Some examples of monothetic groups

by

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Abstract. In this paper we discuss in Section 1 the finest group topology on $\boldsymbol{Z}$ such that $\{2^n\}_{n\in N}$ is a sequence converging to zero. We prove that $\boldsymbol{Z}$ is a complete topological group with respect to that topology. The proof is almost entirely self-contained.

In section 2 we give an example of a monothetic group (that is a topological group which contains a dense copy of $\boldsymbol{Z}$) which is complete metrizable, totally disconnected and which has no continuous characters except 0. The example is constructed by starting with a complete metrizable totally disconnected monothetic group and factoring out a discrete subgroup which is dense in the Bohr compactification. The main difficulty here is in proving discreteness. Section 2 relies heavily on a trick developed in Section 1, namely the use of a kind of "coordinates" for 2-adic numbers. Section 0 contains a survey of related results already existing in the literature.

0. Historical comments. We anticipate Section 1 by introducing for any sequence $A = \{a_n\}_{n\in N}$ of integers the concept of an A-topology. This is a group topology on $\boldsymbol{Z}$ such that A is a sequence converging to **0**. For any sequence A there exists at least one A-topology namely the trivial one (with Ø and $\boldsymbol{Z}$ as the only open sets). The union of all A-topologies is the finest A-topology. It is nondiscrete. We will call it T_A.

The first example of a group topology on $\boldsymbol{Z}$ in which $\boldsymbol{Z}$ is complete is due to Graev [4]. It is the finest topology on $\boldsymbol{Z}$ such that a sequence $\{0\} \cup \{a_n\}_{n\in N}$ has a given structure of compact set. In [4], $a_n = (n+2)!$, but actually only $|a_{n+1}/a_n| \to \infty$ is required. $\boldsymbol{Z}$ becomes then locally isomorphic to the free abelian group generated by $\{a_n\}$ as a set. See also [10]. If one takes $I \subset \boldsymbol{N}$ and considers B_I-topologies, for $B_I = \{a_n\}_{n\in I}$, and if one lets I run through a suitable filter of subsets of $\boldsymbol{N}$, and takes care to define these B_I-topologies conveniently and compatible with each other, then the union of all B_I-topologies provides another example. If $\boldsymbol{Z}$ is complete with respect to all these B_I-topologies, one can take any filter provided it contains only infinite sets. If $\boldsymbol{Z}$ is not complete

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with respect to the chosen collection of B_I-topologies, one must choose the filter more carefully, see also [7]. However, the quoted references are not very simple. The example presented here is rather self-contained, and moreover, the given sequence A does not increase so fast.

In section 2 we construct a complete metrizable monothetic group without continuous characters, except 0. Such an example gives a partial answer to the question: what has compactness to do with being monothetic? Though the first examples of monothetic groups were compact, like p-adic integers, tori, solenoids etc., and though Weil proved that for nondiscrete monothetic groups local compactness implies compactness, a complete monothetic group need not necessarily be compact. This follows in essence from an example given by Bohr [2], of a function f on $\boldsymbol{R}$ that he called "periodenartig", because it had the property that

$$P_\varepsilon = \{x \in \boldsymbol{R}\colon\ |f(x+y)-f(y)| \leqslant \varepsilon \text{ for all } y \in \boldsymbol{R}\}$$

is unbounded. Such a function, Bohr showed, need not be "fast periodisch"; the latter means that P_ε is more or less uniformly distributed over $\boldsymbol{R}$ (more precisely: for all ε, there exists an L such that for any interval I of length exceeding L, $I \cap P_\varepsilon \neq \emptyset$).

When a function f on $\boldsymbol{R}$ is used to obtain an invariant metric Δ_f defined by

$$\Delta_f(x, y) = \sup\{|f(x+z)-f(y+z)|\colon\ z \in \boldsymbol{R}\},$$

the sets P_ε will form a neighborhood basis at 0 for a group topology on $\boldsymbol{R}$. If f is merely uniformly continuous, the ordinary topology on $\boldsymbol{R}$ will result. If f is "periodenartig" some other group topology, weaker than the usual one, will result. If f is almost periodic (fast periodisch) a finite number of translates of any neighborhood of 0 will suffice to cover $\boldsymbol{R}$. Only in that case the completion is compact. So Bohr's example shows the existence of a complete metrizable not locally compact topological group that contains a dense copy of $\boldsymbol{R}$. The whole argument works as well for $\boldsymbol{Z}$, of course.

