The box product of countably many metrizable spaces need not be normal

by

Eric K. van Douwen (Delft)

Abstract. The box product of the family {irrationals} $\cup \{T_n|\ T_n$ is a convergent sequence} is not normal. If $\mathfrak{X} = \{X_\alpha|\ \alpha \in A\}$ is a family of metrizable spaces, the subspace $\Xi_p = \{x_\alpha\ |x_\alpha \neq p_\alpha$ for at most finitely many $\alpha\}$ of the box product of this family is stratifiable, $p \in \Pi_\alpha X_\alpha$ arbitrary. If the family $\mathfrak{X}$ is countable and all finite subproducts are paracompact, Ξ_p is paracompact.

1. Introduction. If $\{X_\alpha|\ \alpha \in A\}$ is a family of spaces, we denote the usual product space by $\Pi_\alpha X_\alpha$ and the box product (see [5, p. 107]) by $\boldsymbol{B}_\alpha X_\alpha$. Stone asked whether $\boldsymbol{B}_\alpha X_\alpha$ is normal if each X_α is metrizable, [6]. A partial answer has been given by Rudin, who showed that the continuum hypothesis implies that $\boldsymbol{B}_n X_n$ is paracompact provided each X_n is a locally compact metrizable space, [10]. (Actually this was stated under the additional hypothesis that the X_n are σ-compact. However, a locally compact metrizable space X_n is the union of a disjoint open family $\{X_{n\alpha}|\ \alpha \in A_n\}$ consisting of locally compact σ-compact subspaces, cf. [11], so $\boldsymbol{B}_n X_n$, being the union of the disjoint open family

$$\{\boldsymbol{B}_n X_{n\alpha(n)}|\ \alpha(n) \in A_n \text{ for } n \in N\},$$

is paracompact.)

We show that the product of countably many separable metrizable spaces need not be normal, even if all factors but one are compact (the noncompact factor is the space of irrationals). Quite surprisingly the proof that our space is not normal, resembles Michael's proof that the product of the irrationals and a certain space is not normal, [7]. This negative solution of Stone's question also solves a question of Borges, whether a box product of metrizable spaces is stratifiable, [2], in the negative and kills a conjecture of Vaughan, that a product of linearly stratifiable spaces is paracompact [12]. As a byproduct we show that a box product of metrizable spaces cannot be hereditarily normal if infinitely many factors are nondiscrete.

Certain subspaces of box products are behaving better. For $p \in \boldsymbol{B}_\alpha X_\alpha$ let Ξ_p be the subspace $\{x \in \boldsymbol{B}_\alpha X_\alpha |\ x_\alpha \neq p_\alpha$ for at most finitely many $\alpha\}$. (The component of p is contained in Ξ_p, provided the X_α's are regular T_1, [6, p. 51]. If each X_α is a group, with identity p_α, Ξ_p is the so-called direct sum.) Then Ξ_p is stratifiable if the X_α's are metrizable. If $\{X_n |\ n \in N\}$ is a countable family of spaces, Ξ_p is paracompact if each finite subproduct is paracompact.

It is shown in [6] that $\boldsymbol{B}_\alpha X_\alpha$ is T_i iff each factor is T_i for $i = 0, 1, 2, 3, 3\frac{1}{2}$. N is the set of positive integers.

2. Non-normal products. For $n \in N$ let T_n be the space $T = \{x |\ x = 0$ or $1/x \in N\}$, let $B = \boldsymbol{B}_n T_n$ and let P be the discrete open subspace $\{x \in B |\ x_n \neq 0$ for $n \in N\}$.

