On the product of derivatives

by

Richard J. Fleissner (Milwaukee, Wis.)

Abstract. In this note it is shown that the product of a continuous function of bounded variation with a derivative is a derivative. An example is then given of a discontinuous function whose product with every derivative is a derivative.

Introduction. Let A denote the class of real-valued functions defined on the closed interval $[0,1]$ whose product with every derivative is a derivative. James Foran [1] has shown that every absolutely continuous function belongs to A. In the present note, we show that A includes all continuous functions of bounded variation. An example of a discontinuous function which belongs to A is also presented.

We recall that if $-\infty < F'(x) = f(x) < +\infty$ for each point $x \in [a, b]$, then $f(x)$ is Denjoy-integrable on $[a, b]$ in both the wide and the restricted sense. For a proof of this, see Theorem (10.5), Saks [2, p. 235], and the descriptive definitions of the D and D_* integrals, Saks [2, p. 241]. (In this note I will use the wide sense D-integral, although the restricted sense D_*-integral may be used throughout.) Noting then that a function is a derivative if, and only if, it is the derivative of its indefinite D-integral, we are ready to proceed.

LEMMA 1. *If $F(x)$ is a continuous function of bounded variation on $[a, b]$ and $g(x)$ is D-integrable on $[a, b]$, then $F(x)g(x)$ is D-integrable on $[a, b]$ and*

$$(1) \qquad (D)\int_a^b F(x)g(x)\,dx = G(b)F(b) - G(a)F(a) - \int_a^b G(x)\,dF(x)$$

where $G(x) = (D)\int_a^x g(t)\,dt$ and $\int_a^b G(x)\,dF(x)$ denotes the Lebesgue-Stieltjes integral of $G(x)$ with respect to $F(x)$.

Proof. This is a special case of Theorem (2.5), Saks [2, p. 246].

THEOREM. *If $F(x)$ is a continuous function of bounded variation on $[0,1]$ and $G'(x) = g(x)$ for each x in $[0,1]$, then $F(x)g(x)$ is the derivative of its indefinite Denjoy integral on $[0,1]$.*

Proof. It suffices to prove the theorem in the case where $F(x)$ is a continuous increasing function on $[0,1]$. Given x_0 in $[0,1]$, we shall show that

$$L = \lim_{x\to x_0} (x-x_0)^{-1}\Big(G(x)F(x)-G(x_0)F(x_0)-\int_{x_0}^{x} G(t)\,dF(t)\Big) - F(x_0)g(x_0)$$

exists and equals 0. It follows from this and (1) that

$$\lim_{x\to x_0} (x-x_0)^{-1}(D)\int_{x_0}^{x} F(t)g(t)\,dt = F(x_0)g(x_0).$$

Adding and subtracting $F(x)G(x_0)(x-x_0)^{-1}$ yields

$$L = \lim_{x\to x_0} (x-x_0)^{-1}\big(F(x)G(x)-F(x)G(x_0)\big)-F(x_0)g(x_0)+$$
$$+\lim_{x\to x_0} (x-x_0)^{-1}\Big([F(x)-F(x_0)]G(x_0)-\int_{x_0}^{x} G(t)\,dF(t)\Big)$$

provided both limits exist. Since $F(x)$ is continuous and $G'(x_0) = g(x_0)$,

$$\lim_{x\to x_0} (x-x_0)^{-1}\big(F(x)G(x)-F(x)G(x_0)\big)-F(x_0)g(x_0) = 0 .$$

Hence, we need only show that

$$L^* = \lim_{x\to x_0} (x-x_0)^{-1}\Big([F(x)-F(x_0)]G(x_0)-\int_{x_0}^{x} G(t)\,dF(t)\Big)$$

exists and equals 0. Since $F(x)$ is continuous and increasing $F(x)-F(x_0) = \int_{x_0}^{x} dF(t)$. Therefore, L^* becomes

$$\lim_{x\to x_0} (x-x_0)^{-1}\int_{x_0}^{x} \big(G(x_0)-G(t)\big)\,dF(t)$$
$$= \lim_{x\to x_0} (x-x_0)^{-1}\int_{x_0}^{x} \big(G(x_0)-G(t)\big)(x_0-t)^{-1}(x_0-t)\,dF(t) .$$

