The proofs of (5.1) and (5.2) are quite analogous to the ones of Lemma 3, Theorem 4, and Theorem 5 on p. 54 ff as well as Corollary 1 on p. 53 of [1], and establish the fact that $i_r(X, \varphi, U)$ satisfies the homotopy and additivity axioms. The proof of the counterpart of Theorem 5 in [1, p. 59/60] requires that two usc acyclic multifunctions δ_0 and δ_1 which are related by an usc acyclic homotopy induce the same homomorphism

$$\delta_0^* = \delta_1^*\colon H^n(R^n, R^n \setminus 0) \to H^n\big(V, V \setminus S(\varphi, U)\big).$$

That this is true can be shown as in [3, Theorem 3]. It would be of interest to check (but it is not needed in this paper) to what extent $i_r(X, \varphi, U)$ satisfies other axioms often associated with a fixed point index, and how it extends to more general spaces than polyhedra. Some modifications will arise, e.g. in the commutativity axiom, as the composite of two acyclic multifunctions need not be acyclic.

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Partition topologies for large cardinals

by

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Abstract. Two topologies are introduced on the power set of a large cardinal. Partition theorems in the style of Kleinberg-Shore are obtained for the first topology and ones in the style of Galvin-Prikry for the second.

1. Introduction. Let $\varkappa, \lambda, \nu$ be cardinal numbers and $u, v, s, t \subseteq \varkappa$. $\overline{u}$ is the order type of u and $[s, v]^\lambda = \{u \mid s \subseteq u \subseteq s \cup v \wedge \overline{u} = \lambda\}$. $[s, v]^{<\lambda}$ is defined in the same way except $\overline{u} = \lambda$ in the definition of $[s, v]^\lambda$ is replaced by $\overline{u} < \lambda$. $[\emptyset, v]^\lambda$ will be written as $[v]^\lambda$ where $\emptyset$ is the empty set. We define two topologies on $[\varkappa]^\lambda$ where $\omega \leqslant \lambda \leqslant \varkappa$ and ω is the first infinite cardinal. The *classical topology* (*c-topology*) is generated by a basis consisting of $[s, \varkappa - t]^\lambda$ where $s, t \in [\varkappa]^{<\omega}$. If $\varkappa$ is measurable let $D \subseteq [\varkappa]^\varkappa$ be a $\varkappa$-complete normal ultrafilter. The *measure topology* (*d-topology*) is generated by a basis consisting of $[s, u]^\lambda$ where $s \in [\varkappa]^{<\omega}$ and $u \in D$. When we speak of a topology without the c or d prefix we mean either topology. $S \subseteq [\varkappa]^\lambda$ is *Borel* if it is generated from the open sets by complementation and $< \varkappa$ intersections. It is *meager* if it is the union of $< \varkappa$ nowhere dense sets and is *Baire* if its symmetric difference with an open set is meager. $S \subseteq [\varkappa]^\lambda$ is *Ramsey* if there is a $u \in [\varkappa]^\varkappa$ such that $[u]^\lambda \subseteq S$ or $[u]^\lambda \subseteq [\varkappa]^\lambda - S$. Such a u is called *homogeneous* for S.

THEOREM 1. *If $\varkappa$ is a Ramsey cardinal and $S \subseteq [\varkappa]^\lambda$ is c-Borel then S is Ramsey.*

THEOREM 2. *If $\varkappa$ is a measurable cardinal and $S \subseteq [\varkappa]^\lambda$ is d-Baire then S is Ramsey.*

Our proof of Theorem 1 is based on the work of Kleinberg–Shore [3] and that of Theorem 2 on the work of Galvin–Prikry [2] and of the author [1].

