_i \) for \( i = 1, 2, ... \) be projections such that \( p_i = p_i \circ p_i \) for \( i = 2, ... \). Since \( U_n \) is open in \( N \in \operatorname{AR}(3) \), there exist an integer \( \tilde{u} \) and a map \( \tilde{u} : K_n \to U \) such that \( \tilde{u} : p_n \approx u \).
Let \( K_a \) be homotopically equivalent to \( K \times W^n \). Hence there exist maps \( f : K_a \to K \) and \( g : K \to K_a \) such that \( g \circ f \simeq 1_{K_a} \). Take \( \varphi = a \circ g \). Since \( X \) is \( W^n \)-movable, there exists a \( \varphi' : K \to U \) satisfying (3.11). Thus \( \varphi' \circ f \cdot p_{a} \simeq a \circ g \cdot f \cdot p_{a} \simeq a \in U \) and \( \varphi' \circ f \cdot p_{a} \subseteq U \). Then \( X \) is \( \mathcal{R} \)-movable, and thus by (3.4) \( X \) is \( n \)-movable.

Let \( W^n \) be a one-point compactification of a disjoint union of polyhedra belonging to \( W^n \). The family \( \mathcal{M} \) is \( \mathcal{M} \)-dominated by the family consisting of the single element \( W^n \). Combining this with Theorem 17 in [4] and with Theorem (3.10) we get the following

(3.12) COROLLARY. The following conditions are equivalent:

(a) \( X \) is \( n \)-movable,
(b) \( X \) is \( \mathcal{A} \)-movable for every compactum \( A \) of dimension \( < n \),
(c) \( X \) is \( W^n \)-movable.

By Corollary (2.12) we can replace "dim\( A \)" by "\( \mathcal{F}(\mathcal{A}) \)" in condition (b).

(3.13) THEOREM. If a compactum \( X \) is \( n \)-movable and \( \mathcal{F}(\mathcal{A}) < n \), then \( X \) is movable.

Proof. Since \( \mathcal{F}(\mathcal{A}) < n \), there exists a compactum \( Y \) such that \( \mathcal{S}(Y) = \mathcal{S}(X) \) and \( \dim(Y) < n \). Hence there exist polyhedra \( Y_i \) for \( i = 1, 2, \ldots \) and maps \( p_{Y_i} : Y_i \to Y \) such that \( \dim(Y_i) < n \) for \( i = 1, 2, \ldots \) and \( Y = \lim(X_i, p_{Y_i}) \). \( Y \) is \( n \)-movable ([3], p. 860). Then it follows by Theorem (3.10) that \( Y \) is \( W^n \)-movable. Let \( \mathcal{A} = (Y_i : i = 1, 2, \ldots) \). By Theorem (2.5), \( Y \) is \( \mathcal{R} \)-movable. Finally, by Lemma (3.5), \( Y \) is movable. Movable is a shape property ([1], p. 142), and thus \( X \) is movable.

It is easy to see that if a compactum \( X \) is \( \mathcal{R} \)-movable, then \( X \) is \( \mathcal{A} \)-movable for every \( A \in \mathcal{A} \). But the converse implication fails.

(3.14) EXAMPLE. There exist a family \( \mathcal{A} \) of compacta and a compactum \( X \) which is \( \mathcal{A} \)-movable for every \( A \in \mathcal{A} \), but is not \( \mathcal{R} \)-movable.