This remark was actually made by Anzai and Kakutani [1] and in slightly improved form it was mentioned in a footnote to an article by Comfort and Ross [3]. We call this Example A. Example B (not metrizable) is the one by Graev [4], cited above. Example C is by Hinrichs [6]. It is a Hausdorff ring topology on $\boldsymbol{Z}$, such that small neighborhoods do not contain ideals, while compact Hausdorff rings containing a dense copy of $\boldsymbol{Z}$ must have a basis of neighborhoods consisting of ideals. One may however dispense with the "ideal" argument, see [10]. Rolewicz [11] showed that the subgroup of the infinite dimensional torus consisting of these elements whose coordinates converge to 0, is monothetic, if provided with the proper metrization. This is Example D. The author [7] showed that, again with proper metrization, the infinite dimensional torus contains closed totally disconnected unbounded monothetic groups (Example E).

One might ask now: do continuous characters of complete monothetic groups separate points? In abelian topological groups the continuous characters separate points if they are locally compact; without this condition this need not hold, as the example of L^p-spaces with $0<p<1$ (Hewitt and Ross [5]) shows. For monothetic groups, one can factor a discrete subgroup out of Example D, [9], and out of Example E (Section 2 below), such that the quotient has no continuous characters except 0.

1. A simple example of a complete non-discrete infinite cyclic topological group.

1.1. Notations and terminology, outline of proofs. $\boldsymbol{Z}$ is the group of integers and $\boldsymbol{N}$ is the set of natural numbers including zero. If S is a sequence of elements of $\boldsymbol{Z}$ and S converges to zero with respect to some group topology on $\boldsymbol{Z}$ then we will call that topology an *S-topology*.

So for $A = \{2^n\}_{n\in N}$, we can say that the 2-adic topology on $\boldsymbol{Z}$ is an *A-topology*. The finest A-topology we will call T_A. We will prove that $\boldsymbol{Z}$ is complete with respect to T_A.

Generally when a metric topology is given by a metric m we will speak about the *m-topology*, *m-ball*, *m-open* etc. By the way, metrics on abelian groups are supposed to be functions satisfying

$$m(x) = 0 \Leftrightarrow x = 0,$$

$$m(-x) = m(x),$$

$$m(x)+m(y) \geqslant m(x+y).$$

We will use the notation $\|\cdot\|$ if it is clear which metric is being used. We will concentrate our attention to a class $\mathfrak{C}$ of metrics, namely all those for which $\|2^n\|$ is a decreasing zero sequence. $\mathfrak{C}$ will be proved to be rich enough to generate T_A.

We will identify metrics of $\mathfrak{C}$ with the topologies on $\boldsymbol{Z}$ they generate, in the sense that we will say "topology from $\mathfrak{C}$" instead of "topology defined by a metric from $\mathfrak{C}$".

Summations, unless written otherwise, are always taken for the variable (which will be evident) running through N, but we always suppose a finite number of summands is unequal zero. When we say a sequence p_n is decreasing we mean $p_n \geqslant p_{n+1}$ for all n.

The proof of completeness of T_A will proceed as follows. Every T_A-Cauchy net in $\boldsymbol{Z}$ is of course a Cauchy net with respect to each of those metrics. Convergence with respect to any of these metrics implies 2-adic convergence (the 2-adic topology is a coarsest Hausdorff A-topology). Then, for a given T_A-Cauchy net in Z, not convergent to an element of Z we will find a metric from $\mathcal{C}$ in which it will not converge, thus arriving at a contradiction.

1.2. The class $\mathcal{C}$. For any decreasing sequence $p = \{p_n\}_{n \in N}$ of positive non-zero real numbers, we define a metric d_p by

$$d_p(x) = \inf\left(\sum |a_n| p_n\right), \tag{1.2.1}$$

where the infimum is taken over all finite sequences a_n of integers such that $x = \sum a_n 2^n$.

