Partition topologies for large cardinals

by

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Abstract. Two topologies are introduced on the power set of a large cardinal.
Partition theorems in the style of Kleiheberg-Shore are obtained for the first
topology and most in the style of Galvin-Priddy for the second.

1. Introduction. Let $\kappa$, $\lambda$, and $\nu$ be cardinal numbers and $a, b, c, d \subseteq \kappa$.
$c$ is the order type of $c$ and $[a, b] = \{a \leq c \leq b \}$ and $\nu = \{b \in \kappa : c \leq \nu \nu \}$. $[a, b]^c$ is
defined in the same way except $c = \lambda$ in the definition of $[a, b]^c$, replaced
by $\nu < \kappa$. $[\nu, \kappa]^c$ will be written $[\nu]^c$ where $\nu$ is the empty set.
We define two topologies on $[\nu]^c$ where $\alpha < \lambda < \kappa$ and $\omega$ is the first infinite
continuous. The classical topology (c-topology) is generated by a basis consisting
of $[\alpha, \beta]$ where $\alpha, \beta \subseteq [\nu]^c$. If $x$ is measurable let $D \subseteq [\nu]^c$ be a $\nu$-complete
$[\nu]^c$ is a measure topology if it is generated from the open sets by complementation
and $< \kappa$ intersections. It is meager if it is the union of $< \kappa$ nowhere dense
sets and is Baire if its symmetric difference with an open set is meager.
$[\nu]^c$ is Ramsey if there is a $\alpha \subseteq [\nu]^c$ such that $[a, b] \subseteq S$ or $[a, b] \subseteq [\nu]^c - S$.
Such a $\alpha$ is called homogeneous for $S$.

THEOREM 1. If $x$ is a Ramsey cardinal and $S \subseteq [\nu]^c$ is c-Borel then $S$ is Ramsey.

THEOREM 2. If $x$ is a measurable cardinal and $S \subseteq [\nu]^c$ is d-Baire then $S$ is Ramsey.

Our proof of Theorem 1 is based on the work of Kleiheberg-Shore [3] and that of
Theorem 2 on the work of Galvin-Priddy [2] and of the author [1].

2. Details. Write $a < b$ if every element of $a$ is strictly less than
$\kappa$ and every element of $b$. $[a, b]^c = \{a \subseteq b \subseteq \kappa : \lambda \in a, \lambda < b - \lambda\}$
is defined in the same way except $a = \lambda$ in the definition of $[a, b]^c$ is

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replaced by \( u < \lambda \). If \( f: \alpha \to \aleph_1 \) let \( f''(u) = f'(u) \) if \( u \in A \) and \( |u| = \text{the cardinal of } u. \) \( u \in \text{a Ramsey cardinal if } \).

\[ (1) \quad u < \kappa \wedge f: [x]^{<\kappa} \to y \implies \left( \forall u < \kappa \right) \left( \forall v < \omega \right) \left| f''(u)^v \right| = \lambda . \]

Such a \( u \) is called homogeneous for \( f \). For further notice assume that \( u \) is a Ramsey cardinal and \( S \subseteq [x]^\kappa \). As in [3] we say that \( S \) is regular if there is a \( u < \kappa \) and an \( f: [x]^{<\kappa} \to y \) such that every \( u \in [x]^\kappa \) which is homogeneous for \( f \) is also homogeneous \( S \).