2. Details. Write $u < v$ if every element of u is strictly less than every element of v. $(s, v)^\lambda = \{u \mid s \subseteq u \subseteq s \cup v \wedge \overline{u} = \lambda \wedge s < u - s\}$. $(s, v)^{<\lambda}$ is defined in the same way except $\overline{u} = \lambda$ in the definition of $(s, v)^\lambda$ is

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replaced by $\bar{u} < \lambda$. If $f\colon A \to B$ let $f^*(u) = \{f(x) \mid x \in u \cap A\}$ and $|u| =$ the cardinal of u. $\varkappa$ is a *Ramsey cardinal* if

$$(1) \qquad \nu < \varkappa \wedge f\colon [\varkappa]^{<\omega} \to \nu \text{ implies } (\exists u \in [\varkappa]^{\varkappa})(\forall n < \omega)\,|f^*([u]^n)| = 1\,.$$

Such a u is called *homogeneous* for f. Until further notice assume that $\varkappa$ is a Ramsey cardinal and $S \subseteq [\varkappa]^\lambda$. As in [3] we say that S is *regular* if there is a $\nu < \varkappa$ and an $f\colon [\varkappa]^{<\omega} \to \nu$ such that every $u \in [\varkappa]^{\varkappa}$ which is homogeneous for f is also homogeneous for S.

Lemma 1. *Every c-open set is regular.*

Proof. Assume S is c-open and define $f\colon [\varkappa]^{<\omega} \to 2$ by $f(s) = 0$ if $(s, \varkappa)^\lambda \subseteq S$ and $f(s) = 1$ otherwise. Let $u \in [\varkappa]^{\varkappa}$ be homogeneous for f. Case 1). There is an $n < \omega$ such that $f^*([u]^n) = \{0\}$. If $v \in [u]^\lambda$ let s be the first n elements of v (recall that $\omega \leqslant \lambda$). Then $v \in (s, \varkappa)^\lambda \subseteq S$ and hence $[u]^\lambda \subseteq S$. Case 2). $(\forall n < \omega)\, f^*([u]^n) = \{1\}$. We claim that $[u]^\lambda \subseteq [\varkappa]^\lambda - S$. If not there is a $v \in [u]^\lambda \cap S$. Since S is c-open there are $s, t \in [\varkappa]^{<\omega}$ such that $v \in [s, \varkappa - t]^\lambda \subseteq S$. Let $\alpha \in u$ be an element larger than every element of $s \cup t$. Then $(s \cup \{\alpha\}, \varkappa)^\lambda \subseteq [s, \varkappa - t]^\lambda \subseteq S$ and hence $f(s \cup \{\alpha\}) = 0$. But $s \cup \{\alpha\} \subseteq u$. Contradiction. Q.E.D.

Lemma 2. *The complement of a regular set is regular.*

Lemma 3. *The intersection of $< \varkappa$ regular sets is regular.*

Proof. Let $(S_\alpha \mid \alpha < \gamma)$ be a sequence of regular sets where $\gamma < \varkappa$. For each $\alpha < \gamma$ there is a $\nu_\alpha < \varkappa$ and $f_\alpha\colon [\varkappa]^{<\omega} \to \nu_\alpha$ such that every $u \in [\varkappa]^{\varkappa}$ which is homogeneous for f_α is also homogeneous for S_α. Define $g\colon [\varkappa]^{<\omega} \to \Pi_\alpha \nu_\alpha$ (the direct product of the ν_α) by letting the αth component of $g(s)$ be $f_\alpha(s)$. Note that $|\Pi_\alpha \nu_\alpha| < \varkappa$ because $\varkappa$ is strongly inaccessible and that any $u \in [\varkappa]^{\varkappa}$ which is homogeneous for g is simultaneously homogeneous for each S_α. It readily follows that any such u is homogeneous for $\bigcap_\alpha S_\alpha$. Q.E.D.

Proof of Theorem 1. Every c-Borel set is regular and the partition property (1) immediately gives a $u \in [\varkappa]^{\varkappa}$ which is homogeneous for S. Q.E.D.

It is possible to somewhat strengthen Theorem 1. $f\colon [\varkappa]^\lambda \to \nu$ is a *Borel function* if $f^{-1}(\alpha)$ is Borel for each $\alpha < \nu$. If $u \in [\varkappa]^{\varkappa}$ and $|f^*([u]^\lambda)| = 1$ then u is *homogeneous* for f. Finally f is *regular* if there is $\nu' < \varkappa$ and a $g\colon [\varkappa]^{<\omega} \to \nu'$ such that every $u \in [\varkappa]^{\varkappa}$ which is homogeneous for g is also homogeneous for f.