We prove that $\|x\| = d_p(x)$ is a metric from $\mathcal{C}$ such that $d_p(2^n) = p_n$.

(i) If x is not divisible by 2^{n+1}, then $\|x\| \geqslant p_n$.

Observe that one can restrict oneself to sequences a_n such that $|a_n| \leqslant 1$ for all n.

(ii) If $x = 2^n$ then $\|x\| \leqslant p_n$, hence $\|2^n\| = p_n$.

(iii) $\|x\| + \|y\| = \inf\left(\sum (|a_n| + |b_n|) p_n\right)$ where $\sum a_n 2^n = x$, $\sum b_n 2^n = y$; this is larger or equal $\inf\left(\sum |a_n + b_n| p_n\right)$ where $\sum (a_n + b_n) 2^n = x + y$; hence $\|x\| + \|y\| \geqslant \|x + y\|$ follows.

(iv) Clearly $\|-x\| = \|x\|$ and $\|0\| = 0$.

Let now be given any invariant metric d on Z such that $d(x)$ is continuous in x with respect to the finest A-topology on Z. Then $d(2^n)$ is a zero sequence. We define a sequence of numbers p_n by

$$p_n = \max_{m \geqslant n} d(2^m). \tag{1.2.2}$$

Clearly p_n is decreasing to zero.

Using this sequence $p = \{p_n\}$, define d_p by (1.2.1), so $d_p \in \mathcal{C}$. We observe $d(2^n) \leqslant d_p(2^n) = p_n$, so by definition of d_p it follows that $d(x) \leqslant d_p(x)$.

Remark. Actually $p_{n+1} \geqslant 2p_n$ would have sufficed for the proof that d_p satisfies $d(2^n) = p_n$. Moreover, d_p is the largest metric that satisfies these equations. Also, for $|x| \geqslant 1$, we may restrict the collection of a_n over which the inf is taken, to those for which: $a_n = 0$ if $2^{n+1} | x$, $a_n = 0$ if $n-1 \geqslant {}^2\log x$. So the inf here is actually a minimum.

We indicate now how to prove that the topology T_A is the union of all topologies from $\mathcal{C}$. T_A certainly contains the union of all topologies from $\mathcal{C}$.

Let U be a T_A-neighborhood of $0 \in Z$. Let V_0 be a symmetric neighborhood contained in U. Let V_1, V_2, etc. be such that for all i: $V_{i+1} + V_{i+1} \subset V_i$, V_i symmetric and $\bigcap_{i \in N} V_i = \{0\}$.

The V_i form then a neighborhood basis for a Hausdorff topology D on Z; D is a metrizable A-topology, let d be a metric that induces D. So U contains a d-ball around 0. If we pass to d_p, then it is seen that U contains a d_p-ball. So we have proved that an arbitrary T_A-open set contains an open set from the union of all topologies from $\mathcal{C}$. So indeed T_A is the union of topologies from $\mathcal{C}$.

Remark. The above mentioned metric d can be defined as follows: let f on Z be defined by $f(x) = 1$ for $x \notin V_1$ and $f(x) = 2^{-i}$ for $x \in V_i \setminus V_{i+1}$, $i \geqslant 1$ and $f(0) = 0$. Then let d be defined by $d(x) = \inf\left(\sum f(x_i)\right)$ where infimum is taken over all finite sequences $\{x_i\}$ such that $\sum x_i = x$.

1.3. Coordinates. In the following we restrict ourselves to metrics d_p, for some sequence p. We observe that if $2^n | x$, then $\|x\| \geqslant p_n$, hence convergence with respect to a topology from $\mathcal{C}$ implies 2-adic convergence.