**Lemma 1.** Every \( e \)-open set is regular.

**Proof.** Assume \( S \) is \( e \)-open and define \( f: [x]^{<\kappa} \to 2 \) by \( f(s) = \emptyset \) if \( (s, \gamma) \subseteq S \) and \( f(s) = 1 \) otherwise. Let \( u \in [x]^\kappa \) be homogeneous for \( f \). Case 1. There is a \( n < \omega \) such that \( f''(u) = \emptyset \). If \( v \in [u]^\kappa \) let \( s \) be the first \( n \) elements of \( v \) (recall that \( u < \lambda \)). Then \( v \in (s, \gamma) \subseteq S \) and hence \( [u]^\kappa \subseteq S \). Case 2. \( (\forall u < \omega) f''(u)^v = (1) \). We claim that \( [x]^{<\kappa} \subseteq S \). If not there is a \( u \in [x]^\kappa \) such that \( v \in (s, \gamma) \subseteq S \). Since \( S \) is \( e \)-open there are \( s, t \in [x]^{<\kappa} \) such that \( v \in (s, \gamma) \subseteq S \). Let \( a, b \) be an element larger than every element of \( v \). Then \( (s, a) \cup (b, \gamma) \subseteq (s, \gamma) \subseteq S \) and hence \( s \cup \{a\} \in \omega \). Contradiction. Q.E.D.

**Lemma 2.** The complement of a regular set is regular.

**Lemma 3.** The intersection of \( \kappa \)-regular sets is regular.

**Proof.** Let \( \langle S_i \mid a < \gamma \rangle \) be a sequence of regular sets where \( \gamma < \kappa \). For each \( a < \gamma \) there is a \( v_a < \kappa \) and \( f_a: [x]^{<\kappa} \to 2 \) such that every \( u \in [x]^\kappa \) which is homogeneous for \( f_a \) is also homogeneous for \( S_a \). Define \( g: [x]^{<\kappa} \to \bigcap S_a \). (The direct product of the \( v_a \)) by letting the \( a \)-the component \( g(a) \) be \( f_a(s) \). Note that \( \bigcap S_a \) is a regular element and that any \( u \in [x]^\kappa \) which is homogeneous for \( g \) is simultaneously homogeneous for each \( S_a \). It readily follows that any \( u \in [x]^\kappa \) is homogeneous for \( \bigcap S_a \). Q.E.D.

**Proof of Theorem 1.** Every \( e \)-Borel set is regular and the partition property (1) immediately gives \( u \in \bigcap S_a \) which is homogeneous for \( S \). Q.E.D.

It is possible to somewhat strengthen Theorem 1: \( f: [x]^{<\kappa} \to y \) is a Borel function if \( f''(a) \) is Borel for each \( a < \nu \). If \( u \in [x]^\kappa \) and \( f''(u)^v = (1) \) then \( u \) is homogeneous for \( f \). Finally \( f \) is regular if there is a \( \nu' < \kappa \) and \( g: [x]^{<\kappa} \to \nu' \) such that every \( u \in [x]^\kappa \) which is homogeneous for \( g \) is also homogeneous for \( f \).

**Lemma 4.** If \( u < \kappa \) and \( f: [x]^{<\kappa} \to y \) is Borel then \( f(u) \) is regular.

**Proof.** \( f''(a) \) is Borel and hence regular for each \( a < \nu \). Hence there are \( v_a < \kappa \) and \( f_a: [x]^{<\kappa} \to v_a \) such that every \( u \in [x]^\kappa \) which is homogeneous for \( f_a \) is also homogeneous for \( f''(a) \). Define \( g \) as in the proof of Lemma 3. Then just as in that proof it follows that any \( u \in [x]^\kappa \) which is homogeneous for \( g \) is also homogeneous for \( f \). Q.E.D.

**Corollary 1.** If \( \alpha \) is a Ramsey cardinal, \( \omega < \lambda < \alpha \), \( \gamma < \kappa \), \( f: [x]^{<\kappa} \to y \) is a Borel function, \( u < [x]^{<\kappa} \) and \( u \in [x]^{<\kappa} \) then there is an \( \alpha \in [x]^{<\kappa} \) such that \( |f''(\alpha)^u| = 1 \).

**Proof.** Define \( g \) on \( [x]^{<\kappa} \) by \( g(v) = s \cup \nu \). Let \( \gamma < \kappa \) be greater than every element of \( s \) and let \( h \) be a strictly increasing function mapping \( \gamma \) onto \( u \). Then \( g \cdot h : [x]^{<\kappa} \to [x]^{<\kappa} \) is \( e \)-continuous making \( f \cdot g \cdot h \) \( e \)-Borel. By Lemma 4 there is an \( \nu' \in [x]^{<\kappa} \) which is homogeneous for \( f \cdot g \cdot h \). Then \( \nu' \in h''(u) \) is easily seen to satisfy the corollary. Q.E.D.

