It is easily verified that the union of members of a chain in $(R, \subseteq)$ is also in $R$. (Here $\subseteq$ is the set-theoretic inclusion.) By Zorn's lemma, therefore, $(R, \subseteq)$ has a maximal element, say $J$. We show that $J$ is an ideal. In view of (iii) this only requires showing that $x \in J$, $y \notin J \to xy \in J$. Thus if we write $K$ for the set of $y \notin J$ such that $xy \notin J$ for some $x \in J$ then we need to show that $K$ is empty.

Suppose $K$ is not empty and let $k \in K$, $j \in J$, $jk \notin J$. Then $j \vee k \notin J$, by (ii). Since $j \vee k = j(jk)$ it follows that $jk \in K$. Consider the two subsets of $B$:

$$J_1 = \{x;\ x \leqslant y \vee k,\ y \in J\},$$
$$J_2 = \{x;\ x \leqslant y \vee jk,\ y \in J\}.$$

Since $k, jk \notin J$ the sets $J_1$, $J_2$ both properly contain $J$. We complete the proof of the lemma by showing that at least one of the two sets $J_1$, $J_2$ satisfies (i), (ii), (iii), and thus arriving at a contradiction to the maximality of $J$.

Clearly $a \in J_1, J_2$. Suppose that $b \in J_1, J_2$, so that $b \leqslant y_1 \vee k$, $y_2 \vee jk$ for some $y_1, y_2 \in J$. If we write $z = y_1 \vee y_2 \vee j$ then $b \leqslant z \vee k$, and by $(\alpha_1)$, $(\alpha_5)$, $b \leqslant z \vee zk = zk$. By $(\alpha_2)$, then, $zb \geqslant z(zk) = z \vee k \geqslant b$. Hence $b \vee zb = zb$, which by $(\alpha_3)$ gives $zb = 1$. Then $z(zb) = z1 = z$, by (2). Hence $b \vee z = z$. By (ii) this implies $b \in J$, a contradiction. Hence $b$ does not belong to both $J_1$ and $J_2$, and one of $J_1, J_2$ satisfies (i), say $J_1$. It is easy to see that $J_1$ satisfies (ii). If $xy, y \in J_1$, then by (ii), $(\alpha_3)$ and $(\alpha_4)$ we have $1 = y \vee xy \in J_1$ and $J_1 = B$. This however is not possible since $b \notin J_1$. Hence $xy \in J_1 \to y \notin J_1$. Also, by (ii) and $(\alpha_5)$, $xy \in J_1 \to x \in J_1$. Hence $J_1 \in R$ and the lemma is proved.

Now the proof of Theorem 3 can be concluded as follows: By $(\beta_3)$ of Lemma 2 and Lemma 3, $(B, .)$ is embeddable in a cartesian power of the two element Boolean groupoid. But cartesian powers and subgroupoids of Boolean groupoids are themselves Boolean. Hence $(B, .)$ is a Boolean groupoid.

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# A characterization of locally compact fields II

by

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**Abstract.** Let $(K, \mathfrak{T})$ be a non-discrete topological field. Define the Krull topology in the group $G(K)$ of all its continuous automorphisms, i.e. take for a base of the zero neighbourhoods all groups $G(K) \cap G(K/M)$ for finitely generated extensions $M$ of the fixed field of $G(K)$. It is shown that $K$ is locally compact if and only if $K$ is locally bounded and complete and, for every closed subfield $F$ of $K$, $G(F)$ is compact in its Krull topology.

**0.** In my previous paper [15] I gave a characterization of locally compact fields of zero characteristic. The aim of this paper is to give a characterization of all locally compact fields. At first let us recall some definitions. For any topological field $(F, \mathfrak{T})$ we write $G(F)$ for the group of all its continuous automorphisms. Let $L/K$ be a field extension and let us denote by $G(L/K)$ the Galois group of $L$ over $K$. If $G$ is a subgroup of $G(L/K)$ we shall introduce a group topology in $G$ taking for a base of the zero neighbourhoods in $G$ all sets of the form $G \cap G(L/M)$, where $M$ is a finitely generated extension of the fixed field $K'$ of $G$, i.e. $M = K'(X_1, X_2, \ldots, X_s)$, $X_j \in L$ for $j = 1, 2, \ldots, s$ (algebraic over $K'$ or not). We shall call such topology in $G$ the *Krull topology* in $G$. Let $(K, \mathfrak{T})$ be a topological field. A field topology $\mathfrak{T}$ is said to be *locally bounded* if there exists a bounded neighbourhood $A$ of zero, i.e. if for every neighbourhood $U$ of zero there exists another one, $V$, such that $AV \subset U$.