Assertion. *P is not an F_σ-subset of B.*

Proof. Let $F_n \subset P$ be a closed subset of B. Define a sequence $x_1, x_2, \ldots$ by induction as follows: $x_1 = 1$. Assume that $x_i \in T \setminus \{0\}$ is known for $1 \leqslant i \leqslant n$ and that a strictly positive function ε_i on N is known if $1 \leqslant i \leqslant n-1$ such that $F_i \cap U_i = \emptyset$, where

$$U_i = \prod_{j=1}^{i} \{x_j\} \times \prod_{j=i+1}^{\infty} \{t \in T_j |\ t < \varepsilon_i(j)\}.$$

The point $(x_1, x_2, \ldots, x_n, 0, 0, \ldots, 0, \ldots)$ does not belong to P, hence there is a strictly positive function ε_n on N such that $U_n \cap F_n = \emptyset$. Pick an $x_{n+1} \in T \setminus \{0\}$ such that $x_{n+1} < \varepsilon_i(n+1)$ for $1 \leqslant i \leqslant n$. This completes the definition. Let x be the point $(x_1, x_2, \ldots, x_n, \ldots)$. Then $x \in P$ and $x \notin F_n$ for $n \in N$, hence $P \neq \bigcup \{F_n |\ n \in N\}$.

Corollary. *A box product with infinitely many nondiscrete metrizable spaces is not hereditarily normal.*

Proof. Such a product contains a (closed) subspace homeomorphic to B. B and $T \times B$ are homeomorphic. By a theorem of Katětov a product $X \times Y$ is hereditarily normal only if every countable subset of X is closed or Y is perfectly normal [4]. Hence $T \times B$ is not hereditarily normal (1).

Example. A box product of countably many metrizable spaces which is not normal.

Let B^* be the space $\Pi_n T_n$ and let P^* be the subspace $\{x \in B^* |\ x_n \neq 0$ for $n \in N\}$, let d be a metric for B^*. Then B and B^* have the same underlying set, and so do P and P^*. Observe that P^* is homeomorphic to the irrationals. Let $T_0 = P^*$, then $P^* \times B$ and $\boldsymbol{B}\{T_n |\ n = 0$ or $n \in N\}$ are the same space. We claim that $P^* \times B$ is not normal.

(1) K. Kunen has independently found a different proof that B is not hereditarily normal, *Some comments on box products*, Coll. Math. Soc. János Bolyai 10, Keszthely, Hungary, 1973.

$F = \{(x, x) |\ x \in P\}$ and $G = P^* \times (B \setminus P)$ are disjoint closed subsets of $P^* \times B$. Let U and V be neighborhoods of F and G respectively. For $x \in P^*$ and $\varepsilon > 0$ define $S(x, \varepsilon) = \{y \in P^* |\ d(x, y) < \varepsilon\}$. Define

$$P_n = \{x \in P |\ S(x, 1/n) \times \{x\} \subset U\}.$$

Then $P = \bigcup_n P_n$, hence by the assertion there is a $q \in B \setminus P$ and an $n \in N$ such that $q \in P_n^-$ in the space B, hence also in B^*. Pick a $p \in P^*$ such that $d(p, q) < 1/2n$. There are an $\varepsilon > 0$ and a neighborhood W of q in B such that $S(p, \varepsilon) \times W \subset V$.

Choose an $r \in P_n$ such that $r \in W$ and $d(q, r) < 1/2n$. Then $S(r, 1/n) \times \{r\} \subset U$, hence $(p, r) \in U$ since $d(p, r) < d(p, q) + d(q, r) < 1/n$. But also $(p, r) \in V$ since $(p, r) \in S(p, \varepsilon) \times W$. Consequently $U \cap V \neq \emptyset$, so $P^* \times B$ is not normal.

The space T can be embedded as a closed subspace in the irrationals. It follows that $\boldsymbol{B}\{X_n |\ X_n = \{\text{irrationals}\}\}$ is not normal.

Problem. For what kind of metrizable spaces X is the product $\boldsymbol{B}\{X_n |\ X_n = X\}$ normal or paracompact? Are these problems the same, cf. [9]?

In [10] conditions are given under which a box product is normal.