Now assume that \( \alpha \) is a measurable cardinal and \( D \subseteq [x]^{<\kappa} \) is a \( \kappa \)-complete normal ultralfilter. The normality of \( D \) allows us to prove

(2) If \( u \in D \) for \( \gamma < \kappa \) then \( (s, a) \not\in \bigcap D \).

(3) \( u \in [x]^{<\kappa} \to \gamma \) implies \( (\forall u \in D) \left( \forall u < \omega \right) \left| f''(u)^v \right| = 1 \).

Proofs of these results as well as a comprehensive discussion about measurable cardinals can be found in [4]. For the following definitions let \( S \subseteq [x]^{<\kappa} \) be fixed. If \( s \in [x]^{<\kappa} \) and \( u \in D \) then \( u \) accepts \( s \) if \( (s, a) \subseteq S \) and \( u \) rejects \( s \) if for no \( v \in [x]^{<\kappa} \) \( D \) does \( v \) accept \( s \) (cf. [2]).

**Lemma 5** (cf. [2]), (i) \( u \) accepts (rejects) \( s \) if and only if \( (a \in \bigcap u) \) (accepts) \( s \). (ii) \( u \) accepts \( (\forall s) \) \( \gamma \) then \( \gamma \in \bigcap D \). (iii) For any \( s \in [x]^{<\kappa} \) and \( u \in D \) there is a \( v \in [x]^{<\kappa} \) \( D \) which either accepts or rejects \( s \).

**Lemma 6.** There is a \( u \in D \) which either accepts or rejects every \( s \in [x]^{<\kappa} \).

**Proof.** By Lemma 5 and the \( \kappa \)-completeness of \( D \) for each \( \gamma < \kappa \) \( u \in D \) accepts or rejects every \( s \in [x]^{<\kappa} \) \( \gamma \). Let \( u = \langle a < \kappa \mid (a \in \bigcap u) \rangle \). By (2) \( u \in D \). Let \( s \in [x]^{<\kappa} \) have \( \gamma \) as its largest element. Then \( u \) accepts or rejects \( s \) and if \( a \in u \) and \( a > \gamma \) then \( a \in u \). Hence \( u \) accepts or rejects \( s \). Q.E.D.

**Lemma 7.** There is an \( u \in D \) such that if \( u \) rejects \( \emptyset \) then \( u \) rejects every \( s \in [x]^{<\kappa} \).

**Proof.** Define three sets contained in \( [x]^{<\kappa} \) as follows.

\[ A := \{ s \in [x]^{<\kappa} \mid (\forall v \in D) \text{ accepts } s \} \]
\[ B := \{ s \in [x]^{<\kappa} \mid (\forall v \in D) \text{ rejects } s \} \]
\[ C := [x]^{<\kappa} - (A \cup B) \]

If \( s \in A \cup B \) then there exist \( \gamma \) \( \rho \in D \) such that \( s \) is rejected \( \gamma \) \( \rho \) and \( \rho \). Then \( \nu \gamma \in D \) and by Lemma 5 both accepts and rejects \( s \). Contradiction. Thus \( A \cap B = \emptyset \) so \( (A, B, C) \) is a partition of \( [x]^{<\kappa} \). By (3) we can find an \( u \in D \) which is homogeneous for this partition, i.e., \( \left( \forall v < \omega \right) \left| f''(u)^v \right| = 1 \).
Lemma 8. If \( S \) is \( d \)-open then there is a \( u \subseteq D \) such that \((u) [\subseteq \mathcal{S} \) or \([u] [\subseteq \mathcal{S} \) for \( S \).