**1.** The aim of this paper is to prove the following

THEOREM. *Let $(K, \mathfrak{T})$ be a non-discrete topological field. Then the following conditions are equivalent:*

(1) *$K$ is a locally bounded, complete field and, for every closed subfield $F$ of $K$, $G(F)$ is compact in its Krull topology,*

(2) *$K$ is a locally compact field,*

(3) *$K$ is a finite extension either of the reals $\mathbf{R}$, of a $p$-adic number field $\mathbf{Q}_p$, or of some formal power series field over the prime field $\mathbf{Z}_p$ (i.e. a finite extension either of $\mathbf{Z}_p\langle x\rangle$ or $\mathbf{Z}_p\{x\}$).*

P r o o f  o f  t h e  t h e o r e m. The equivalence (2) $\Leftrightarrow$ (3) is the classical theorem of Pontryagin–Kowalsky–van Dantzig (see [6]).

(3) ⇒ (1). Suppose at first that $K$ has zero characteristic. Since every automorphism of $\boldsymbol{R}$ and $\boldsymbol{Q}_p$ is trivial, $G(K)$ is finite as a subgroup of the Galois group $G(K/\boldsymbol{R})$ (resp. $G(K/\boldsymbol{Q}_p)$), $G(K)$ is compact in its (discrete) Krull topology. Moreover, $K$ is complete in a locally bounded field topology induced by a real norm which extends either the absolute value $|a|$ from $\boldsymbol{R}$ or the $p$-adic norm from $\boldsymbol{Q}_p$.

Suppose now that $K$ has characteristic $p \neq 0$. Then $K$ is a finite extension of some Laurent series field over $\boldsymbol{Z}_p$. But then $G(K) = \mathrm{Aut}(K)$, where by $\mathrm{Aut}(K)$ we mean the group of all automorphisms of $K$. Moreover, $G(K)$ is compact in its Krull topology, as follows from [11] (Corollary 2 from Theorem 2).

It remains to show that (1) ⇒ (3).

Case I. $K$ is of zero characteristic.

Since $K$ is complete in a locally bounded field topology, then it follows from [8] (Theorem 3) that the closure of $Q$ in $K$ either equals $\boldsymbol{R}, \boldsymbol{Q}_p$ or is a discrete subfield of $K$. If $\boldsymbol{R}$ is a subfield of $K$, then $K = \boldsymbol{R}$ or $K = \boldsymbol{C}$, since $\boldsymbol{R}$ and $\boldsymbol{C}$ are the only locally bounded extensions of $\boldsymbol{R}$ ([8], Theorem 5).

It remains to consider the case $\boldsymbol{Q}_p \subset K$. Indeed, the case where $K$ contains a discrete subfield $F$ never arises. Suppose, to the contrary, that $F$ is such a field. Then $K$ is not an algebraic extension of $F$, since otherwise $\mathcal{T}$ should be discrete on every finitely generated extension of $F$ contained in $K$, and, finally, discrete on $K$, since $K$ is the union of such extensions. A contradiction. Hence let $x \in K$ be transcendental over $F$. By [15] (Lemma 3) the closure $L$ of $F(x)$ in $\mathcal{T}$ is a Laurent series field in $y$, $y = x^{-1}$ or $y = p(x)$, $p(x) \in F[x]$ — irreducible over $F$. This implies that the topology $\mathcal{T}|_L$ is induced by a non-Archimedean norm, say $|a|$. Let us note that $G(L) = \mathrm{Aut}(L)$, i.e. every automorphism of $L$ is continuous. Otherwise, if $\varphi$ were any discontinuous automorphism of $L$, then $L$ should be complete in the norm $|a|_\varphi$ defined as $|a|_\varphi = |\varphi(a)|$ for all $a \in L$, since $L$ is complete in the norm $|a|$. Since $L$ is not algebraically closed, Schmidt's theorem [13] should give the equivalence of the norms $|a|_\varphi$ and $|a|$, contradicting the discontinuity of $\varphi$. But $G(L)$ is not a compact group in its Krull topology, since for an infinite set $A \subset F$, $0 \notin A$, the net of automorphisms $\varphi_a \in G(L)$, $\varphi_a(x) = ax$, has no convergent subnets.

