A Nielsen number for fixed points and near points of small multifunctions

by

Helga Schirmer * (Ottawa)

Abstract. A Nielsen number $N(\varphi)$ is defined for usc multifunctions $\varphi\colon |K| \to |K|$ on a polyhedron $|K|$ if φ is small, i.e. if each $\varphi(x)$ is contained in the star of a vertex of $|K|$. We show that, for suitable $\varepsilon > 0$, the set $S(\varphi, \varepsilon) = \{x \in |K| \mid d(x, \varphi(x)) \leqslant \varepsilon\}$ of ε-near points of φ has at least $N(\varphi)$ path-components, and construct within the small homotopy class of φ functions ψ with exactly $N(\psi)$ path-components of $S(\psi, \varepsilon)$. For acyclic multifunctions $S(\varphi, \varepsilon)$ can be replaced by the fixed point set $S(\varphi) = S(\varphi, 0)$ in these statements. The proof in this case uses a fixed point index for acyclic usc multifunctions.

1. Fixed points and ε-near points of multifunctions. Self-maps of spaces with a minimal set of fixed points have been studied for some time. Nielsen's work [4], concerned with fixed points on surfaces, is now almost 50 years old. The Nielsen number $N(f)$ provides a lower bound on the number of fixed points of maps in the homotopy class of f. Self-maps of certain polyhedra with exactly $N(f)$ fixed points were constructed by Wecken [9], and on more general polyhedra by Shi [6]. The easiest introduction to current results is found in R. F. Brown's book [1], which we shall frequently use as reference.

We attempt here to extend some of these results from single-valued to multi-valued functions. A Nielsen number $N(\varphi)$ for multifunctions $\varphi\colon |K| \to |K|$ on a connected polyhedron is defined in § 2 if φ is small in the sense of [5]. It equals the usual one if φ is single-valued, and is invariant under small homotopies. As a small multifunction is often fixed point free, we relate $N(\varphi)$ in § 3 to minimal sets of ε-near points rather than fixed points. If φ is acyclic, then its fixed point behaviour resembles more closely that of single-valued mappings, and we therefore study minimal fixed point sets of acyclic small multifunctions in § 4. The fixed point sets and ε-near point sets of multifunctions can usually not be expected to consist of isolated points, hence their path-components

* This research was partially supported by the National Research Council of Canada (Grant A 7579).

rather than single points are investigated. All proofs rely heavily on the existence of a single-valued simplicial map approximating a small multifunction which was established in [5]. In the case of acyclic multifunctions we also make use of a fixed point index for such multifunctions. The definition and necessary properties of such an index are sketched in an appendix.

In order to ensure that fixed point sets are closed, and to be able to apply the results about small multifunctions from [5], we only consider usc (*upper semi-continuous*) multifunctions $\varphi\colon X \to Y$, which means that for every open set V in Y with $\varphi(x) \subset V$ there exists an open neighbourhood U of x such that $\varphi(U) \subset V$. We will also assume, without explicitly stating it later on, that all usc multifunctions are *point-closed*, i.e. that each $\varphi(x)$ is closed in Y. By a map we always mean a single-valued continuous function.

It is well-known that an usc multifunction $\varphi\colon X \to X$ need not have a fixed point even if X has the fixed point property. A typical example is the multifunction $\varphi\colon I \to I$ on the unit interval $I = [0, 1]$ given, for any $0 < k \leqslant 1$, by

$$\varphi(x) = \begin{cases} \frac{1}{2}(1+k) & \text{if} \quad x < \frac{1}{2}, \\ \{\frac{1}{2}(1-k), \frac{1}{2}(1+k)\} & \text{if} \quad x = \frac{1}{2}, \\ \frac{1}{2}(1-k) & \text{if} \quad x > \frac{1}{2}. \end{cases}$$

Obviously its *fixed point set* $S(\varphi) = \{x \in X \mid x \in \varphi(x)\}$ is empty. But we notice that φ comes close to having a fixed point if k is very small, and we shall find this idea worth pursuing. So we shall study, for multifunctions $\varphi\colon X \to X$ on a space with a metric d, the *ε-near point set* $S(\varphi, \varepsilon) = \{x \in X \mid d(x, \varphi(x)) \leqslant \varepsilon\}$. ($\varepsilon$-near points have also been called, by various authors, ε-invariant points or ε-fixed points; see e.g. [8, p. 98 ff.] and the literature mentioned there.) Obviously the structure of $S(\varphi, \varepsilon)$ depends on the choice of ε. We have $S(\varphi, 0) = S(\varphi)$, and in our example $S(\varphi, \varepsilon)$ is empy, consists of one point or of a closed interval if $\varepsilon < \frac{1}{2}k$, $\varepsilon = \frac{1}{2}k$, $\varepsilon > \frac{1}{2}k$ respectively, with k equal to the supremum of the diameters $\operatorname{diam}\varphi(x)$. Thus the example illustrates two properties of $S(\varphi, \varepsilon)$.

PROPOSITION 1.1. *If (X, d) is a metric space, $\varepsilon \geqslant 0$ arbitrary, and $\varphi\colon X \to X$ usc, then $S(\varphi, \varepsilon)$ is closed.*

The proof is straightforward. Now let $\operatorname{Int} A$ denote the interior of A.