We first find a sequence a_i in $x = \sum a_i 2^i$ such that $\|x\| = \sum |a_i| \|2^i\|$. Firstly $|a_i| \leqslant 1$ for all i, evidently. Then, $a_j = 1 = a_{j+1}$ does not have to occur for any j. Indeed, $2^j + 2^{j+1} = -2^j + 2^{j+2}$, so if $a_j = 1 = a_{j+1}$ we could find an equivalent or better choice a'_i by putting $a'_i = a_i$ for $i \notin \{j, j+1, j+2\}$, and put $a'_j = -1$, $a'_{j+1} = 0$ and $a'_{j+2} = a_{j+2} + 1$. Furthermore $a_j = -1$, $a_{j+1} = +1$ certainly does not occur, as $-2^j + 2^{j+1} = +2^j$. So an optimal sequence a_i can be chosen to satisfy: $a_j \neq 0$ implies $a_{j-1} = 0 = a_{j+1}$. Let us call this property for a sequence of numbers equal to 0, +1 or −1, the *property O*.

We prove now that O characterizes a_i completely. So let a_i and b_i be sequences consisting of 0, +1, −1 and having the property O, and suppose $x = \sum a_i 2^i = \sum b_i 2^i$. Then $0 = \sum (a_i - b_i) 2^i$. Suppose there exists a lowest index such that $a_i - b_i \neq 0$, say j, then $|a_j - b_j| \leqslant 2$, but as $(a_j - b_j) 2^j = -\sum_{i > j} (a_i - b_i) 2^i$, we cannot have $|a_j - b_j| = 1$. Therefore, we may suppose that $a_j - b_j = 2$, so both a_j and b_j are unequal zero. Hence $a_{j+1} = b_{j+1} = 0$ and we have $2 \cdot 2^j = -\sum_{i \geqslant j+2} (a_i - b_i) 2^i$ which is a contradiction, as the left hand member is not divisible by 2^{j+2} and the right hand member is. We have proved that we can write $x = \sum a_i 2^i$ in a unique way such that the a_i form a sequence with the property O, moreover that $\|x\| = \sum |a_i| \|2^i\|$ if $\|x\| = d_p(x)$ for some sequence p.

The above argument also shows that when $2^n | (x-y)$, then the first $n-1$ coordinates of x and y are the same.

1.4. Cauchy nets in the T_A-topology. Let now $(x_\lambda)_{\lambda \in A}$ be a Cauchy net in the T_A-topology. Suppose it does not converge to an element of Z. It converges in the 2-adic topology to a 2-adic integer $x = \sum a_i 2^i$, where the a_i form a sequence with the property O. Let $\{j_1, j_2, \ldots\}$ be the sequence of indices j for which $a_j \neq 0$, $j_1 < j_2 < j_3 < \ldots$ Let $j_0 = -1$.

Define q_i by $q_i = 1/n$ for $j_{n-1} < i \leqslant j_n$ for all $n \geqslant 1$. Let $m = d_q$, with $q = \{q_i\}_{i \in N}$. As $(x_\lambda)_{\lambda \in \Lambda}$ is a Cauchy net in the m-topology, we may suppose it to be m-bounded, moreover the coordinates of x_λ converge to the coordinates of x. But this gives a contradiction, because if the coordinates of x_λ equal the coordinates of x up to and including the j_nth coordinate, it follows that $m(x_\lambda) \geqslant \sum_{k=1}^{n} 1/k \geqslant \log n$. So $m(x_\lambda)$ is not bounded.

1.5. Characters. Let G be a monothetic group such that the powers of 2 in the embedded copy of Z converge to 0. Then any character that is not continuous in the 2-adic topology of Z is not continuous on G either. This follows from the fact that $2^n a \bmod 1$ is a zero sequence if and only if a is rational with denominator a power of 2. The characters of Z that are continuous with respect to the 2-adic topology are also continuous with respect to the topology constructed in the preceding sections.