For every natural \( n \) let \( T_n \) be the orientable surface with \( n \) handles. Put \( \mathfrak{A} = (T_n : n = 1, 2, \ldots) \) and let \( X \) be a non-movable continuum described by K. Borsuk in [2]. The compactum \( X \) can be obtained as an inverse limit of a sequence \( (T_n, p_{m,n}) \) satisfying the condition: for \( m < n \) there exists a point \( x_m \in T_n \) for which \( p_{m,n}(x_m) \) is an embedding. By Lemma (3.5), \( X \) is not \( \mathcal{R} \)-movable. It remains to prove that \( X \) is \( T_2 \)-movable for every \( T_2 \in \mathcal{A} \). Let \( m \) be an integer and let \( u \neq 0 \) be greater than \( n \) and \( k \). Take \( \alpha = x_m \) and \( \beta \) a carry \( T_n \) into \( T_{2n} \). Since the number of handles of \( T_n \) is greater than \( n \), it is homotopic to a map \( \beta \) with values in \( T_{2n} \). Define \( \alpha : T_{2n} \to T_2 \) by \( \alpha(x) = \beta \cdot z_{m,n}(\beta(x)) \); thus (3.3) is satisfied. By Lemma (3.2) \( X \) is \( T_2 \)-movable.

4. Some properties of \( \mathcal{R} \)-movability.

(4.1) EXAMPLE. For \( n = 2, 3, \ldots \), there exists a continuum \( X_n \) which is \( (n-1) \)-movable but not \( n \)-movable.

Let \( X_n \) be an inverse limit of a sequence \( (S_k, p_{2k}) \), where \( S_k \) is a \( n \)-dimensional sphere for \( k = 1, 2, \ldots \) and the maps \( p_{2k} : S_k \to S_{2k} \) for \( k > k \) are such that \( deg(p_{2k}) > 1 \). \( X_n \) is a solenoid and \( X_n \) is the suspension of a solenoid. Since the homotopy classes of the maps \( p_{2k} \) are given, the shape of \( X_n \) is completely determined. \( X_n \) is non-movable ([8], p. 692); therefore by Theorem (3.13), \( X_n \) is not \( \mathcal{R} \)-movable. By Corollary (3.4) and Theorem (3.10), \( X_n \) is \( (n-1) \)-movable. This example is an answer to the Problem (4.6) from [3], p. 864.

For a family \( \mathcal{A} \) of compacts and for an arbitrary binary operation \( \circ \) in the family of all compacts, the following two problems arise:

1° Is it true that if \( X \) and \( Y \) are \( \mathcal{R} \)-movable, then \( X \circ Y \) is \( \mathcal{R} \)-movable?

2° Is it true that if \( X \) is \( \mathcal{A} \)-movable and \( \mathbb{B} \)-movable, then \( X \) is \( \mathcal{A} \circ \mathbb{B} \)-movable?

First, we are going to answer these two questions for \( \circ \) being the Cartesian product. By a slight change of the proof that if \( X \) and \( Y \) are movable, then \( X \times Y \) is movable ([1], p. 142) one proves the following

(4.2) THEOREM. \( X \) and \( Y \) are \( \mathcal{R} \)-movable if and only if \( X \times Y \) is \( \mathcal{R} \)-movable.

(4.3) COROLLARY. \( X \) and \( Y \) are \( n \)-movable if and only if \( X \times Y \) is \( n \)-movable.

By Example (4.1), for \( n = 1, 2, \ldots \) there exists a \( n \)-movable compactum which is not \( (n+1) \)-movable. Therefore

(4.4) If \( X \) is \( n \)-movable and \( Y \) is \( m \)-movable, then \( X \times Y \) is \( min(n, m) \)-movable and the last number cannot be increased in general.

The statement (4.4) is an answer to Problem (1.6) from [3], p. 860.

It is not true that if \( X \) is \( \mathcal{A} \)-movable and \( \mathbb{B} \)-movable, then \( X \) is \( \mathcal{A} \circ \mathbb{B} \)-movable.

(4.5) EXAMPLE. There exists a compactum \( X \) which is \( S^m \)-movable but is not \( S^m \times S^m \)-movable. Furthermore the non-movable compactum \( X \) is \( \mathcal{R} \)-movable, where \( \mathcal{R} \) is a family of shapes of all dimensions.