PROPOSITION 1.2. *Let (X, d) be a metric space, $\varphi\colon X \to X$ usc, and k the supremum of $\operatorname{diam}\varphi(x)$ for all $x \in X$. If there exists an $\varepsilon' \geqslant 0$ such that $S(\varphi, \varepsilon') \neq \emptyset$, then $\operatorname{Int} S(\varphi, \varepsilon) \neq \emptyset$ for all $\varepsilon > \varepsilon' + k$.*

Proof. If $S(\varphi, \varepsilon') \neq \emptyset$ and $\varepsilon > \varepsilon' + k$, select $x_0 \in S(\varphi, \varepsilon')$. Write $U(A, \delta) = \{x \in X \mid d(x, A) < \delta\}$. As φ is usc, there exists a $\delta > 0$ such that $x \in U(x_0, \delta)$ implies $\varphi(x) \subset U(\varphi(x_0, \varepsilon_1))$ with $\varepsilon_1 = \frac{1}{2}(\varepsilon - \varepsilon' - k) > 0$. If $0 < \varepsilon_2 \leqslant \delta$ and $0 < \varepsilon_2 \leqslant \varepsilon_1$, then we have for all $x \in U(x_0, \varepsilon_2)$

$$\begin{aligned} d(x, \varphi(x)) &\leqslant d(x, x_0) + d(x_0, \varphi(x_0)) + \operatorname{diam}\varphi(x_0) + d(\varphi(x_0), \varphi(x)) \\ &\leqslant \varepsilon_2 + \varepsilon' + k + \varepsilon_1 \leqslant \varepsilon . \end{aligned}$$

Hence $U(x_0, \varepsilon_2) \subset S(\varphi, \varepsilon)$ and $\operatorname{Int} S(\varphi, \varepsilon) \neq \emptyset$.

We shall later use a special case of Proposition 1.2.

COROLLARY 1.3. *Let (X, d) be a metric space and $f\colon X \to X$ be a map with a fixed point. Then $\operatorname{Int} S(f, \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$.*

Note that it is not possible in Proposition 1.2 to replace the condition $\varepsilon > \varepsilon' + k$ by $\varepsilon > \varepsilon'$; counterexamples are easy to construct.

2. A Nielsen number for small multifunctions. We shall now define a Nielsen number for multifunctions in the case that $\operatorname{diam}\varphi(x)$ does not exceed a certain size, and relate this number to minimal sets of fixed points and ε-near points. More precisely, we shall work with the small multifunctions studied in [5]. We repeat some of the definitions.

Let $|K|$ be a polyhedron underlying the finite simplicial complex K. We shall assume throughout this paper that $|K|$ is connected. Let $\operatorname{st}(v; K)$ denote the *star* of the vertex v of K, which is the open subset of $|K|$ consisting of the union of the interiors of all simplexes which have v as a vertex. A multifunction $\varphi\colon X \to |K|$ is called *small* (with respect to the simplicial structure K of $|K|$) if

$$\varphi(x) \subset \operatorname{st}(v; K) \quad \text{for all } x \in X ,$$

where $v = v(x)$ is a suitable vertex of K. A *small homotopy* between two usc multifunctions $\varphi_0, \varphi_1\colon X \to |K|$ is a small and usc multifunction $\Phi\colon X \times I \to |K|$ with $\Phi(x, 0) = \varphi_0(x)$ and $\Phi(x, 1) = \varphi_1(x)$ for all $x \in X$. Our proofs shall depend on the fact that a small usc multifunction $\varphi\colon |K| \to |K|$ has a *simplicial approximation* $f\colon |K'| \to |K|$, i.e., that there exists a simplicial map f from a suitable subdivision of K into K for which

$$\varphi(\operatorname{st}(v; K')) \subset \operatorname{st}(f(v); K)$$

for all vertices $v \in K'$ [5, Theorem 2.4].

We define the *Nielsen number* $N(\varphi)$ of a small usc multifunction $\varphi\colon |K| \to |K|$ by $N(\varphi) = N(f)$, where $N(f)$ is the Nielsen number of any map [1, p. 87] related to φ by a small homotopy. That $N(\varphi)$ is independent of the choice of f follows from the facts that two maps which are both related to φ by a small homotopy are homotopic [5, Theorem 3.4], and that $N(f)$ is homotopy invariant [1, p. 95]. $N(\varphi)$ is always defined as it equals the Nielsen number of any simplicial approximation of φ, and is of course equal to the usual one if φ is single-valued. It is also homotopy invariant:

Proposition 2.1. *If two small* usc *multifunctions* $\varphi_0, \varphi_1\colon |K| \to |K|$ *are related by a small homotopy, then* $N(\varphi_0) = N(\varphi_1)$.

This follows immediately from [5, Theorem 3.1], as the simplicial approximations of φ_0 and φ_1 are homotopic.

3. Minimal sets of ε-near points of small multifunctions. We now attempt to relate the Nielsen number $N(\varphi)$ of a small usc multifunction $\varphi\colon |K| \to |K|$ to its ε-near point set $S(\varphi, \varepsilon)$ for suitable values of ε. The example in § 1 shows that the choice of ε is important if we want to obtain results resembling those for minimal fixed point sets of maps: if ε is too small, then $S(\varphi, \varepsilon)$ might be empty; if ε is too large, then $S(\varphi, \varepsilon)$ might equal all of $|K|$. Nevertheless we shall show, in Theorem 3.1, that there always exist values of $\varepsilon > 0$ for which $S(\varphi, \varepsilon)$ has at least $N(\varphi)$ path-components. Then we construct, within the small homotopy class of φ, a map f for which $S(f, \varepsilon)$ has exactly $N(\varphi)$ path-components (Theorem 3.3) and a small usc multifunction ψ for which $S(\psi, \varepsilon)$ consists of exactly $N(\varphi)$ points (Theorem 3.4).