Lemma 4. *If $\nu < \varkappa$ and $f\colon [\varkappa]^\lambda \to \nu$ is c-Borel then f is regular.*

Proof. $f^{-1}(\alpha)$ is c-Borel and hence regular for each $\alpha < \nu$. Hence there are $\nu_\alpha < \varkappa$ and $f_\alpha\colon [\varkappa]^{<\omega} \to \nu_\alpha$ such that every $u \in [\varkappa]^{\varkappa}$ which is homogeneous for f_α is also homogeneous for $f^{-1}(\alpha)$. Define g as in the proof of Lemma 3. Then just as in that proof it follows that any $u \in [\varkappa]^{\varkappa}$ which is homogeneous for g is also homogeneous for f. Q.E.D.

Corollary 1. *If $\varkappa$ is a Ramsey cardinal, $\omega \leqslant \lambda \leqslant \varkappa$, $\nu < \varkappa$, $f\colon [\varkappa]^\lambda \to \nu$ is a c-Borel function, $s \in [\varkappa]^{<\omega}$ and $u \in [\varkappa]^{\varkappa}$ then there is a $v \in [u]^{\varkappa}$ such that $|f^*([s, v]^\lambda)| = 1$.*

Proof. Define g on $[\varkappa]^\lambda$ by $g(v) = s \cup v$. Let $\gamma < \varkappa$ be greater than every element of s and let h be a strictly increasing function mapping $\varkappa$ onto $u - \gamma$. Then $g \circ h^*\colon [\varkappa]^\lambda \to [\varkappa]^\lambda$ is c-continuous making $f \circ g \circ h^*$ c-Borel. By Lemma 4 there is a $v' \in [\varkappa]^{\varkappa}$ which is homogeneous for $f \circ g \circ h^*$. Then $v = h^*(v')$ is easily seen to satisfy the corollary. Q.E.D.

Now assume that $\varkappa$ is a measurable cardinal and $D \subseteq [\varkappa]^{\varkappa}$ is a $\varkappa$-complete normal ultrafilter. The normality of D allows us to prove

$$(2) \qquad \text{if } u_\gamma \in D \text{ for } \gamma < \varkappa \text{ then } \{\alpha < \varkappa \mid \alpha \in \bigcap_{\gamma < \alpha} u_\gamma\} \in D,$$

$$(3) \qquad \nu < \varkappa \wedge f\colon [\varkappa]^{<\omega} \to \nu \text{ implies } (\exists u \in D)(\forall n < \omega)\,|f^*([u]^n)| = 1.$$

Proofs of these results as well as a comprehensive discussion about measurable cardinals can be found in [4]. For the following definitions let $S \subseteq [\varkappa]^\lambda$ be fixed. If $s \in [\varkappa]^{<\omega}$ and $u \in D$ then u *accepts* s if $(s, u)^\lambda \subseteq S$ and u *rejects* s if for no $v \in [u]^{\varkappa} \cap D$ does v accept s (cf. [2]).

Lemma 5 (cf. [2]). (i) *u accepts (rejects) s if and only if $\{\alpha \in u \mid s < \{\alpha\}\}$ accepts (rejects) s.* (ii) *If u accepts (rejects) s then so does every $v \in [u]^{\varkappa} \cap D$.* (iii) *For any $s \in [\varkappa]^{<\omega}$ and $u \in D$ there is a $v \in [u]^{\varkappa} \cap D$ which either accepts or rejects s.*

Lemma 6. *There is a $u \in D$ which either accepts or rejects every $s \in [\varkappa]^{<\omega}$.*

Proof. By Lemma 5 and the $\varkappa$-completeness of D we can find for each $\gamma < \varkappa$ a $u_\gamma \in D$ which accepts or rejects every $s \in [\gamma+1]^{<\omega}$. Let $u = \{\alpha < \varkappa \mid \alpha \in \bigcap_{\gamma \in \alpha} u_\gamma\}$. By (2) $u \in D$. Let $s \in [\varkappa]^{<\omega}$ have γ as its largest element. Then u_γ accepts or rejects s and if $\alpha \in u$ and $\alpha > \gamma$ then $\alpha \in u_\gamma$. Hence u accepts or rejects s. Q.E.D.