Let \( (a_k) \) and \( (x_k) \) be sequences of prime numbers greater than \( 1 \). Let \( S_k \) and \( S_k' \) be circles for \( k = 1, 2, \ldots \) and let \( a_k \) be \( S_k \) and \( a_k' \) be \( S_k' \). Denote \( (S_k \times S_k') \cup ((a_k \times S_k') \cup S_k \times S_k') \), \( X_k \). Let \( S_k \cup S_k' \), \( k = 2, 3, \ldots \)
and $S_1 \times S'_1$ be pairwise disjoint sets. Put $X_n = S_1 \times S'_1$ and $X_{n+1} = S_1 \times S'_1 \cup \bigcup_{k=1}^{n} S_k \cup S'_k$ for $n \geq 2$. Define maps $p_{n+1} : X_{n+1} \to X_n$ by

$$p_{n+1}(x) = \begin{cases} \rho_{n+1}(x) & \text{for } x \in S_1 \times S'_1, \\ (p_n, \rho_n)(x) & \text{for } x \in S_1 \times S'_1', \\ x & \text{for } x \in \bigcup_{k=1}^{n} S_k \cup S'_k, \end{cases}$$

where the map $\rho_n : S_1 \cup S'_1 \to S_1 \cup S'_1$ is a homeomorphism and maps $p_n : S_1 \to S_1$ and $p'_n : S'_1 \to S'_1$ are such that $\deg p_n = \lambda_n$ and $\deg p'_n = \lambda'_n$ for $n = 1, 2, \ldots$. Put $p_{n+1} = p_{n+1} \circ \cdots \circ p_{n+1} : X_{n+1} \to X_n$ for $n > n$. Let $X = \lim \{X_n, p_n\}$. Let $X$ be a family of spheres of all dimensions. We will prove that $X$ is $n$-movable. Let $n$ be an integer and put $n_0 = n$. Take $\alpha = n$ and a map $a : S^n \to X_{n_0}$. If $m > 1$, then $a$ is homotopic to a constant map. Let $a' : S^m \to X_{n_0}$ be a constant map such that the sets $p_{n_0} \circ a(S^m)$ and $a(S^m)$ are both included in the same component of $X_{n_0}$. Then condition (3.3) is satisfied. In the case of $n = n_0$, we put $a' = a$; then condition (3.3) is also satisfied. Consider $m = 1$ and $n > n_0$. If $a(S^1) \cup S_1 \cup S_1' \supset S_2$, then define $a' : S^1 \to X_{n_0}$ by $a'(x) = a(x)$ for $x \not\in S_1$. Thus $p_{n_0} \circ a = a = a$. If $a(S^1) \subset S_1 \times S_1'$, then $a$ is homotopic to some map $\tilde{a} : S^1 \to X_{n_0}$ with values in $S_1 \cup S'_1$. Then define $a' : S^1 \to X_{n_0}$ by $a'(x) = \tilde{a}^{-1}(G(a(x)))$. Thus $p_{n_0} \circ a = a = a$. Assume that a map $a' : S^1 \to X_{n_0}$ is connected, $a'(S^1) \subset S_1 \times S_1'$ or $a'(S^1) \subset S_1' \times S_1$. In the first case, since $\lambda_1$ and $\lambda_1'$ are greater than 1, $p_{n+1} \circ a$ and $p_{n+1} \circ a'$ are not homotopic. In the second case, $p_{n+1} \circ a'(S^1) \subset S_1 \times S_1'$ or $S_1' \times S_1$. But $p_{n+1} \circ a$ is homotopic to a map with values in a proper subset of $S_1 \times S_1'$. By Lemma (3.2) we infer that $X$ is not $n$-movable. Thus $X$ is nonmovable. Since $X$ is $n$-movable, $X$ is $n$-movable for $n = 1, 2, \ldots$.

Example (4.5) is now the same as Problem 19 from [4].