We shall denote by $\mu(K')$ the *mesh* of the star-covering of a subdivision K' of the polyhedron $|K|$, i.e. the supremum of the diameters of the set of stars with vertices of K'. By m we mean the minimum of the dimensions of the maximal simplexes of K, where a simplex is *maximal* if it is not a proper face of any other simplex.

Theorem 3.1. *For every polyhedron $|K|$ there exists an $\varepsilon = \varepsilon(|K|) > 0$ with the following property: If K' is a subdivision of K with $\mu(K') < \varepsilon$ and $\varphi\colon |K| \to |K|$ is any* usc *multifunction which is small with respect to K', then $S(\varphi, \varepsilon)$ has at least $N(\varphi)$ path-components of dimension $\geqslant m$.*

Proof. We know from [1, p. 39] that there exists a $\delta = \delta(|K|) > 0$ for which the following is true: If

$$W = \{(x, x') \in |K| \times |K| \,|\, d(x, x') < \delta\},$$

then there exists a map $h\colon W \times I \to |K|$ such that

$$h(x, x', 0) = x, \quad h(x, x', 1) = x',$$
$$h(x, x, t) = x \quad \text{for all } t \in I.$$

Hence if $p\colon I \to |K|$ is a path in $|K|$ and $f\colon |K| \to |K|$ a self-map with $f \circ p(0) = p(0)$, $f \circ p(1) = p(1)$ and $d(x, f(x)) < \delta$ for all $x \in p(I)$, then the paths p and $f \circ p$ are homotopic in $|K|$ modulo the end-points.

Now take $\varepsilon = \frac{1}{3}\delta$. According to [5, Theorem 2.4] there exists a simplicial approximation $f\colon |K''| \to |K'|$ of φ, i.e. a simplicial map from a subdivision K'' of K into K' for which

$$\varphi(\mathrm{st}(v;\, K'')) \subset \mathrm{st}(f(v);\, K') \quad \text{for all vertices } v \in K''.$$

We shall relate $S(f)$ to $S(\varphi, \varepsilon)$.

(i) $S(f) \subset S(\varphi, \varepsilon)$: Take any $x \in S(f)$. Select a vertex $v \in K''$ with $x \in \mathrm{st}(v;\, K'')$. As f is a simplicial approximation of φ we have

$$\varphi(x) \subset \varphi(\mathrm{st}(v;\, K'')) \subset \mathrm{st}(f(v);\, K').$$

As f is simplicial, $f(\mathrm{st}(v;\, K'')) \subset \mathrm{st}(f(v);\, K')$, and hence

$$x = f(x) \in f(\mathrm{st}(v;\, K'')) \subset \mathrm{st}(f(v);\, K').$$

Therefore $d(x, \varphi(x)) < \varepsilon$, so that $x \in S(\varphi, \varepsilon)$.

(ii) If $x_i, x_j \in S(f)$ lie in different fixed point classes, then they lie in different path-components of $S(\varphi, \varepsilon)$: Assume to the contrary that there exists a path $p\colon I \to S(\varphi, \varepsilon)$ with $p(0) = x_i$ and $p(1) = x_j$. Take any point $x = p(t)$, $t \in I$, and select a vertex $v \in K''$ with $x \in \mathrm{st}(v;\, K'')$. As f is a simplicial approximation of φ we have

$$\varphi(x) \subset \varphi(\mathrm{st}(v;\, K'')) \subset \mathrm{st}(f(v);\, K').$$

From this and

$$f(x) \in f(\mathrm{st}(v;\, K'')) \subset \mathrm{st}(f(v);\, K')$$

it follows that $d(f(x), \varphi(x)) < \varepsilon$. As $x \in S(\varphi, \varepsilon)$ we have $d(x, \varphi(x)) \leqslant \varepsilon$, and as φ is small with respect to K', we see that $\operatorname{diam}\varphi(x) < \varepsilon$. Therefore

$$d(x, f(x)) \leqslant d(x, \varphi(x)) + \operatorname{diam}\varphi(x) + d(\varphi(x), f(x)) < 3\varepsilon = \delta,$$

so that p and $f \circ p$ are homotopic modulo x_i and x_j. But this is impossible if x_i and x_j lie in different fixed point classes of f.

In consequence of (i) and (ii) the set $S(\varphi, \varepsilon)$ must have at least $N(\varphi)$ path-components, as f has $N(f) = N(\varphi)$ essential fixed point classes. It remains to show that every path-component of $S(\varphi, \varepsilon)$ which is related to an essential fixed point class of f in the manner of (i) is at least m-dimensional.

We saw in (i) that $x = f(x) \in \mathrm{st}(f(v);\, K')$ and $\varphi(x) \subset \mathrm{st}(f(v);\, K')$ for some $v \in K''$. As $\mathrm{st}(f(v);\, K')$ is open and φ is usc, there exists a $\delta' > 0$ such that $d(x, x') < \delta'$ implies $\varphi(x') \subset \mathrm{st}(f(v);\, K')$ and $x' \in \mathrm{st}(f(v);\, K')$. But then $d(x', \varphi(x')) < \varepsilon$. Let $|\tau|$ be a maximal simplex of K which has the carrier simplex of x as its (proper or improper) face. Then $U(x, \delta') \cap |\tau|$ is at least m-dimensional, is contained in $S(\varphi, \varepsilon)$, and is path-connected. Hence it is contained in the same path-component of $S(\varphi, \varepsilon)$ as x, and Theorem 3.1 follows.

We state the special case of Theorem 3.1 in which φ is a map.

