The box product of countably many metrizable spaces need not be normal

by

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Abstract. The box product of the family (irrationals) ∪ {P₃} T₃ is a convergent sequence is not normal. If $X = \{X_\alpha \mid \alpha \in A\}$ is a family of metrizable spaces, the subspace $\mathbb{F}_n = \{x_\alpha \mid x_\alpha \neq x_\beta \text{ for at most finitely many } \alpha \}$ of the box product of this family is stratifiable, $\mathbb{P} \times \Pi_{\alpha} X_\alpha$ arbitrary. If the family $X$ is countable and all finite subproducts are paracompact, $\mathbb{F}_n$ is paracompact.

1. Introduction. If $\{X_\alpha \mid \alpha \in A\}$ is a family of spaces, we denote the usual product space by $\Pi\alpha X_\alpha$ and the box product (see [5, p. 107]) by $B\alpha X_\alpha$. Stone asked whether $B\alpha X_\alpha$ is normal if each $X_\alpha$ is metrizable, [6]. A partial answer has been given by Rudin, who showed that the continuum hypothesis implies that $B\alpha X_\alpha$ is paracompact provided each $X_\alpha$ is a locally compact metrizable space, [10]. (Actually this was stated under the additional hypothesis that the $X_\alpha$ are $\sigma$-compact. However, a locally compact metrizable space $X_\alpha$ is the union of a disjoint open family $\{X_{\alpha n} \mid \alpha \in A_\alpha\}$ consisting of locally compact $\sigma$-compact subspaces, cf. [11], so $B\alpha X_\alpha$, being the union of the disjoint open family

$$\{B\alpha X_{\alpha n} \mid \alpha \in A_\alpha \text{ for } n \in \mathbb{N}\},$$

is paracompact.)

We show that the product of countably many separable metrizable spaces need not be normal, even if all factors but one are compact (the noncompact factor is the space of irrationals). Quite surprisingly the proof that our space is not normal, resembles Michael’s proof that the product of the irrationals and a certain space is not normal, [7]. This negative solution of Stone’s question also solves a question of Borges, whether a box product of metrizable spaces is stratifiable, [2]; in the negative and kills a conjecture of Vaughan, that a product of linearly stratifiable spaces is paracompact [12]. As a byproduct we show that a box product of metrizable spaces cannot be hereditarily normal if infinitely many factors are nondiscrete.
Certain subspaces of box products are behaving better. For \( p \in B \), let \( \mathcal{E}_p \) be the subspace \( \{ x \in B \mid x_n = p_n \text{ for at most finitely many } n \} \). (The component of \( p \) is contained in \( \mathcal{E}_p \), provided the \( X_i \)'s are regular \( T_1 \), [6, p. 51].) If each \( X_i \) is a group, with identity \( p_i \), \( \mathcal{E}_p \) is the so-called direct sum. Then \( \mathcal{E}_p \) is stratifiable if the \( X_i \)'s are metrizable. If \( (X_n) \subseteq N \) is a countable family of spaces, \( \mathcal{E}_p \) is paracompact if each finite subspace is paracompact.

It is shown in [6] that \( B \times X_i \) is \( T_i \) iff each factor is \( T_i \) for \( i = 1, 2, 3, 3 \), \( N \) is the set of positive integers.

2. Non-norm products. For \( n \in N \) let \( T_n \) be the space \( T = \{ x \mid x_n = 0 \} \) or \( \{ x \in X \} \), let \( B = B \times T_n \) and let \( P \) be the discrete open subspace \( \{ x \in B \mid x_n \neq 0 \text{ for at most finitely many } n \} \).