Now consider the join of two spaces as the operation $\ast$. The join $X \ast Y$ of two compacta $X, Y$ is the quotient space $(X \times Y) \cup \{0\}$, where $\Theta$ is the decomposition of $X \times Y \times \{0, 1\}$ into sets of the form $(a) \times X \times \{1\}$ or $(b) \times Y \times \{0\}$ where $a \in X$ and $b \in Y$ and into single points. The shape of $X \ast Y$ depends only upon $Sh(X)$ and $Sh(Y)$ ([10]), p. 854).

In general for the operation of the join the answers to questions 1 and 2 are negative. Indeed, the join $S^n \ast S^n$ of a solenoid $S^n$ and a $n$-dimensional sphere $S^n$ (i.e., the space $S^n$ in Example (4.1)) is not movable. $S^n \ast S^n$ is a inverse limit of a sequence of $n+2$-dimensional spheres. By Lemma (3.3) $S^n \ast S^n$ is not $n^{n+3}$-movable, while by Corollary (3.4) $S^n$ and $S^n$ are $n^{n+3}$-movable. Also it is easy to see that the join $A \ast B$ of two two-point spaces $A$ and $B$ is a circle $S^1$. By Corollary (3.4) a solenoid is $A$-movable and $B$-movable, but is not $S^n$-movable.

(4.6) Theorem. Let $\mathcal{K}$ be a family of compacta such that if $A \ast \mathcal{K}$ and a compaction $B$ is the closure of an open subset of $A$, then $B \in \mathcal{K}$. If compacta $X$ and $Y$ are $\mathcal{K}$-movable, then the join $X \ast Y$ is $\mathcal{K}$-movable.

Proof. Let $Q$ and $Q'$ be the Hilbert cubes. Assume that $X \ast Q$ and $Y \ast Q'$ are $\mathcal{K}$-movable in $Q$ and $Q'$ respectively. Let $U = Q \ast Q' \in \mathcal{K}$ and let $U'$ be a neighborhood of $X \ast Y$ in $M$. There exists a neighborhood $U_1$ of $X$ in $Q$ and a neighborhood $U_2$ of $Y$ in $Q'$ and a number $\varepsilon \in (0, 1)$ such that the sets $U_1 \ast U_2 = \{(x, y, t) \in M_1 \times U_1, y \in U_2, K(U_1, U_2) = \{(x, y, t) \in M_1 \times U_1, y \in U_2, 1 - \varepsilon < t < 1\}$. Then $U'$ is a subset of $U_1 \ast U_2$.

Since $X$ and $Y$ are $\mathcal{K}$-movable, for $U_1$ and $U_2$ there exist neighborhoods: $U_1' \subseteq U_1$ in $Q$ and $U_2' \subseteq U_2$ in $Q'$ satisfying required conditions of the definition of the $\mathcal{K}$-movability. Then $U_1' \ast U_2' = \{(x, y, t) \in M_1 \times U_1', y \in U_2', K(U_1', U_2') = \{(x, y, t) \in M_1 \times U_1', y \in U_2', 1 - \varepsilon < t < 1\}$ and $U_1' \ast U_2' \subseteq U_1 \ast U_2$.

Define maps $p : b \ast U \to b$ by $p(a) = a(a)$ for $a \in b'$. Since $X$ and $Y$ are $\mathcal{K}$-movable, there exists a homotopy $F : b \times [0, 1] \to U^2$ and there exists a homotopy $F' : b' \times [0, 1] \to U^2$ satisfying conditions: $F(b, 0) = b(b)$ and $F(b, 1) = b'$. Define a map $q : b \ast U \to b'$ by $q(a) = a(a)$ for $a \in b'$. Since $X$ and $Y$ are $\mathcal{K}$-movable, there exists a homotopy $F : b \times [0, 1] \to U^2$ and there exists a homotopy $F' : b' \times [0, 1] \to U^2$ satisfying conditions: $F(b, 0) = b(b)$ and $F(b, 1) = b'$. Define a map $r : b \times [0, 1]$ by $r = (x, y, t) \in b \times [0, 1]$ and $r(a) = a(a)$ for $a \in b'$. Let $a_1 \in b_1$ such that $a_1(0)$ is a subset of $U_1$ and $a_1(1)$ is a subset of $U_2$ and $a_1(t) = F(a_1(a))(t)$ for $t \in [0, 1]$.
Define a homotopy $H: A \times [0,1] \to U$ by the formula:

$$H(a,s) = \begin{cases} p_{2} (a(s)), & 0 < s < \frac{1}{2} \text{ and } \alpha(a) \leq \frac{1}{2}, \\ p_{1} (a(s)), & 0 < s < \frac{1}{2} \text{ and } \alpha(a) > \frac{1}{2}, \\ 1, & s = \frac{1}{2}, \\ q_{3} (a(s)) = q(a(s)), & s > \frac{1}{2} \text{ and } q(a(s)) < 1, \\ q_{4} (a(s)) = q(a(s)), & s > \frac{1}{2} \text{ and } q(a(s)) \geq 1. \\ \end{cases}$$

This homotopy satisfies conditions: $H(a,0) = q(a)$ and $H(a,1) = q_{3} \circ q^{4} \in U$ for every $a \in A$. Thus $X \times Y$ is $\mathcal{A}$-movable.

**Remark.** If the compacta $X$ and $Y$ are $n$-movable, then the join $X \star Y$ is $n$-movable.

Proof. $n$-movability is equivalent to $\mathcal{A}$-movability, where $\mathcal{A}$ is a family of all compacta of the dimension $\leq n$ (cf. (2.4)). If $A \in \mathcal{A}$ and a compactum $B \subset A$, then dim $B \leq n$, then $B \in \mathcal{A}$. Hence satisfies the assumption of Theorem (4.8).

**Example.** There exist compacta $X^{1}, X^{2}, A^{1}$ and $A^{2}$ such that $X^{1} \times X^{2} = \{x_{i}^{1}, X^{1} \times A^{2} = \{y_{i}^{2}\}$, $X^{1}$ is $A^{1} \times A^{2}$-movable and $X^{2} \times X^{1}$ is $A^{2} \times A^{1}$-movable for $i = 1, 2$, but $X^{1} \times X^{2}$ is not $A^{1} \times A^{2}$-movable.

The main idea of this example is due C. Cox [5]. Let $i = 1, 2$. Let $\{j_{k}\}$ be sequences of prime number different from 1. For $k = 1, 2, ..., \{k_{j_{k}}\}$ let $S_{j_{k}}^{1}$ be pairwise disjoint $k$-dimensional spheres, except the pair $S_{j_{k}}^{1}$, $S_{j_{k}+1}^{1}$ with the point $a_{k}$ in common. Let $f_{j_{k}}^{1}: S_{j_{k}}^{1} \to S_{j_{k}+1}^{1}$ be a map such that deg $f_{j_{k}}^{1} = 1$ and $f_{j_{k}}^{1}(a_{k}) = a_{k}$ for $j_{k} = 1, 2, ..., \text{ and let } h_{j_{k}}^{1}: S_{j_{k}}^{1} \to S_{j_{k}}^{1}$ be a homeomorphism for $k = 1, 2, ...$. Put $X_{j_{k}}^{1} = \bigcup_{k=1}^{\infty} S_{j_{k}}^{1}$. Define $p_{n+1}^{1}: X_{j_{k}}^{1} \to X_{j_{k}}^{1}$ by

$$p_{n+1}^{1}(x) = \begin{cases} f_{j_{k}}^{1}(x) & \text{ for } x \in S_{j_{k}}^{1}, \\ x & \text{ for } x \in S_{j_{k}+1}^{1}. \end{cases}$$