Corollary 3.2. *For every polyhedron $|K|$ there exists an $\varepsilon = \varepsilon(|K|) > 0$ such that the ε-near point set of every self-map f of $|K|$ has at least $N(f)$ path-components of dimension $\geqslant m$.*

Our next task is to show that the lower bound for $S(\varphi, \varepsilon)$ established in Theorem 3.1 can actually be achieved. First we relate φ by a small homotopy to a map f for which $S(f, \varepsilon)$, for a suitable $\varepsilon = \varepsilon(\varphi)$, is minimal. As in the corresponding case for fixed points of maps (see [1, p. 140]) we have to assume that the polyhedron $|K|$ is of *type S*, which means that the dimension of $|K|$ is at least three and that, for each vertex v of $|K|$, the boundary $\mathrm{Bd\,st}(v; K)$ is connected.

THEOREM 3.3. *Let $|K|$ be a polyhedron of type S and $\varphi\colon |K| \to |K|$ be* usc *and small with respect to K. Then there exists an $\varepsilon = \varepsilon(\varphi) > 0$, and a map $f\colon |K| \to |K|$ related to φ by a small homotopy, such that $S(f, \varepsilon)$ has exactly $N(\varphi)$ path-components, and each is homeomorphic to an m-ball.*

Proof. Choose a simplicial approximation $g\colon |K'| \to |K|$ of φ. Then if follows from [1, p. 140 f] that there exists a map $g'\colon |K| \to |K|$ with exactly $N(g) = N(\varphi)$ fixed points x_j $\big(j = 1, 2, \ldots, N(\varphi)\big)$, and such that each x_j lies in a maximal simplex of dimension $\geqslant m$.

If $N(\varphi) = 0$, then g' is fixed point free, and therefore $d\big(x, g'(x)\big) \geqslant \delta$ for some $\delta > 0$. Hence $f = g'$ and any ε with $0 < \varepsilon < \delta$ will satisfy Theorem 3.3.

If $N(\varphi) \neq 0$, then we select for each x_j a maximal simplex $|\tau_j|$ of dimension m. (The $|\tau_j|$ need not be distinct.) As $|K|$ is of type S, we can by repeated use of Lemma 6 in [1, p. 135] find a map $g''\colon |K| \to |K|$ which is homotopic to g' and has precisely $N(\varphi)$ fixed points y_j, with $y_j \in |\tau_j|$. For each y_j choose a $\delta_j > 0$ such that (with Cl denoting the closure)

$$\mathrm{Cl}\, U(y_j, \delta_j) \subset |\tau_j|\,, \quad g''\big(\mathrm{Cl}\, U(y_j, \delta_j)\big) \subset |\tau_j|$$

and all $\mathrm{Cl}\, U(y_j, \delta_j)$ are distinct. Select $\varepsilon > 0$ such that

$$d\big(x, g''(x)\big) > \varepsilon \quad \text{ for all } x \notin U(y_j, \delta_j)\,.$$

Then $S(g'', \varepsilon) \subset \bigcup \{\mathrm{Cl}\, U(y_j, \delta_j) |\ j = 1, 2, \ldots, N(\varphi)\}$.

We now construct $f\colon |K| \to |K|$. If $x \notin U(y_j, \delta_j)$ for any j, let $f(x) = g''(x)$. If $x \in U(y_j, \delta_j) \setminus \{y_j\}$ for some j, then $x \in |\tau_j|$ and $g''\big(\mathrm{Cl}\, U(y_j, \delta_j)\big) \subset |\tau_j|$. Let $y \in |\tau_j|$ be the unique point in which the ray from y_j to x intersects $\mathrm{Bd}\, U(y_j, \delta_j)$, and define $f(x) \in |\tau_j|$ by

$$\overrightarrow{y_j f(x)} = \overrightarrow{y_j x} + d(y_j, x)/\delta_j \overrightarrow{y g''(y)}\,.$$

Finally put $f(y_j) = y_j$. This defines f continuously over all of $|K|$.

We have $S(f, \varepsilon) \subset \bigcup \{U(y_j, \delta_j) |\ j = 1, 2, \ldots, N(\varphi)\}$. For each j the set $S_j(f, \varepsilon) = S(f, \varepsilon) \cap U(y_j, \delta_j)$, which contains y_j, is path-connected, and therefore these $N(\varphi)$ sets are the path-components of $S(f, \varepsilon)$. Each $S_j(f, \varepsilon)$ is a closed subset of an m-simplex, and a homeomorphism onto an m-ball can be obtained by selecting, for each $z \in \mathrm{Bd}\, U(y_j, \delta_j)$, the point y on the segment $[y_j, z]$ for which $d\big(y, f(y)\big) = \varepsilon$, and mapping $[y_j, y]$ linearly onto $[y_j, z]$. Hence Theorem 3.3 holds.

In particular Theorem 3.3 can again be applied to single-valued maps to obtain a result about minimal sets of ε-near points within a homotopy class of maps. On the other hand, we can construct a function with only isolated ε-near points if we stay within the framework of small multifunctions. This is done in the next theorem.