Lemma 7. *There is a $u \in D$ such that if u rejects $\emptyset$ then u rejects every $s \in [u]^{<\omega}$.*

Proof. Define three sets contained in $[\varkappa]^{<\omega}$ as follows.

$$A = \{s \in [\varkappa]^{<\omega} \mid (\exists v \in D)\ v \text{ accepts } s\}\,,$$

$$B = \{s \in [\varkappa]^{<\omega} \mid (\exists v \in D)\ v \text{ rejects } s\}\,,$$

$$C = [\varkappa]^{<\omega} - (A \cup B)\,.$$

If $s \in A \cap B$ then there exist $v, v' \in D$ such that v accepts s and v' rejects s. Then $v \cap v' \in D$ and by Lemma 5 both accepts and rejects s. Contradiction. Thus $A \cap B = \emptyset$ so (A, B, C) is a partition of $[\varkappa]^{<\omega}$. By (3) we can find a $u \in D$ which is homogeneous for this partition, i.e., $(\forall n < \omega)[u]^n$

$\subseteq A$ or $[u]^n \subseteq B$ or $[u]^n \subseteq C$. Since D is closed under intersection we may assume by Lemma 6 that u accepts or rejects each $s \in [\varkappa]^{<\omega}$. Now suppose that u rejects $\emptyset$ but does not reject some $s \in [u]^n$ where $n < \omega$. Then u accepts s and $s \in A$. If $t \in [u]^n$ and u does not accept t then u rejects t and $t \in B$. But u is homogeneous for (A, B, C). Contradiction. Hence u accepts t, i.e., u accepts every $s \in [u]^n$. If $v \in (\emptyset, u)^\lambda$ let s be the first n elements of v. Then $s \in [u]^n$ so $v \in (s, u)^\lambda \subseteq S$, i.e., $(\emptyset, u)^\lambda \subseteq S$. But then u accepts $\emptyset$. Contradiction. Q.E.D.

LEMMA 8. *If S is d-open then there is a $u \in D$ such that $[u]^\lambda \subseteq S$ or $[u]^\lambda \subseteq [\varkappa]^\lambda - S$.*

Proof. Let $u \in D$ satisfy Lemmas 6 and 7. If u accepts $\emptyset$ then $[u]^\lambda \subseteq S$ and we are done. Otherwise u rejects every $s \in [u]^{<\omega}$. We claim that $[u]^\lambda \cap S = \emptyset$. If not then there is a $v' \in [u]^\lambda \cap S$. Since S is d-open there must be an $s \in [\varkappa]^{<\omega}$ and a $v \in D$ such that $v' \in [s, v]^\lambda \subseteq S$. But then $s \subseteq u$ and $(s, v)^\lambda \subseteq S$ so $u \cap v$ accepts s and hence u cannot reject s. Contradiction. Q.E.D.

LEMMA 9. *If S is d-meager then there is a $u \in D$ such that $[u]^\lambda \subseteq [\varkappa]^\lambda - S$.*

Proof. If S is d-nowhere dense then so is its closure $\bar{S}$. By Lemma 8 there is a $u \in D$ such that $[u]^\lambda \subseteq \bar{S}$ or $[u]^\lambda \subseteq [\varkappa]^\lambda - \bar{S}$. Since the former cannot occur we have $[u]^\lambda \subseteq [\varkappa]^\lambda - S$. Now let $(S_\alpha | \ \alpha < \gamma)$ be a sequence of d-nowhere dense sets of length $\gamma < \varkappa$ whose union is S. For each $\alpha < \gamma$ choose $u_\alpha \in D$ so that $[u_\alpha]^\lambda \subseteq [\varkappa]^\lambda - S_\alpha$. If $u = \bigcap_{\alpha<\gamma} u_\alpha$ then we clearly have $u \in D$ and $[u]^\lambda \subset [\varkappa]^\lambda - S$. Q.E.D.