Let $p_{n}^{1} = p_{n+1}^{1} \circ ... \circ p_{n+1}^{1}$ for $n < n'$ and $p_{n}^{1} = \text{id}_{X_{j_{k}}^{1}}$. Let $n$ be an integer and put $n_{0} = n$. Put $n > n_{0}$. Define a map $r_{j_{k}}^{1}: X_{j_{k}}^{1} \to X_{j_{k}}^{1}$ by

$$r_{j_{k}}^{1}(x) = \begin{cases} x & \text{ for } x \in S_{j_{k}}^{1}, \\ (h_{j_{k}}^{1})^{-1}(x) & \text{ for } x \in S_{j_{k}+1}^{1}. \end{cases}$$

Then $p_{n}^{1} \circ r_{j_{k}}^{1} \circ p_{m}^{1} = \text{id}_{X_{j_{k}}^{1}}$. By Lemma (3.1) $X_{j_{k}}^{1}$ is movable; then $X_{j_{k}}^{1}$ is $B$-movable for every compactum $B$. Let $A^{1} = S_{j_{k}}^{1} \times A^{2}$ and $A^{2} = S_{j_{k}}^{1}$. It is easy to see that a compactum $X = X^{1} \times X^{2}$ is $A^{1}$-movable and $A^{2}$-movable. It remains to prove that $X$ is not $S_{j_{k}}^{1} \times S_{j_{k}}^{1}$-movable. Let $S_{j_{k}}^{1} \times S_{j_{k}}^{1} = A$. We have $X = \lim_{n \to \infty} (X_{j_{k}}^{1}, a_{n})$, where $X_{j_{k}}^{1} = X_{j_{k}}^{1} \times X_{j_{k}}^{1}$ and $a_{n}(x) = p_{m}^{1}(x)$ for $x \in X_{j_{k}}^{1}$. Take $\varepsilon = 1$ and let $n_{0} = 1$. Put $n = n_{0} + 1$ and let $a: A \to X_{j_{k}}^{1}$ be an inclusion map. Take a map $a^{1}: A \to X_{j_{k}}^{1}$. For $i = 1, 2$, $S_{j_{k}}^{1}$ is reeled $(s_{0}^{1}, ..., s_{m-1}^{1})$ times in $S_{j_{k}}^{1}$ by $p_{m}^{1}(x)$ for $x \in A$. Since $A$ is connected, $a^{1}(A)$ is contained in some component of $X_{j_{k}}^{1}$. If $a^{1}(A) \subset S_{j_{k}}^{1}$ for $k > 1$, then $a^{1}(A)$ is not homotopic to a constant map. If $a^{1}(A) \subset S_{j_{k}}^{1} \times S_{j_{k}}^{1}$, then for $i = 1, S_{j_{k}}^{1}$ is reeled $s_{0}^{1}$ times in $S_{j_{k}}^{1}$ by $a^{1}(A)$ for some integer $s_{0}^{1}$. Since $s_{0}^{1} > 1$, $S_{j_{k}}^{1}$ and $S_{j_{k}}^{1} \times S_{j_{k}}^{1}$ are different. Thus $p_{m}^{1} \circ a$ and $p_{m}^{1} \circ a^{1}$ are not homotopic. By Lemma (3.2), $X$ is not $A^{1} \times A^{2}$-movable.

(4.9) **Theorem.** If every component of a compactum $X$ is $\mathcal{A}$-movable, then $X$ is $\mathcal{A}$-movable.

Proof. Assume that $X \times N \in \mathcal{A}(\mathcal{B})$. Let $U$ be a neighborhood of $X$ in $N$. As in the proof of a similar theorem for movability (113, p. 140) we can choose a free system of components $X_{1}, ..., X_{n}$ of $X$ and pairwise disjoint open sets $U_{1}, ..., U_{n}$ satisfying three conditions:

$U_{i}$ is a neighborhood of $X_{i}$ in $N$ for $i = 1, 2, ..., n$, $U_{n} = \bigcup_{i=1}^{n} U_{i}$ is a neighborhood of $X$ in $N$, $U_{i} \subset U_{i+1}$ for $i = 1, 2, ..., n$, for every neighborhood $U_{i}$ of $X_{i}$ in $N$ and for every $A \in \mathcal{A}$ every map $a: A \to U_{i}$ is homotopic in $U$ to a map with values in $U_{i}$.