We note that $\big(G(x_0)-G(t)\big)(x_0-t)^{-1} = g(x_0)+\varepsilon(t)$, where $\varepsilon(t)\to 0$ as $t\to x_0$. For $x \neq x_0$, let $\hat{\varepsilon}(x) = \sup_t |\varepsilon(t)|$, where $0 < |t-x_0| \leqslant |x-x_0|$. Then $\hat{\varepsilon}(x)\to 0$ as $x\to x_0$ and if t is in the interval from x_0 to x, $|G(x_0)-G(t)|\,|x_0-t|^{-1} \leqslant |g(x_0)|+\hat{\varepsilon}(x)$. Hence,

$$\overline{\lim_{x\to x_0}}\, |x-x_0|^{-1}\Big|\int_{x_0}^{x} |G(x_0)-G(t)|\,|x_0-t|^{-1}|x_0-t|\,dF(t)\Big|$$
$$\leqslant \overline{\lim_{x\to x_0}}\, |x-x_0|^{-1}|x-x_0|\big(|g(x_0)+\hat{\varepsilon}(x)\big)\Big|\int_{x_0}^{x} dF(t)\Big|$$
$$= \overline{\lim_{x\to x_0}}\, \big(|g(x_0)|+\hat{\varepsilon}(x)\big)|F(x)-F(x_0)| = 0 . \qquad \text{Q.E.D.}$$

If $F(x)$ has a bounded derivative on $[0,1]$, then $F(x)$ is of bounded variation and, consequently, belongs to A. To show that differentiability is not a sufficient condition, we give the following example. For $x\in(0,1]$, let

$$F(x) = x^2\sin(x^{-4}), \qquad g(x) = x^{-2}\sin(x^{-4}) ,$$
$$G(x) = \tfrac{1}{4}x^3\cos(x^{-4})-\tfrac{3}{4}\int_0^x t^2\cos(t^{-4})\,dt ,$$
$$J(x) = x^5\sin(x^{-4})\cos(x^{-4})$$

and

$$k(x) = 5x^4\sin(x^{-4})\cos(x^{-4})-4 .$$

Let $F(0) = G(0) = g(0) = F(0)g(0) = J(0) = 0$ and let $k(0) = -4$. It is easily verified that $F(x)$, $G(x)$ and $J(x)$ are differentiable, that $g(x)$ is the derivative of $G(x)$, that $k(x)$ is continuous and is, therefore, a derivative, and that $J'(x)-k(x)-8F(x)g(x) = 0$ for $x\in(0,1]$. However, $J'(0)-k(0)-8F(0)g(0) = 4$. Because this function does not possess the Darboux property, and since $J'(x)$ and $k(x)$ are derivatives, it follows that $F(x)g(x)$ cannot be a derivative.

The example of a discontinuous function which belongs to A requires two lemmas.

LEMMA 2. *For an interval $I = [a,b]$, let $c = \frac{1}{2}(a+b)$ and let $h_I(x) = 2(b-a)^{-1}(x-a)$ if $x\in[a,c]$ and $h_I(x) = -2(b-a)^{-1}(x-b)$ if $x\in[c,b]$. (Geometrically, the graph of $h_I(x)$ consists of the two equal sides of an isosceles triangle whose base is I and whose altitude is 1.) If $g(x)$ is D-integrable, then $g(x)h_I(x)$ is D-integrable by Lemma 1. Moreover, letting $G(x) = (D)\int_a^x g(t)\,dt$ and $H(x) = (D)\int_a^x g(t)h_I(t)\,dt$, we have*

$$(2) \qquad O(H,I) \leqslant 4O(G,I) , \quad \text{where} \quad O(F,I) = \sup_{\alpha,\beta\in I} F(\alpha)-F(\beta) .$$

Proof. If x_0 and x_1 are in $[a,c]$, it follows from (1) that

$$\Big|(D)\int_{x_0}^{x_1} g(x)h_I(x)\,dx\Big| = \Big|G(x_1)h_I(x_1)-G(x_0)h_I(x_0)-\int_{x_0}^{x_1} G(x)\,dh_I(x)\Big|$$
$$= \Big|G(x_1)h_I(x_1)-\int_a^{x_1} G(x)\,dh_I(x)+\int_a^{x_0} G(x)\,dh_I(x)-G(x_0)h_I(x_0)\Big|$$
$$\leqslant \Big|G(x_1)h_I(x_1)-\int_a^{x_1} 2(b-a)^{-1}G(x)\,dx\Big|+$$
$$+\Big|\int_a^{x_0} 2(b-a)^{-1}G(x)\,dx-G(x_0)h_I(x_0)\Big|$$