3. Stratifiable direct sums. A T_1-space X is said to be *stratifiable* if there is a function G: $\{\text{closed subsets}\} \times N \to \{\text{open subsets}\}$ such that (a) $F = \bigcap_{n \in N} G(F, n) = \bigcap_{n \in N} G(F, n)^-$ and (b) $G(E, n) \subset G(F, n)$ whenever $E \subset F$, see [1], where a dual formulation is used. The following useful characterization and proof are due to Heath [3].

Lemma. *A T_1-space X is stratifiable iff there is a function g: $X \times N \to \{$open subsets$\}$ such that* (a) *$x \in g(x, n)$ and* (b) *given any closed subset M of X and any point $q \in X \setminus M$, there is an n such that $q \notin (\bigcup \{g(x, n) |\ x \in M\})^-$.*

Proof. Given G, define $g(x, n) = G(\{x\}, n)$, and given g, define $G(F, n) = \bigcup \{g(x, n) |\ x \in F\}$.

Theorem. *If $\{X_\alpha |\ \alpha \in A\}$ is a family of metrizable spaces, Ξ_p is stratifiable for each $p \in \boldsymbol{B}_\alpha X_\alpha$.*

Proof. Let d be a metric on each X_α (this will cause no confusion) and let E be the set of all strictly positive real-valued functions on A. Basic neighborhoods of $x \in \Xi_p$ are $S(x, \varepsilon)$ for $\varepsilon \in E$, where

$$S(x, \varepsilon) = \{y \in \Xi_p |\ d(x_\alpha, y_\alpha) < \varepsilon(\alpha) \text{ for } \alpha \in A\}.$$

For $x \in \Xi_p$ and $\varepsilon \in E$ define

$$A(x) = \{\alpha \in A |\ x_\alpha \neq p_\alpha\},$$
$$\Delta(x) = \min\big(\{d(x_\alpha, p_\alpha) |\ \alpha \in A(x)\} \cup \{1\}\big),$$
$$\Delta(x; \varepsilon) = \min\big(\{\varepsilon(\alpha) |\ \alpha \in A(x)\} \cup \{1\}\big).$$

Define $\mu \in E$ by $\mu(\alpha) = 1$ for $\alpha \in A$. Then the function $g\colon \Xi_p \times N \to \{\text{open subsets}\}$ of the lemma can be defined by

$$g(x, n) = S\big(x, (\Delta(x)/n)\cdot\mu\big).$$

Let M be a closed subset of Ξ_p and let q be a point of $\Xi_p \setminus M$. There is an $\varepsilon \in E$ such that $S(q, \varepsilon) \cap M = \emptyset$. Define $\delta \in E$ by

$$\delta(\alpha) = \tfrac{1}{2}\varepsilon(\alpha).$$

Fix $n \in N$ so that $1/n \leqslant \frac{1}{2}\Delta(q, \varepsilon)$ (then $n \geqslant 2$). We claim that $S(q, \delta) \cap g(x, n) = \emptyset$ for $x \in \Xi_p \setminus S(q, \varepsilon)$. This shows that g satisfies (b).

Let x be any point of $\Xi_p \setminus S(q, \varepsilon)$. Then $d(x_\alpha, q_\alpha) \geqslant \varepsilon(\alpha)$ for some α.

Case 1. $\alpha \in A(q)$. If $z \in S(q, \delta)$ then

$$d(q_\alpha, z_\alpha) < \delta(\alpha) = \tfrac{1}{2}\varepsilon(\alpha)$$

and if $z \in g(x, n)$ then

$$d(z_\alpha, x_\alpha) < 1/n \leqslant \tfrac{1}{2}\Delta(q, \varepsilon) \leqslant \tfrac{1}{2}\varepsilon(\alpha).$$

Therefore $S(q, \delta) \cap g(x, n) = \emptyset$.