LEMMA 10. *If S is d-Baire then there is a $u \in D$ such that $[u]^\lambda \subseteq S$ or $[u]^\lambda \subseteq [\varkappa]^\lambda - S$.*

Proof. Let $S = S_0 \Delta S_1$ where S_0 is d-meager, S_1 is d-open, and Δ is symmetric difference. By Lemma 9 we can find a $u \in D$ such that $[u]^\lambda \subseteq [\varkappa]^\lambda - S_0$ and by Lemma 8 a $v \in D$ such that $[v]^\lambda \subseteq S_1$ or $[v]^\lambda \subseteq [\varkappa]^\lambda - S_1$. If the former occurs then $[u \cap v]^\lambda \subseteq S$ while in the latter case $[u \cap v]^\lambda \subseteq [\varkappa]^\lambda - S$. Our proof is complete since $u \cap v \in D$. Q.E.D.

Proof of Theorem 2 is now an immediate consequence of Lemma 10 since $D \subseteq [\varkappa]^\varkappa$. Q.E.D.

This result may be strengthened as in Corollary 1. f: $[\varkappa]^\lambda \to \nu$ is a *Baire function* if $f^{-1}(\alpha)$ is Baire for each $\alpha < \nu$.

COROLLARY 2. *If $\varkappa$ is a measurable cardinal, $D \subseteq [\varkappa]^\varkappa$ is a $\varkappa$-complete normal ultrafilter and the d-topology is defined as above, $\omega \leqslant \lambda \leqslant \varkappa$, $\nu < \varkappa$, f: $[\varkappa]^\lambda \to \nu$ is a d-Baire function, $s \in [\varkappa]^{<\omega}$ and $u \in D$ then there is a $v \in [u]^\varkappa \cap D$ such that $|f^*([s, v]^\lambda)| = 1$.*

Proof. Define g on $[\varkappa]^\lambda$ by $g(v) = s \cup v$. We easily see that g is d-continuous so that $f \circ g$ is d-Baire. Now $(f \circ g)^{-1}(\alpha)$ is d-Baire for each $\alpha < \nu$ so by Lemma 10 we may choose a $v_\alpha \in D$ so that $[v_\alpha]^\lambda \subseteq (f \circ g)^{-1}(\alpha)$ or $[v_\alpha]^\lambda \subseteq [\varkappa]^\lambda - (f \circ g)^{-1}(\alpha)$. Then $v = \bigcap_{\alpha<\nu} v_\alpha \in D$ and is homogeneous for $f \circ g$. It then follows that $|f^*([s, v]^\lambda)| = 1$. $u \cap v$ satisfies our corollary. Q.E.D.

THEOREM 3. *S is d-Baire if and only if for each $s \in [\varkappa]^{<\omega}$ and $u \in D$ there is a $v \in [u]^\varkappa \cap D$ such that $[s, v]^\lambda \subseteq S$ or $[s, v]^\lambda \subseteq [\varkappa]^\lambda - S$* (cf. [1]).

Proof. If S is d-Baire then so is its characteristic function. Corollary 2 then gives us a $v \in [u]^\varkappa \cap D$ such that $[s, v]^\lambda \subseteq S$ or $[s, v]^\lambda \subseteq [\varkappa]^\lambda - S$. Conversely if for every $s \in [\varkappa]^{<\omega}$ and $u \in D$ there is a $v \in [u]^\varkappa \cap D$ such that $[s, v]^\lambda \subseteq S$ or $[s, v]^\lambda \subseteq [\varkappa]^\lambda - S$ then S minus its interior is d-nowhere dense making S d-Baire. Q.E.D.

We conclude with a word as to why in our definitions of Borel (Baire) we used $< \varkappa$ unions rather than $\leqslant \varkappa$ unions. At first sight the latter seems to be the correct generalization of the case where $\varkappa = \gamma = \omega$; however the following facts were pointed out in [3]. Let $\varkappa$ be strongly inaccessible and $\lambda = \omega$. Then $[\varkappa]^\lambda$ has cardinality $\varkappa$. Since every $u \in [\varkappa]^\lambda$ is the intersection of $\varkappa$ c-open sets (neighborhoods in fact) this implies that every $S \subseteq [\varkappa]^\lambda$ would belong to the class of sets generated from the c-open sets by complementation and $\leqslant \varkappa$ intersections. Then we could not hope to obtain Theorem 1 because (as stated in [3]) every $S \subseteq [\varkappa]^\lambda$ is Ramsey violates the axiom of choice.

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