THEOREM 3.4. *Let $|K|$ be a polyhedron of type S and $\varphi\colon |K| \to |K|$ be* usc *and small with respect to K. Then there exists an $\varepsilon' = \varepsilon'(\varphi) > 0$ and a small* usc *multifunction $\psi\colon |K| \to |K|$ related to φ by a small homotopy such that $S(\psi, \varepsilon')$ consists of exactly $N(\varphi)$ points.*

Proof. If $N(\varphi) = 0$, then Theorem 3.4 is a consequence of Theorem 3.3. If $N(\varphi) \neq 0$, we modify the map f constructed in the proof of Theorem 3.3 in the following way to obtain ψ: Let $\psi(x) = f(x)$ if $x \notin S_j(f, \varepsilon)$ for any j. If $x \in S_j(f, \varepsilon) \setminus \{y_j\}$, let $y \in |\tau_j|$ be the unique point on the ray from y_j to x for which $d\big(y, f(y)\big) = \varepsilon$, and define $\psi(x) = z$ as the point in $|\tau_j|$ for which

$$\overrightarrow{y_j z} = \overrightarrow{y_j x} + \tfrac{1}{2}\big(\lambda(x) + 1\big)\overrightarrow{y f(y)} \quad \text{ with } \quad \lambda(x) = d(y_j, x)/d(y_j, y)\,.$$

Finally let

$$\psi(y_j) = \{z |\ \overrightarrow{y_j z} = \tfrac{1}{2}\overrightarrow{y f(y)} \text{ for some } y \in S_j(f, \varepsilon) \text{ with } d\big(y, f(y)\big) = \varepsilon\}\,.$$

It is easy to check that then $S(\psi, \frac{1}{2}\varepsilon) = \{y_j |\ j = 1, 2, \ldots, N(\varphi)\}$, and that ψ and $\varepsilon' = \frac{1}{2}\varepsilon$ satisfy Theorem 3.4.

We note that the multifunction ψ which we have just constructed is not acyclic (see Definition 4.1 below) if $N(\varphi) \neq 0$. This is necessary; Theorem 3.4 cannot be satisfied with an acyclic multifunction ψ if ε is sufficiently small in consequence of Proposition 1.2 (with $\varepsilon = 0$) and Theorem 4.1. In particular, Theorem 3.4 has no equivalent for single-valued maps.

4. Minimal sets of fixed points of acyclic small multifunctions. We now turn our attention to acyclic multifunctions. In this case we study the fixed point set, as results by Eilenberg and Montgomery [2] and others indicate that acyclic multifunctions have fixed point properties similar to those of maps. A frequent tool in the proof of known fixed point theorems for acyclic multifunctions is the Vietoris–Begle mapping theorem [7, p. 344]. We employ this theorem indirectly, as we require the existence of a fixed point index, and the definition of a fixed point index for acyclic usc multifunctions described in the appendix uses the Vietoris-Begle mapping theorem.

Acyclic multifunctions are defined in the literature by different authors in more than one way. For our purposes the most convenient definition seems to be the following.

DEFINITION 4.1. A multifunction φ on a space X is *acyclic* if $\tilde{H}^q(\varphi(x)) = 0$ for all $x \in X$ and all integers q, where $\tilde{H}$ denotes reduced Čech cohomology over the rationals.

For acyclic small multifunctions, we can prove the analogue to Theorem 3.1 for fixed point sets.

THEOREM 4.2. *For every polyhedron $|K|$ there exists an $\varepsilon = \varepsilon(|K|) > 0$ with the following property: If K' is a subdivision of K with $\mu(K') < \varepsilon$ and if $\varphi\colon |K| \to |K|$ is any multifunction which is* usc, *acyclic, and small with respect to K', then $S(\varphi)$ has at least $N(\varphi)$ path-components.*

Proof. Let $f\colon |K''| \to |K'|$ be a simplicial approximation of φ, and let S_j $(j = 1, 2, \ldots, N(\varphi))$ be the essential fixed point classes of f. Choose $\varepsilon(|K|) > 0$ as in Theorem 3.1. Write

$$\boldsymbol{S}(f, \varepsilon) = \{x \in K \mid d(x, f(x)) < \varepsilon\}.$$

(The set $\boldsymbol{S}(f, \varepsilon)$ is open and contained in $\operatorname{Int} S(f, \varepsilon)$, but in general $\boldsymbol{S}(f, \varepsilon) \neq \operatorname{Int} S(f, \varepsilon)$.) For each j define

$$U_j = \{x \in |K| \mid \text{there exists a path } p\colon I \to \boldsymbol{S}(f, \varepsilon) \text{ with } p(0) \in S_j \text{ and } p(1) = x\}.$$

The proof of Theorem 4.2 will be accomplished by comparing the fixed point indices of φ and f over U_j. Therefore we establish the necessary properties of the sets U_j. Remember that a path $p\colon I \to \boldsymbol{S}(f, \varepsilon')$ with $f \circ p(0) = p(0)$ and $f \circ p(1) = p(1)$ is homotopic in $|K|$ to $f \circ p$ modulo its end points if $\varepsilon' \leqslant \delta$, where δ is chosen as in the proof of Theorem 3.1.

(i) The U_j are open: This is simple to check.

(ii) $U_j \cap U_k = \emptyset$ if $j \neq k$: Assume that $x \in U_j \cap U_k$. Then we have paths $p_j\colon I \to \boldsymbol{S}(f, \varepsilon)$ and $p_k\colon I \to \boldsymbol{S}(f, \varepsilon)$ with $p_j(0) = x_j \in S_j$, $p_k(0) = x_k \in S_k$, and $p_j(1) = p_k(1) = x$. The path $p_j \circ p_k^{-1}$ is a path from x_j to x_k in $\boldsymbol{S}(f, \varepsilon)$, which implies that x_j and x_k lie in the same fixed point class. But this is impossible if $j \neq k$.