Case 2. $\alpha \in A \setminus A(q)$. Then $\alpha \in A(x)$. So if $z \in S(q, \delta)$ then

$$d(q_\alpha, z_\alpha) < \delta(\alpha) = \tfrac{1}{2}\varepsilon(\alpha) \leqslant \tfrac{1}{2}d(x_\alpha, q_\alpha)$$

and if $z \in g(x, n)$ then

$$d(z_\alpha, x_\alpha) < (1/n)\Delta(x) \leqslant \tfrac{1}{2}d(x_\alpha, p_\alpha) = \tfrac{1}{2}d(x_\alpha, q_\alpha).$$

Therefore $S(q, \delta) \cap g(x, n) = \emptyset$.

This can be used for an easy to describe nonmetrizable countable stratifiable space without isolated points. If $p_n = 0$ for $n \in N$, the subspace Ξ_p of $\boldsymbol{B}_n\{Q_n |\ Q_n = \{\text{rationals}\} \text{ for } n \in N\}$ has all properties required. Of course Ξ_p is a topological group under coordinatewise addition.

Question. Is the theorem valid if one merely assumes that the X_α's are stratifiable?

4. Countable direct sums. If $\{X_n |\ n \in N\}$ is a family of spaces, $p \in \boldsymbol{B}_n X_n$, then let R_n be the subspace $\{x \in \Xi_p |\ x_k = p_k \text{ for } k > n\}$ of Ξ_p. Then R_n is a retract of Ξ_p, which is closed if the X_n are T_1 (a retraction $r_n\colon \Xi_p \to R_n$ can be naturally defined by $\big(r_n(x)\big)_k = x_k$ if $k \leqslant n$, $\big(r_n(x)\big)_k = p_k$ if $k > n$). Obviously $\{R_n |\ n \in N\}$ is a countable cover of Ξ_p. Therefore Ξ_p often has a property if each finite subproduct $\prod_{i=1}^{n} X_i$ has, e.g. the properties Lindelöf, hereditarily Lindelöf and the property of being perfect (i.e. open sets are F_σ) and T_1. It is also easy to see that Ξ_p is cosmic/a σ-space if each factor is (see [8] for definitions and references). The following theorem is a bit less trivial.

Theorem. *Ξ_p is a paracompact Hausdorff iff $\prod_{i=1}^{n} X_i$ is paracompact Hausdorff for $n \in N$ (we only consider countable direct sums).*

Proof. Only the sufficiency requires proof. Each X_n is regular T_1, hence so is Ξ_p. Let $\mathcal{U}$ be an open cover of Ξ_p. For each $n \in N$ there is a locally finite open cover $\mathcal{V}_n$ of the subspace R_n which refines $\{U \cap R_n |\ U \in \mathcal{U}\}$. For $n \in N$ the family $\{r_n^{-1}(V) |\ V \in \mathcal{V}_n\}$ is locally finite in Ξ_p: For $x \in \Xi_p$ there is a neighborhood W of $r_n(x)$ in R_n which intersects only finitely many members of $\mathcal{V}_n$. Then $r_n^{-1}(W)$ is a neighborhood of x in Ξ_p which intersects only finitely many members of $\{r_n^{-1}(V) |\ V \in \mathcal{V}_n\}$. For $V \in \mathcal{V}_n$ choose a $c_n(V) \in \mathcal{U}$ such that $V \subset c_n(V)$. Then

$$\bigcup\big\{\{r_n^{-1}(V) \cap c_n(V) |\ V \in \mathcal{V}_n\} |\ n \in N\big\}$$

is a σ-locally finite refinement of $\mathcal{U}$. Consequently Ξ_p is paracompact.