This fact and the above observation allows us to conclude: the characters $x \to px/2^n \bmod 1$, $p \in Z$, $n \in N$, are the only continuous characters of Z with the topology T_A. The Bohr compactification of Z with the topology T_A consists of the 2-adic integers. We have proved:

THEOREM 1.6. *There exists a finest group topology on Z such that 2^n converges to zero. Z is complete with respect to this topology. The Bohr compactification of Z with this topology consists of the 2-adic integers.*

2. A totally disconnected monothetic group without continuous characters. We introduce now a subclass $\mathfrak{C}_1$ of $\mathfrak{C}$. $\mathfrak{C}_1$ consists of all those metrics from $\mathfrak{C}$ that satisfy $\|x\| = \sum |a_i| \|2^i\|$ where a_i are the coordinates of x. The reader will observe that $\mathfrak{C}_1$ consists of metrics $\|\cdot\|$ that are maximal among all metrics m in $\mathfrak{C}$ that satisfy $m(2^n) = \|2^n\|$. A 2-adic number whose first coordinate is ± 1 will be called *odd*. The smallest number l such that the coordinates c_i of a number $c \in Z$ are 0 if $i > l$ will be denoted by $l(c)$. Let x be a given 2-adic integer with coordinates c_i. When $c_i = 0$ for $p \leqslant i \leqslant q$ but $c_{p-1} \neq 0$ and $c_{q+1} \neq 0$ then we will say that the interval $[p, q] \subset N$ is a *gap* in the coordinates of x. The number $q-p+1$ will be called the *length* of the gap. The number of non-zero coordinates of $x \in Z$ will be called the *weight* of x. It follows that for any odd $x \in Z$ with $w(x) > 1$ the coordinates of x must have at least one gap $[p, q]$ of length $\geqslant \dfrac{t - w(x)}{w(x) - 1}$, and $q < t$ if the t-th coordinate of x is non-zero ($t > 0$).

Let $x = \sum_{i=0}^{\infty} c_i 2^i$ be a 2-adic integer with coordinates c_i. Then we define

$$P_k x = \sum_{i=0}^{k} c_i 2^i \quad \text{and} \quad P^k x = \sum_{i=k}^{\infty} c_i 2^i, \quad k \geqslant 0 .$$

Observe that $x = P_k x + P^{k+1} x$.

We observe that for each $x \in Z$, we have $|P_{k-1}x| \leqslant \frac{1}{3}(2^k - 1)$ and if the kth coordinate of x is not equal to zero, $|P^k x| \geqslant 2^k$, so $|x| = |P_{k-1}x + P^k x| \geqslant \frac{2}{3}2^k + \frac{1}{3}$, hence $|P_{k-1}x| < \frac{1}{2}|x|$. Moreover, if the kth coordinate is the last non-zero coordinate of x $(k = l(x))$ then $|P^k x| = 2^k$ hence $|x| < \frac{4}{3}2^k$. So $|xy| < (\frac{4}{3})^2 2^{l(x)+l(y)}$. But $|xy| > \frac{2}{3}2^{l(xy)}$ so $l(xy) \leqslant l(x) + l(y) + 1$, for all $x \in Z$ and $y \in Z$.

We are now going to consider a special kind of metric in $\mathfrak{C}_1$, characterized by an infinite sequence

$$M = \{m_0, m_1, \ldots\}, \quad m_0 = 0, \quad m_k > (k+2)(m_{k-1}+3)$$

for all $k > 0$. For given M, we let $\|\cdot\|_M$ be the metric from $\mathfrak{C}_1$ defined by $\|2^m\|_M = \dfrac{1}{(i+1)^2}$ for $m_i \leqslant m < m_{i+1}$, and $i \geqslant 0$.

The completion of Z with respect to $\|\cdot\|_M$ is denoted by Z_M. The 2-adic number $\sum_{m \in M} 2^m$ is denoted by a_M. Clearly $a_M \in Z_M$, $\|a_M\| = \pi^2/6$. We will prove a_M generates a discrete subgroup of Z_M, which we are going to call D_M. First we need a lemma.

LEMMA 2.1. *Let $w(n) \leqslant k$. Then $w(n \cdot P_{m_k} a_M) \geqslant k$, if $n > 0$.*

Proof. We denote for all $l \in N$, $P_{m_l} a_M = a_l$. We prove the statement by induction on k. Clearly the statement is true for $k = 1$ or 2. We may suppose n odd. We distinguish two cases. Case (i) is: all coordinates of n in the interval $[(k+1)(m_{k-1}+3), m_k - 1]$ equal zero, case (ii) is the contrary case: some coordinate in that interval is not equal to zero.