(iii) If $x \in \operatorname{Bd} U_j$, then $d(x, f(x)) = \varepsilon$: If $d(x, f(x)) < \varepsilon$, then there exists a path-connected neighbourhood $N(x)$ with $N(x) \subset \boldsymbol{S}(f, \varepsilon)$. If also $x \in \operatorname{Cl} U_j$, then $N(x)$ contains a point $x' \in U_j$ connected to a point in S_j by a path p' in $\boldsymbol{S}(f, \varepsilon)$. But then every point in $N(x)$ can be connected to a point in S_j by the composite of p' and a path in $N(x)$, hence $N(x) \subset U_j$ and $x \in \operatorname{Int} U_j$. So $d(x, f(x)) \geqslant \varepsilon$ for $x \in \operatorname{Bd} U_j$, and that $d(x, f(x)) > \varepsilon$ cannot hold follows from the continuity of f.

(iv) $S(f) \cap \operatorname{Cl} U_j = S_j$: By definition of U_j we have $S_j \subset U_j$, and from (iii) we see that $\operatorname{Bd} U_j \cap S(f) = \emptyset$. As every fixed point in U_j is joined to a fixed point in S_j by a path in $\boldsymbol{S}(f, \varepsilon)$, we have $(S(f) \setminus S_j) \cap U_j = \emptyset$.

(v) There exists a small acyclic homotopy from φ to f which is fixed point free on $\operatorname{Bd} U_j$ for all j: Take the small homotopy $\Phi\colon |K| \times I \to K$ constructed in the proof of Theorem 2.4 in [5]. It is a small multifunction for which $\Phi(x, 0) = \varphi(x)$ and $\Phi(x, 1) = f(x)$ for all $x \in |K|$, is acyclic if φ is acyclic as $\Phi(x, t)$ is homeomorphic to $\varphi(x)$ for all $t \in [0, 1)$, and has the property that for all $x \in |K|$ there exists a vertex w of K' with

$$\Phi(x, t) \subset \operatorname{st}(w; K') \quad \text{for all } t \in I.$$

Hence $x \in \Phi(x, t)$, for some $t \in I$, implies $x \in \boldsymbol{S}(f, \varepsilon)$ and therefore, by (iii), $x \notin \operatorname{Bd} U_j$.

Now consider, for every j, the index $i_r(|K|, \varphi, U_j)$ defined in § 5. It follows from (i) and (iii) that this index can be defined, and from (v) as well as (5.1) that $i_r(|K|, \varphi, U_j) = i_r(|K|, f, U_j)$. But $i_r(|K|, f, U_j)$ is the index of the essential fixed point class S_j and hence non-zero [1, p. 87]. Therefore $i_r(|K|, \varphi, U_j) \neq 0$, and (5.2) shows that φ has a fixed point on U_j. So we obtain a fixed point y_j of φ on each of the U_j, and as the U_j are distinct, this yields $N(\varphi)$ fixed points.

If y is any point in $S(\varphi)$, then

$$d(y, f(y)) \leqslant \operatorname{diam}\varphi(y) + d(\varphi(y), f(y)) < 2\varepsilon < \delta.$$

From this we see that y_j and y_k $(j \neq k)$ must belong to different path-components of $S(\varphi)$: otherwise the composite $p = p_j \circ q \circ p_k^{-1}$ of a path $p_j\colon I \to \boldsymbol{S}(f, \varepsilon)$ from a point $x_j \in S_j$ to y_j, a path $q\colon I \to \boldsymbol{S}(f, 2\varepsilon)$ from y_j to y_k and a path $p_k\colon I \to \boldsymbol{S}(f, \varepsilon)$ from a point $x_k \in S_k$ to y_k would give a path from x_j to x_k which lies in $\boldsymbol{S}(f, 2\varepsilon)$ and hence is (as $2\varepsilon < \delta$) end-point homotopic to $f \circ p$. This contradicts the choice of x_j and x_k in different fixed point classes. So we see that $S(\varphi)$ has at least $N(\varphi)$ path-components.

Remarks. (i) The only reason why the proof of Theorem 4.2 requires that φ is acyclic is that it uses a fixed point index for φ which satisfies the homotopy and additivity axioms. But non-acyclic multifunctions would usually not satisfy Theorem 4.2, as can e.g. be seen from the example in § 1. Therefore it seems doubtful that a fixed point index can be defined for small usc multifunctions in a meaningful way.

(ii) We cannot strengthen Theorem 4.2 with a statement about the dimension of the path-components of $S(\varphi)$, as it is not difficult to construct examples where the dimensions of the various path-components assume all values $\geqslant 0$.

Two corollaries follow easily from Theorem 4.2.

COROLLARY 4.3. *Let $|K|$ be a polyhedron of type S which has the fixed point property. Then there exists a subdivision K' of K such that $|K|$ has the fixed point property for all multifunctions which are* usc, *acyclic, and small with respect to K'.*

Proof. Select K' as in Theorem 4.2, and let $\varphi\colon |K| \to |K|$ be usc, acyclic, and small with respect to K'. If f is a simplicial approximation of φ then $N(f) \neq 0$ [1, p. 146] and hence $N(\varphi) \neq 0$. As Theorem 4.2 implies that $S(\varphi)$ has at least $N(\varphi)$ path-components, $S(\varphi)$ must be non-empty.

The second corollary is a special case of a fixed point theorem by Eilenberg and Montgomery [2].