Remark. There are many topologies on the set $\Pi_n X_n$ between the usual topology and the box topology: If $\mathcal{F}$ is a collection of subsets of N such that $\bigcup \mathcal{F} = N$ and $F \cup G \in \mathcal{F}$ whenever $F, G \in \mathcal{F}$, then

$$\{\Pi_n U_n |\ U_n \text{ open in } X_n,\ \{n \in N |\ U_n \neq X_n\} \in \mathcal{F}\}$$

is a base for a topology $\tau(\mathcal{F})$ on $\Pi_n X_n$. This topology is T_i iff each factor is T_i for $i = 0, 1, 2, 3, 3\frac{1}{2}$. The subspace Ξ_p is paracompact Hausdorff if each finite subproduct is, for all these topologies $\tau(\mathcal{F})$.

Questions. Is Ξ_p (hereditarily) normal if all finite subproducts are (hereditarily) normal? Is Ξ_p paracompact if $\{X_\alpha |\ \alpha \in A\}$ is an uncountable family of spaces such that all finite subproducts (i.e. $\prod\{X_\alpha |\ \alpha \in F\}$ for finite subsets F of A) are paracompact Hausdorff?

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DELFT INSTITUTE OF TECHNOLOGY
Delft, Netherlands
UNIVERSITY OF PITTSBURGH
Pittsburgh, Pa.

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On the approximate Peano derivatives

by

S. N. Mukhopadhyay (Burdwan, West Bengal)

Abstract. It is known that a kth approximate Peano derivative belongs to Baire class 1. In the present paper it is shown that the other properties of the ordinary kth Peano derivative are also possessed by the kth approximate Peano derivative.

Introduction. Let a function f be defined in some neighbourhood of the point x_0. If there exist numbers $a_1, a_2, \ldots, a_r$, depending on x_0 but not on h such that

$$(*) \qquad \lim_{h\to 0} \operatorname{ap} \frac{r!}{h^r}\left\{f(x_0+h)-f(x_0)-\sum_{k=1}^{r}\frac{h^k}{k!}a_k\right\}=0\,,$$

where lim ap denotes the approximate limit [13, p. 218], then a_r is called the *approximate Peano derivative* of f at x_0 of order r and is denoted by $f_{r,a}(x_0)$ (see [4]). The definition is such that if $f_{r,a}(x_0)$ exists then all the previous derivatives $f_{k,a}(x_0)$ also exist and $a_k = f_{k,a}(x_0)$, $1 \leqslant k < r$. It is convenient to write $a_0 = f_{0,a}(x_0) = f(x_0)$.

Let us now suppose that for a fixed r, $f_{r,a}(x_0)$ exists. Writing

$$\frac{h^{r+1}}{(r+1)!}\Phi_{r+1}(f;\, x_0, h) = f(x_0+h) - \sum_{k=0}^{r}\frac{h^k}{k!} f_{k,a}(x_0)\,,$$

$$\limsup_{h\to 0}\Phi_{r+1}(f;\; x_0, h) = \bar{f}_{r+1}(x_0)\,, \qquad \liminf_{h\to 0}\Phi_{r+1}(f;\; x_0, h) = \underline{f}_{r+1}(x_0)\,,$$

$$\limsup_{h\to 0}\operatorname{ap}\Phi_{r+1}(f;\; x_0, h) = \bar{f}_{r+1,a}(x_0)\,, \qquad \liminf_{h\to 0}\operatorname{ap}\Phi_{r+1}(f;\; x_0, h) = \underline{f}_{r+1,a}(x_0)$$

where lim sup ap denotes the approximate upper limit [13; p. 218], $\bar{f}_{r+1}(x_0)$ and $\underline{f}_{r+1}(x_0)$ will be called the *upper* and the *lower Peano derivates* of f at x_0, while $\bar{f}_{r+1,a}(x_0)$ and $\underline{f}_{r+1,a}(x_0)$ will be called the *upper* and the *lower approximate Peano derivates* of f at x_0 of order $r+1$. (The upper and the lower Peano derivates as defined in [14, 1, 2, 3] are different from those defined here. For, in the former cases the existence of the Peano derivatives $f_r(x_0)$ was required. However, the upper and the lower Peano derivates in the former sense are also the upper and the lower Peano