$$= |G(x_1)h_I(x_1) - 2(b-a)^{-1}(x_1-a)G(\xi)| + |2(b-a)^{-1}(x_0-a)G(\xi') - G(x_0)h_I(x_0)|$$
$$= |G(x_1)-G(\xi)|\,|h_I(x_1)| + |G(\xi')-G(x_0)|\,|h_I(x_0)|$$
$$\leqslant 2O(G,[a,c])$$

since x_0, x_1, ξ and ξ' are in $[a,c]$ and $0 \leqslant h_I(x) \leqslant 1$. The demonstration that $O(H,[c,b]) \leqslant 2O(G,[c,b])$ is identical and, since H and G are continuous on I, $O(H,I) \leqslant 4O(G,I)$.

LEMMA 3. *Let $I_n = [a_n, b_n]$ be a sequence of closed intervals contained in $(0,1]$ such that*

(i) $$a_{n+1} < b_{n+1} < a_n < b_n, \quad n = 1,2,3,\ldots,$$

(ii) $$\lim_{n\to\infty} b_n = 0,$$

(iii) $$\frac{b_n - a_n}{a_n b_n} < R, \quad R > 0,\ n = 1,2,3,\ldots$$

Let $G(x)$ be a continuous function defined on $(0,1]$ such that $G(0)=0$ and $G'(0)$ exists. Let $O(G,I_n)b_n^{-1} = \varepsilon_n$. Then

(3) $$\lim_{n\to\infty} \varepsilon_n = 0.$$

Proof. Without loss of generality, $G'(0)=0$. For if $G'(0)=h$, letting $F(x) = G(x) - hx$, we have that $F(x)$ is continuous, $F(0) = F'(0) = 0$ and $O(G,I_n) \leqslant O(F,I_n) + O(hx,I_n)$. But condition (iii) implies that the lemma is true for the function $f(x) = hx$. Therefore, proving the result for $F(x)$ would give the result for $G(x)$ and we may assume that $G'(0)=0$.

Since $G(x)$ is continuous on $[a_n,b_n]$, $O(G,I_n) = G(r_n) - G(s_n)$, where $r_n, s_n \in I_n$. Since

$$\lim_{x\to 0}\left|\frac{G(x)}{x}\right| = 0 \quad\text{and}\quad \left|\frac{G(x)}{b_n}\right| \leqslant \left|\frac{G(x)}{x}\right| \quad\text{for}\quad x \leqslant b_n,$$

$$\lim_{n\to\infty}\frac{G(r_n)}{b_n} = \lim_{n\to\infty}\frac{G(s_n)}{b_n} = 0.$$

Hence,

$$\lim_{n\to\infty}\varepsilon_n = \lim_{n\to\infty}\frac{G(r_n)-G(s_n)}{b_n} = 0.$$

EXAMPLE. There exists a discontinuous function which belongs to A.

Construction. Let $[a_n,b_n]$ be a sequence of intervals which satisfy conditions (i), (ii), and (iii) of Lemma 3 and in addition satisfy

(iv) $$\lim_{k\to\infty} a_k^{-1}\sum_{i=k+1}^{\infty} b_i = 0.$$

(For example, we could let $[a_n,b_n] = [1/(2n)!,\ 1/(2n)!+1/(4n)!]$).

Let $f(x) = h_{I_n}(x)$ if $x \in I_n$ and let $f(x)=0$ if $x \in [0,1] - \bigcup_n I_n$. Then $f(x)$ is discontinuous at $x=0$ since $f(0)=0$ and $f(a_n + b_n/2) = 1$.

Let $g(x)$ be a derivative and $G(x) = (D)\int_0^x g(t)\,dt$. Since $G'(x) = g(x)$ for all $x \in [0,1]$, $G(x)$ satisfies the conditions of Lemma 3.