Proof. Let \( u \subseteq D \) satisfy Lemmas 6 and 7. If \( u \neq \emptyset \) then \([u] [\subseteq \mathcal{S} \) and we are done. Otherwise \( u \) does not contain every \( x \in [\mathcal{S} \). We claim that \([u] [\subseteq \mathcal{S} \) for \( S \). If not then there is a \( \psi \in [\mathcal{S} \) such that \( \psi \in [\mathcal{S} \) for \( S \). Since \( S \) is \( d \)-open there must be an \( x \in [\mathcal{S} \) and \( \psi \in [\mathcal{S} \) for \( S \). But then \( \psi \subseteq u \) and \( [\psi] [\subseteq \mathcal{S} \) so \( u \) and \( [\psi] [\subseteq \mathcal{S} \) for \( S \) contradict \( u \). Contradiction. Q.E.D.

Lemma 9. If \( S \) is \( d \)-meager then there is a \( u \subseteq D \) such that \((u) [\subseteq \mathcal{S} \) and \( [u] [\subseteq \mathcal{S} \) for \( S \).

Proof. If \( S \) is \( d \)-nowhere dense then so is its closure \( \overline{S} \). By Lemma 8 there is a \( u \subseteq D \) such that \((u) [\subseteq \mathcal{S} \) and \( [u] [\subseteq \mathcal{S} \). Since the former cannot occur we have \((u) [\subseteq \mathcal{S} \) and \( [u] [\subseteq \mathcal{S} \). Now let \( \mathcal{S} \) and \( \mathcal{S} \) be a sequence of \( d \)-nowhere dense sets of length \( \gamma \) such that \( \mathcal{S} \) and \( \mathcal{S} \) are \( S \). For each \( \gamma \) choose \( u \subseteq D \) so that \((u) [\subseteq \mathcal{S} \) and \( [u] [\subseteq \mathcal{S} \). If \( u = \bigcap_{\gamma} u \gamma \), then we clearly have \( u \subseteq D \) and \( [u] [\subseteq \mathcal{S} \) for \( S \). Q.E.D.

Lemma 10. If \( S \) is \( d \)-Baire then there is a \( u \subseteq D \) such that \((u) [\subseteq \mathcal{S} \) or \([u] [\subseteq \mathcal{S} \) for \( S \).

Proof. Let \( S = S_\alpha \Delta S_\beta \), where \( S_\alpha \) is \( d \)-meager, \( S_\beta \) is \( d \)-open, and \( \Delta \) is symmetric difference. By Lemma 9 we can find \( u \subseteq D \) such that \((u) [\subseteq \mathcal{S} \) or \([u] [\subseteq \mathcal{S} \) for \( S \). If the former occurs then \( [u \cup v] [\subseteq \mathcal{S} \) while in the latter case \( [u \cup v] [\subseteq \mathcal{S} \) for \( S \). Our proof is complete since \( u \cup v \subseteq D \). Q.E.D.

Proof of Theorem 2 is now an immediate consequence of Lemma 10 since \( D \subseteq [\mathcal{S} \). Q.E.D.

This result may be strengthened as in Corollary 1. If \( f : [x] \rightarrow [y] \) is a Baire function, then \( f^{-1}(a) \) is Baire for each \( a \subseteq [y] \).

Corollary 2. If \( f \) is a measurable cardinal, \( D \subseteq [\mathcal{S} \) is a \( v \)-complete normal ultrafilter and the \( d \)-topology is defined as above, then \( \lambda \leq \kappa \), \( \lambda = \kappa \), \( \nu < \kappa \), \( f : [x] \rightarrow [y] \) is a \( d \)-Baire function, \( s \in [\mathcal{S} \) and \( u \subseteq D \) such that \( f^{-1}(s) \subseteq [\mathcal{S} \).

Proof. Define \( g \) on \( [x] \) by \( g(s) = s \cup v \). We easily see that \( g \) is \( d \)-continuous, so that \( f \circ g \) is \( d \)-Baire. Now \( (f \circ g)^{-1}(a) \) is \( d \)-Baire for each \( a \subset [y] \) so by Lemma 10 we may choose a \( \mu \subseteq D \) so that \([\mu] [\subseteq \mathcal{S} \) for \( \mathcal{S} \).

References


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