Case (i). Let $p \leqslant (k+1)(m_{k-1}+3)$ and $q \geqslant m_k$ be such that $n = P_{p-1}n + P^q n$. Then

$$l(a_{k-1} \cdot P_{p-1}n) < m_{k-1} + p - 1 + 1 \leqslant m_{k-1} + (k+1)(m_{k-1}+3) < m_k - 2 .$$

* We write $na_k = a_{k-1} \cdot P_{p-1}n + (a_{k-1} \cdot P^q n + 2^{m_k} n)$. As the lowest non-zero coordinate of the bracketed expression has index at least m_k, it follows

$$w(na_k) = w(a_{k-1} \cdot P_{p-1}n) + w(a_{k-1} \cdot P^q n + 2^{m_k} n) .$$

Now

$$|a_{k-1} \cdot P^q n| < \tfrac{4}{3} 2^{m_{k-1}} \cdot \tfrac{1}{2}|n| = \tfrac{2}{3} 2^{m_{k-1}} |n| < |2^{m_k} n| ,$$

hence the second term in the expression for $w(na_k)$ is unequal zero. The first term exceeds or equals $k-1$ by hypothesis, unless $w(P_{p-1}n) = k$, but then $P^q n = 0$ so the second term equals k. *

Case (ii). We conclude there is a gap $[p, q]$ with $q < m_k$ and with length $\geqslant \dfrac{(k+1)(m_{k-1}+3) - k}{k-1} \geqslant m_{k-1} + 3$, hence $q > p + m_{k-1} + 2$; in that case

$$l(a_{k-1}\cdot P_{p-1}n) \leqslant m_{k-1}+p-1+1 < q-2 < m_k-2 .$$

From this point on we can repeat the above part beginning and ending with $*$. The lemma is proved.

LEMMA 2.2. *The subgroup $D_M \subset Z_M$ generated by a_M is discrete.*

Proof. Let n be a given integer and let k be such that $m_{k-1} \leqslant l(n) < m_k$. We use the notation $P_{m_k}a_M = a_k$ and $P^{m_k}a_M = a^k$. Then $na_M = na_k + na^{k+1}$; furthermore $l(na_k) < m_{k+1}-2$, so

$$\|na_M\|_M = \|na_k\|_M + \|na^{k+1}\|_M = \|na_k\|_M + w(n)\sum_{i=k+1}^{\infty}\frac{1}{(i+1)^2}.$$

If $w(n) > k$, then the second term exceeds $\frac{k+1}{k+2}$ and the first term, which is not zero, exceeds $\frac{1}{k+1}$. In this case it follows that $\|na_M\|_M > 1$. If $w(n) \leqslant k$, then by Lemma 2.1, the first term exceeds or equals $\frac{k}{k+1}$ and the second term exceeds $w(n)\frac{1}{k+2} > \frac{1}{k+1}$ except when $n = 2^m$, but then we see immediately that $\|na_M\|_M > 1$. So in any case $\|na_M\| > 1$.

THEOREM 2.3. *There exists a totally disconnected metrizable group G with a dense infinite cyclic subgroup (identified with $\boldsymbol{Z}$) such that:*

(1) *$\{2^n\}$ is a zero sequence;*

(2) *G does not have continuous characters, except* 0.

Proof. Let M be a sequence as above and define $G = Z_M/D_M$. For any continuous character $\chi\colon G \to \boldsymbol{R}/\boldsymbol{Z}$ the composition $\chi\cdot j$, where $j\colon Z_M \to G$ is the canonical quotient map, is a continuous character on Z_M which sends a_M to zero. But continuous characters on Z_M are of the form $x \to px \bmod 2^k$, $p \in \boldsymbol{Z}$, p odd, so k must be equal to 0, so $\chi = 0$.

Remark 2.4. It follows from [7], 41, that the above group is topologically a one-dimensional space. G is locally isomorphic to Z_M, which, according to [7], § 9, is homeomorphic to a closed subgroup of a properly normed infinite dimensional torus.

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