Let $U_{i}$ be a neighborhood of $X$ in $N$ and take $A \in \mathcal{A}$ and a map $a: A \to U_{i}$. Define $a_{i}: A \to U_{i}$ for $i = 1, 2, ..., n$ by

$$a_{i}(x) = \begin{cases} a(x) & \text{ for } x \in A_{i}, \\ x_{i} & \text{ for } x \not\in A_{i}, \end{cases}$$

where $x_{i}$ is a fixed point of $X_{i}$.

For $i = 1, 2, ..., n$, put $H_{i}: X \times [0,1] \to U$ be a homotopy such that $H_{i}(x,0) = a(x)$ and $H_{i}(x,1) = x$ for every $x \in A$. Define $H: X \times [0,1] \to U$ by $H(x,t) = H_{i}(a(t))$ for $a \in A_{i}$. $H(x,0) = a(x)$ and $H(x,1) = x$ for every $x \in A$, then $X \in \mathcal{A}$-movable.

On the other hand, it is not true that if $X$ is $\mathcal{A}$-movable, then every component of $X$ is $\mathcal{A}$-movable. There exists a movable compactum with a solenoid as a component (K. Borsuk's Example [13], p. 140, also the compactum $X^{1}$ in Example (4.8)). As in Example (4.8), for every compactum $X$ which is not $\mathcal{A}$-movable one can construct an $\mathcal{A}$-movable compactum $Y$ with $X$ as a component.
(4.10) Theorem. If $\mathcal{A}$ is a family of all components of a compactum $A$, then $(\mathcal{A})$ and $\mathcal{A}$ are $M$-equivalent.

Proof. Assume that $X \subseteq Y$ in $N$. Assume that for every $B \in \mathcal{A}$, there exists a homotopy $\varphi_0: B \times [0,1] \rightarrow U$ such that $\varphi_0(a,0) = a$ and $\varphi_0(a,1) \in \hat{U}$ for every $a \in B$. Since a component $B$ is closed in $A$ and $U$ is open in $N$, a homotopy $\varphi_0$ can be extended over a set $B' \times [0,1]$ such that $B'$ is a closed-open neighborhood of $B$. As in the proof of Theorem (4.9), one can choose a finite system of components $B_1, ..., B_n$ such that the sets $B_1', ..., B_n'$ constructed for them are pairwise disjoint and $\bigcup B_i' = A$. Define a homotopy $H: A \times [0,1] \rightarrow U$ by $H(a,t) = \varphi_0(a,t)$ for $a \in B_i'$; then $H(a,0) = a$ and $H(a,1) \in \hat{U}$ for every $a \in A$. Conversely, assume that $B \in \mathcal{A}$, a map $\beta: B \rightarrow U_0$ and that every map $a: A \rightarrow U_0$ is homotopic in $U$ to a map with values in $\hat{U}$. There exists a closed-open neighborhood $B'$ of $B$ in $A$ and a map $\beta': B' \rightarrow \hat{U}$ extending $\beta$. Let $U_0 \in U_0$, and define $a: A \rightarrow U_0$ by

$$a(a) = \begin{cases} \beta'(a) & \text{for } a \in B', \\ U_0 & \text{for } a \in A - B'. \end{cases}$$

Therefore, $\beta = a|_B$ is homotopic in $U$ to a map with values in $\hat{U}$. Thus $\mathcal{A}$-movability and $\mathcal{A}$-movability are equivalent.

The notion of $\mathcal{X}$-movability has recently been studied by Kodama and Watanabe and by Koskiewicz and Segal (see [6] and [7]). They obtained independently the following results contained in the present paper: Theorems (3.6), (3.10) and (3.13), Example (4.11) and Corollary (4.3).

References