COROLLARY 4.4. *Every* usc *and acyclic multifunction of an n-ball into itself has a fixed point.*

Proof. Imbed the n-ball B^n into an n-simplex $|\tau|$. Let $r\colon |\tau| \to B^n$ be a retraction and $i\colon B^n \to |\tau|$ be the imbedding, and consider the acyclic usc multifunction $i \circ \varphi \circ r\colon |\tau| \to |\tau|$. For $|K| = |\tau|$ the value of δ in the proof of Theorem 3.1, and hence of ε in Theorem 4.2, can be arbitrarily large, and it is therefore not necessary to subdivide $|\tau|$ if we want to apply Theorem 4.2 to the multifunction $i \circ \varphi \circ r$. It is small with respect to τ, and the identity map of $|\tau|$ is a simplicial approximation. So $N(i \circ \varphi \circ r) = N(f) = 1$, and $i \circ \varphi \circ r$ must have a fixed point x. As r is a retraction, $x \in B^n$ and hence $x \in \varphi(x)$.

Finally we state the counterpart to Theorems 3.3 and 3.4, which is very easy to obtain.

THEOREM 4.5. *Let $|K|$ be a polyhedron of type S and $\varphi\colon |K| \to |K|$ be* usc, *acyclic, and small with respect to K. Then there exists a map $f\colon |K| \to |K|$ related to φ by an acyclic small homotopy which has exactly $N(\varphi)$ fixed points.*

Proof. Let $g\colon |K'| \to |K|$ be a simplicial approximation of φ, and let $f\colon |K| \to |K|$ be a map which is homotopic to g and has exactly $N(g) = N(\varphi)$ fixed points. (See [1, p. 140] for the existence of f.) The composite of the acyclic small homotopy from φ to g and the homotopy from g to f is an acyclic small homotopy from φ to f.

Corollary 4.4 points to a direction in which the results of this paragraph should be extended: it seems essential for Theorems 4.2 and 4.5 that the multifunctions considered are usc and acyclic, but not that they are small. The smallness of φ only facilitates the proofs which depend on the existence of a single-valued map closely related to φ. It should be a fairly simple and straightforward task to define a Nielsen number of an usc acyclic multifunction in the same way as in the single-valued case, and prove its invariance under acyclic homotopy. This would imply that Theorem 4.2 still holds. An extension of Theorem 4.5 looks more difficult to obtain.

5. Appendix. An index for usc acyclic multifunctions on polyhedra. The proof of Theorem 4.2 uses the existence of a fixed point index for usc acyclic multifunctions on polyhedra. We sketch here the definition of such an index and the derivation of some of its properties. The development is kept parallel to that given in [1, Chapter IV] for the single-valued case, so that details can easily be filled in by the reader.

Let $X = |K|$ be a finite connected polyhedron. It can be imbedded into an Euclidean space R^n, and there exists an open subset W of R^n and a retraction $r\colon W \to X$. Assume that U is open in X and that $\operatorname{Bd} U \cap S(\varphi) = \emptyset$ for an usc acyclic multifunction $\varphi\colon X \to X$. With $V = r^{-1}(U)$, define a multifunction $\delta\colon V \to R^n$ by $\delta(x) = x - \varphi r(x)$. Then δ is usc acyclic, and if $S(\varphi, U) = S(\varphi) \cap U$, then $\delta(V \setminus S(\varphi, U)) \subset R^n \setminus 0$. We proceed to define a homomorphism

$$\delta^*\colon H^n(R^n, R^n\setminus 0) \to H^n(V, V \setminus S(\varphi, U)).$$

For this purpose, denote by G the graph of δ in $V \times R^n$, and by G' the graph in $(V\setminus S(\varphi, U)) \times (R^n \setminus 0)$ of the restriction of δ to $V \setminus S(\varphi, U)$. Let $p_1\colon V \times R^n \to V$ and $p_2\colon V \times R^n \to R^n$ be the first and second projections of $V \times R^n$ onto its factors, and consider the induced homomorphisms

$$H^n(R^n, R^n\setminus 0) \xrightarrow{p_2^*} H^n(G, G') \xleftarrow{p_1^*} H^n(V, V\setminus S(\varphi, U)).$$

In consequence of the Vietoris-Begle mapping theorem [7, p. 344] p_1^* is an isomorphism, hence we can put $\delta^* = (p_1^*)^{-1} p_2^*$.

Now select generators $\mu_n \in H^n(R^n, R^n \setminus 0)$ and $U_n \in H^n(S^n)$ in the same way as in [1, p. 54], and use in analogy to [1, p. 56] the homomorphisms

$$H^n(R^n, R^n\setminus 0) \xrightarrow{\delta^*} H^n(V, V\setminus S(\varphi, U)) \xrightarrow{j^{*-1}} H^n(S^n, S^n\setminus S(\varphi, U)) \xrightarrow{k^*} H^n(S^n),$$

where j^* and k^* are induced by inclusions. Then

$$k^* j^{*-1} \delta^*(\mu_n) = q\nu_n$$

for some rational number q, and we define the *index* of φ on U as

$$i_r(X, \varphi, U) = q.$$

Obviously $i_r(X, \varphi, U)$ is identical with the index used in [1, p. 54 ff] if φ is single-valued, as then δ^* equals the d^* in Brown's definition. We use the following two properties of the index in the proof of Theorem 4.2.

(5.1) If $\Phi\colon X \times I \to X$ is an usc acyclic homotopy between the usc and acyclic multifunctions $\Phi(x, 0) = \varphi_0(x)$ and $\Phi(x, 1) = \varphi_1(x)$ such that $x \notin \Phi(x, t)$ for all $x \in \operatorname{Bd} U$ and $t \in I$, then

$$i_r(X, \varphi_0, U) = i_r(X, \varphi_1, U).$$

(5.2) If $i_r(X, \varphi, U) \neq 0$, then φ has a fixed point on U.