**Theorem.** \( P \) is not an \( F_\sigma \)-subset of \( B \).

Proof. Let \( F_\sigma \subset P \) be a closed subset of \( B \). Define a sequence \( x_1, x_2, \ldots \) by induction as follows: \( x_1 = \lambda \). Assume that \( x_t \in F_\sigma \) is known for \( 1 \leq t < n \) and that there is a strictly positive function \( e_t \) on \( N \) is known that \( 1 \leq t < n - 1 \) such that \( P_t \cap U_t = \emptyset \), where

\[
U_t = \bigcap_{s \neq t} \{ x_{s} \times \prod_{i \neq t} [0, \infty) \} \cdot t < e(s, t) \}.
\]

The point \( (x_1, x_2, \ldots) \) does not belong to \( P \), hence there is a strictly positive function \( e_n \) on \( N \) such that \( U_n \cap F_\sigma = \emptyset \). Pick \( x_{n+1} \in T_n \) such that \( e_{n+1} < e_t(n+1) \). This completes the definition. Let \( x \in P \) be the point \( (x_1, x_2, \ldots) \). Then \( x \in P \) for \( n \in N \), hence \( P \neq \bigcup \{ F_\sigma \mid n \in N \} \).

**Corollary.** A product with infinitely many nondiscrete metrizable spaces is not hereditarily normal.

Proof. Such a product contains a closed subspace homeomorphic to \( B \times B \) and \( T \times B \) are homeomorphic. By a theorem of Katětov a product \( X \times Y \) is hereditarily normal only if every countable subset of \( X \) is closed or \( Y \) is perfectly normal [4]. Hence \( T \times B \) is not hereditarily normal [6].

**Example.** A product of countably many metrizable spaces which is not normal.

Let \( B^* \) be the space \( B \times \prod T_n \) and let \( P^* \) be the subspace \( \{ x \in B^* \mid x_n \neq 0 \text{ for at most finitely many } n \} \). Let \( P \) be a metric for \( B^* \). Then \( B \) and \( B^* \) have the same underlying set, and so do \( P \) and \( P^* \). Observe that \( P^* \) is homeomorphic to the irrationals. Let \( T_n = T_n \), then \( P^* \times B \) and \( B \times (T_n \times 0, 0, \ldots) \) are the same space. We claim that \( P^* \times B \) is not normal.


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\[
P = \{ (x, y) \mid x \in P \} \text{ and } G = P^* \times (R, R) \text{ are disjoint closed subsets of } P^* \times B. \text{ Let } U \text{ and } V \text{ be neighborhoods of } P \text{ and } G \text{ respectively. For } x \in P \text{ and } \epsilon > 0 \text{ define } S(x, \epsilon) = \{ y \in P^* \mid d(x, y) < \epsilon \}. \text{ Define }
\]

\[
P_n = \{ x \in P \mid S(x, 1/n) \times \{ x \} \subseteq U \}.
\]

Then \( P = \bigcup n \subseteq \mathbb{N} \), hence by the assertion there is a \( q \in B \times P \) and an \( n \in N \) such that \( q \in F_{n} \) and \( \epsilon = 1/n \). Then \( S(q, 1/n) \subseteq U \) since \( (p, r) \in U \) if and only if \( (p, r) \in \epsilon \). Consequently \( U \cap V = \emptyset \), so \( P^* \times B \) is not normal.

The space \( T \) can be embedded as a closed subspace in the irrationals. It follows that \( B \times X \) is not metrizable.

**Problem.** For what kind of metrizable spaces \( X \) is the product \( B \times X \) normal or paracompact? Are these problems the same, cf. [9]?
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Theorem. $\mathcal{E}_p$ is a para-compact Hausdorff if $\bigcap X_i$ is para-compact Hausdorff for $n \in \mathbb{N}$ (we only consider countable direct sums).

Proof. Only the sufficiency requires proof. Each $X_i$ is regular, hence so is $\mathcal{E}_p$. Let $\mathcal{U}$ be an open cover of $\mathcal{E}_p$. For each $n \in \mathbb{N}$ there is a locally finite open cover $\mathcal{V}_n$ of the subspace $B_n$ which refines $\mathcal{U} \cap \mathcal{V}_n$. For $n \in \mathbb{N}$ the family $\{r_n^{-1}(V) \mid V \in \mathcal{V}_n\}$ is locally finite in $\mathcal{E}_p$. For $n \in \mathbb{N}$ there is a neighborhood $W_n$ of $r_n$ in $\mathcal{E}_p$ which intersects only finitely many members of $\mathcal{V}_n$. For $V \in \mathcal{V}_n$, choose a $c_0(V) \in \mathcal{V}_n$ such that $V \subset c_0(V)$. Then

$$\bigcup \{r_n^{-1}(V) \cap c_0(V) \mid V \in \mathcal{V}_n\} \cap \mathcal{E}_p$$

is a $\alpha$-locally finite refinement of $\mathcal{U}$. Consequently $\mathcal{E}_p$ is para-compact.