On any interval $[\delta,1]$ where $\delta > 0$, $f(x)$ is a continuous function of bounded variation and, by our theorem, $f(x)g(x)$ is the derivative of its indefinite D-integral on $[\delta,1]$. Therefore, it suffices to show that $(D)\int_0^1 f(x)g(x)\,dx$ exists and that

$$\lim_{h\to 0} h^{-1}(D)\int_0^h f(x)g(x)\,dx = f(0)g(0) = 0.$$

Let $Q = [0,1] - \bigcup_n (a_n,b_n)$. Then Q is a closed set and $f(x)g(x)=0$ for every $x \in Q$. It follows that

(4) $$(D)\int_Q f(x)g(x)\,dx = 0.$$

It follows from condition (iv) that $\sum_{n=1}^{\infty} b_n < \infty$. By (2), $|(D)\int_{a_n}^{b_n} f(x)g(x)\,dx| \leqslant 4O(G,I_n)$. Letting $\varepsilon_n = O(G,I_n)b_n^{-1}$, we have by (3), that $\lim_{n\to\infty}\varepsilon_n = 0$ and

(5) $$\sum_{n=1}^{\infty}\left|(D)\int_{a_n}^{b_n} f(x)g(x)\,dx\right| \leqslant 4\sum_{n=1}^{\infty}\varepsilon_n b_n < +\infty.$$

Letting $H_n(x) = (D)\int_{a_n}^{x} f(t)g(t)\,dt$, (2) implies

(6) $$\lim_{n\to\infty} O(H_n,I_n) \leqslant \lim_{n\to\infty} 4O(G,I_n) = 0.$$

By (5), (6), (7) and Theorem (5.1), Saks [2, p. 257], $f(x)g(x)$ is D-integrable on $[0,1]$ and

(7) $$(D)\int_0^1 f(x)g(x)\,dx = (D)\int_Q f(x)g(x)\,dx + \sum_{n=1}^{\infty}(D)\int_{a_n}^{b_n} f(x)g(x)\,dx.$$

To show that $\lim_{h\to 0} h^{-1}(D)\int_0^h f(x)g(x)\,dx = 0$, we first note that if $h \in [b_N, a_{N-1}]$, then

$$\left|h^{-1}(D)\int_0^h f(x)g(x)\,dx\right| = \left|h^{-1}(D)\int_0^{b_N} f(x)g(x)\,dx\right|$$
$$\leqslant \left|b_N^{-1}\int_0^{b_N} f(x)g(x)\,dx\right|$$

since $f(x)g(x) = 0$ on $[b_N, a_{N-1}]$ and $b_N \leqslant h$. It suffices to consider $h \in [a_N, b_N]$. By (7),

$$\left|h^{-1}(D)\int_0^h f(x)g(x)\,dx\right|$$

$$= \left|h^{-1}\Big(\sum_{k=N+1}^{\infty}(D)\int_{a_k}^{b_k} f(x)g(x)\,dx\Big) + h^{-1}(D)\int_{a_N}^{h} f(x)g(x)\,dx\right|$$

$$\leqslant \left|h^{-1}\sum_{k=N+1}^{\infty} 4O(G, I_k)\right| + |4h^{-1}O(G, I_N)| \quad \text{by (2)}$$

$$= \left|h^{-1}\sum_{k=N+1}^{\infty} 4b_k\varepsilon_k\right| + |4h^{-1}b_N\varepsilon_N|$$

$$\leqslant \left|a_N^{-1}\sum_{k=N+1}^{\infty} 4b_k\varepsilon_k\right| + |4a_N^{-1}b_N\varepsilon_N| \quad \text{since } a_N \leqslant h.$$

Since $\lim_{n\to\infty}\varepsilon_n = 0$, $4\varepsilon_n$ is eventually less than 1 and $|a_N^{-1}\sum_{k=N+1}^{\infty} 4b_k\varepsilon_k| \to 0$ as $N\to\infty$ by condition (iv). By condition (iii), $(b_N - a_N)(b_N a_N)^{-1} < R$. Therefore, $b_N a_N^{-1} - 1 < b_N R$ and $b_N a_N^{-1} \to 1$ as $N\to\infty$. Hence, $|4a_N^{-1}b_N\varepsilon_N| \to 0$ as $N\to\infty$. This completes the proof.

It is easy to show that conditions (iii) and (iv) in the definition of the I_n imply that $x = 0$ is a point of dispersion for the set $\bigcup_{n=1}^{\infty} I_n$. Thus, any function constructed in this fashion will be approximately continuous. This is in agreement with the following result. If $f(x)$ is a bounded function, a necessary and sufficient condition that $f(x)g(x)$ be a derivative for every bounded derivative $g(x)$, is that $f(x)$ be approximately continuous. This theorem is proved in Iosifescu [3]. The example, $F(x) = x^2\sin(x^{-4})$, given earlier shows that this condition is not sufficient for a function to be in A.

References

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UNIVERSITY OF WISCONSIN
Milwaukee

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