The proofs of (5.1) and (5.2) are quite analogous to the ones of Lemma 3, Theorem 4, and Theorem 5 on p. 54 ff as well as Corollary 1 on p. 53 of [1], and establish the fact that $i_r(X, \varphi, U)$ satisfies the homotopy and additivity axioms. The proof of the counterpart of Theorem 5 in [1, p. 59/60] requires that two usc acyclic multifunctions δ_0 and δ_1 which are related by an usc acyclic homotopy induce the same homomorphism

$$\delta_0^* = \delta_1^*\colon\ H^n(R^n, R^n\setminus 0)\to H^n\big(V, V\setminus S(\varphi, U)\big).$$

That this is true can be shown as in [3, Theorem 3]. It would be of interest to check (but it is not needed in this paper) to what extent $i_r(X, \varphi, U)$ satisfies other axioms often associated with a fixed point index, and how it extends to more general spaces than polyhedra. Some modifications will arise, e.g. in the commutativity axiom, as the composite of two acyclic multifunctions need not be acyclic.

References

[1] R. F. Brown, *The Lefschetz Fixed Point Theorem*, Glenview, Ill., 1971.
[2] S. Eilenberg and D. Montgomery, *Fixed point theorems for multi-valued transformations*, Amer. J. Math. 68 (1946), pp. 214–222.
[3] J. W. Jaworowski, *Some consequences of the Vietoris mapping theorem*, Fund. Math. 45 (1958), pp. 261–272.
[4] J. Nielsen, *Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen I, II, III*, Acta Math. 50 (1927), pp. 189–385; 53 (1929), pp. 1–76; 58 (1932), pp. 87–167.
[5] H. Schirmer, *Simplicial approximation of small multifunctions*, to appear in Fund. Math.
[6] Shi Gen-Hua, *On least number of fixed points and Nielsen numbers*, Chinese Math. — Acta 16 (1966), pp. 223–232.
[7] E. H. Spanier, *Algebraic Topology*, New York 1966.
[8] T. van der Walt, *Fixed and almost fixed points*, Mathematical Centre Tracts, v. 1, Mathematisch Centrum Amsterdam, 1963.
[9] F. Wecken, *Fixpunktklassen I, II, III*, Math. Ann. 117 (1941), pp. 659–671; 118 (1941/43), pp. 216–234 and pp. 544–577.

CARLETON UNIVERSITY
Ottawa, Canada

Accepté par la Rédaction le 11. 1. 1974

Partition topologies for large cardinals

by

Erik Ellentuck * (New Brunswick, N. J.)

Abstract. Two topologies are introduced on the power set of a large cardinal. Partition theorems in the style of Kleinberg-Shore are obtained for the first topology and ones in the style of Galvin-Prikry for the second.

1. Introduction. Let $\varkappa, \lambda, \nu$ be cardinal numbers and $u, v, s, t \subseteq \varkappa$. $\overline{u}$ is the order type of u and $[s, v]^\lambda = \{u|\ s\subseteq u \subseteq s\cup v \wedge \overline{u} = \lambda\}$. $[s, v]^{<\lambda}$ is defined in the same way except $\overline{u} = \lambda$ in the definition of $[s, v]^\lambda$ is replaced by $\overline{u} < \lambda$. $[\emptyset, v]^\lambda$ will be written as $[v]^\lambda$ where $\emptyset$ is the empty set. We define two topologies on $[\varkappa]^\lambda$ where $\omega \leqslant \lambda \leqslant \varkappa$ and ω is the first infinite cardinal. The *classical topology* (*c-topology*) is generated by a basis consisting of $[s, \varkappa - t]^\lambda$ where $s, t \in [\varkappa]^{<\omega}$. If $\varkappa$ is measurable let $D \subseteq [\varkappa]^\varkappa$ be a $\varkappa$-complete normal ultrafilter. The *measure topology* (*d-topology*) is generated by a basis consisting of $[s, u]^\lambda$ where $s \in [\varkappa]^{<\omega}$ and $u \in D$. When we speak of a topology without the c or d prefix we mean either topology. $S \subseteq [\varkappa]^\lambda$ is *Borel* if it is generated from the open sets by complementation and $< \varkappa$ intersections. It is *meager* if it is the union of $< \varkappa$ nowhere dense sets and is *Baire* if its symmetric difference with an open set is meager. $S \subseteq [\varkappa]^\lambda$ is *Ramsey* if there is a $u \in [\varkappa]^\varkappa$ such that $[u]^\lambda \subseteq S$ or $[u]^\lambda \subseteq [\varkappa]^\lambda - S$. Such a u is called *homogeneous* for S.

THEOREM 1. *If $\varkappa$ is a Ramsey cardinal and $S \subseteq [\varkappa]^\lambda$ is c-Borel then S is Ramsey.*

THEOREM 2. *If $\varkappa$ is a measurable cardinal and $S \subseteq [\varkappa]^\lambda$ is d-Baire then S is Ramsey.*

Our proof of Theorem 1 is based on the work of Kleinberg–Shore [3] and that of Theorem 2 on the work of Galvin–Prikry [2] and of the author [1].

2. Details. Write $u < v$ if every element of u is strictly less than every element of v. $(s, v)^\lambda = \{u|\ s \subseteq u \subseteq s \cup v \wedge \overline{u} = \lambda \wedge s < u - s\}$. $(s, v)^{<\lambda}$ is defined in the same way except $\overline{u} = \lambda$ in the definition of $(s, v)^\lambda$ is

* Prepared while the author was partially supported by NSF Grant Proposal 28348.