Remark. There are many topologies on the set $\mathcal{H}_p \mathcal{H}_n$ between the usual topology and the box topology: If $\mathcal{F}$ is a collection of subsets of $\mathcal{N}$ such that $\mathcal{F} = \mathcal{N}$ and $\mathcal{F} \cup e \mathcal{F}$ whenever $e \mathcal{F} \in \mathcal{F}$, then

$$\{\mathcal{H}_n \mathcal{U}_n \mid \mathcal{U}_n \text{ open in } \mathcal{N}_n, \mathcal{H}_n \mathcal{U}_n \cap \mathcal{E}_p \} \cap \mathcal{E}_p$$

is a base for a topology $\tau(\mathcal{F})$ on $\mathcal{H}_p \mathcal{H}_n$. This topology is $\mathcal{T}_i$ if each factor is $\mathcal{T}_i$ for $i = 0, 1, 2, 3, 3$.

Questions. Is $\mathcal{E}_p$ (hereditarily) normal if all finite subproducts are (hereditarily) normal? Is $\mathcal{E}_p$ para-compact if $\mathcal{E}_p$ is an uncountable family of spaces such that all finite sub-products (i.e., $\bigcap X_i$) for finite subsets $F$ of $A$) are para-compact Hausdorff.

References

On the approximate Peano derivatives

by

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Abstract. It is known that a kth approximate Peano derivative belongs to Baire class 1. In the present paper it is shown that the other properties of the ordinary kth Peano derivative are also possessed by the kth approximate Peano derivative.

Introduction. Let a function \( f \) be defined in some neighbourhood of the point \( a_0 \). If there exist numbers \( a_1, a_2, \ldots, a_r \), depending on \( a_0 \), but not on \( h \) such that

\[
\limsup_{b \to 0} \frac{|f(a_0 + b) - f(a_0)|}{b} = 0,
\]

where \( \limsup \) denotes the approximate limit [13, p. 218], then \( a_r \) is called the approximate Peano derivative of \( f \) at \( a_0 \) of order \( r \) and is denoted by \( f_{r+1}(a_0) \) (see [4]). The definition is such that if \( f_{r+1}(a_0) \) exists then all the previous derivatives \( f_k(a_0) \) also exist and \( a_k = f_k(a_0), 1 \leq k < r \). It is convenient to write \( a_r = f_r(a_0) = f(a_0) \).

Let us now suppose that for a fixed \( r \), \( f_{r+1}(a_0) \) exists. Writing

\[
\frac{f^{r+1}(a_0)}{(r+1)!} \Phi_{r+1}(f; a_0, h) = f(a_0 + h) - \sum_{k=0}^{r+1} \binom{r+1}{k} f_k(a_0),
\]

\[
\limsup_{b \to 0} \phi_r(f; a_0, b) = f_{r+1}(a_0),
\]

\[
\liminf_{b \to 0} \phi_r(f; a_0, b) = f_{r+1}(a_0),
\]

\[
\limsup_{b \to 0} \phi_{r+1}(f; a_0, b) = f_{r+1}(a_0),
\]

\[
\liminf_{b \to 0} \phi_{r+1}(f; a_0, b) = f_{r+1}(a_0)
\]

where \( \limsup \) denotes the approximate upper limit [13, p. 218], \( f_{r+1}(a_0) \) and \( f_{r+1}(a_0) \) will be called the upper and the lower Peano derivatives of \( f \) at \( a_0 \) of order \( r+1 \). (The upper and the lower Peano derivatives as defined in [14, 1, 2, 3] are different from those defined here. For, in the former cases the existence of the Peano derivatives \( f_r(a_0) \) was required. However, the upper and the lower Peano derivatives in the former sense are also the upper and the lower Peano