Now let us consider the case $\boldsymbol{Q}_p \subset K$. Let us remark that the topology $\mathcal{T}$ is induced in $K$ by a non-Archimedean pseudonorm. Indeed, since $\boldsymbol{Q}_p \subset K$ topologically and $p^n \to 0$ in $\mathcal{T}$ as $n \to \infty$, the set $T$ of all topological nilpotents in $L$ is non-void, whence open (see [14], Lemma 5). By [2] (Theorem 6.1′) the topology $\mathcal{T}$ is induced by a pseudonorm.

If $K$ is algebraic over $\boldsymbol{Q}_p$, there is nothing to prove: $K$, being a pseudonormed complete algebraic extension of the normed field $\boldsymbol{Q}_p$, must be its finite extension in view of [5] (Theorem 9) and in view of Abel's theorem on primitive element (compare [15], Lemma 2).

We claim that the extension $K$ of $\boldsymbol{Q}_p$ is algebraic. Suppose, to the contrary, that $x \in K$ is transcendental over $\boldsymbol{Q}_p$ and put $M = \boldsymbol{Q}_p(x)$. Let $R_p$ be the ring of integers of $\boldsymbol{Q}_p$, i.e. $R_p = \{a \in \boldsymbol{Q}_p\colon |a|_p \leqslant 1\}$. The topology $\mathcal{T}|_M$ is induced by the open $R_p[x]$ — submodules in $M$. Indeed, let $V_\varepsilon = \{m \in M\colon |m| < \varepsilon\}$, $|m|$ — a pseudonorm defining $\mathcal{T}$ in $M$. This pseudonorm must be non-Archimedean as a pseudonorm extending the $p$-adic norm. Suppose that $|x| \leqslant 1$. If $|x| > 1$, let us take any element $y \in M$, transcendental over $\boldsymbol{Q}_p$ and satisfying $|y| \leqslant 1$, and replace $\boldsymbol{Q}_p(x)$ by $\boldsymbol{Q}_p(y)$. Such an element $y$ must exist since the pseudonorm is not trivial on $M$. Since $V_\varepsilon$ is a subgroup of $M$, $y, z \in V_\varepsilon$ implies $y - z \in V_\varepsilon$. Moreover, for $a \in R_p[x]$, $w \in V_\varepsilon$,

$$a = a_0 + a_1 x + \ldots + a_N x^N,$$

$$|aw| = |a_0 w + a_1 wx + \ldots + a_N wx^N| \leqslant \max_{0 \leqslant s \leqslant N} \{|a_s wx^s|\} < \varepsilon .$$

Note that $M$ is the field of fractions of $R_p[x]$. Since $R_p[x]$ is a unique factorization domain, by [3] $\mathcal{T}$ is the supremum of a family of topologies induced by the real non-Archimedean norms in $M$. But $\mathcal{T}$ is locally bounded, and hence this family must be finite [6] (Satz, p. 177). Finally, the approximation theorem for valuations implies that this family consists of a single element (compare [15], p. 150). Consider now the closure $\overline{M}$ of $M$ in $K$. Let us note that $\mathrm{Aut}(\overline{M}) = G(\overline{M})$ (for a proof see page 124 of this paper), and, moreover, $G(\overline{M}) = G(\overline{M}/\boldsymbol{Q}_p)$, since every automorphism of $\boldsymbol{Q}_p$ is trivial. Applying [10] (Proposition (1.3)) one sees that $G(\overline{M})$ is not compact, since $\overline{M}$ is not algebraic over $\boldsymbol{Q}_p$. This contradicts the assumptions of our theorem.

Case II. $K$ is of characteristic $p \neq 0$.

Let us remark at first that there exists an element $x \in K$, transcendental over $\boldsymbol{Z}_p$ and such that the topology $\mathcal{T}_1 = \mathcal{T}|_{Z_p(x)}$ is non-discrete, since otherwise the topology $\mathcal{T}$ would be discrete on $K$ (compare [15], p. 153).

It follows from [15] (Lemma 3) that the topology $\mathcal{T}_1$ is induced in $\boldsymbol{Z}_p(x)$ by a real non-Archimedean norm and $K$ contains as a closed subfield one of the following formal power series fields: either $\boldsymbol{Z}_p\langle x\rangle$ or $\boldsymbol{Z}_p\{x\}$. For brevity, let $k_0 = \overline{\boldsymbol{Z}_p(x)}$. Let $k$ be the field of invariants of the compact group $G(K)$. By Proposition (1.6) of [10] $K$ is an algebraic extension of $k$. Note that the extension $k$ of $k_0$ is also algebraic. Indeed, otherwise there would be an element $y \in k$, transcendental over $k_0$. Let us put $M = k_0(y)$. Let $R_p$ be a ring of the integers of $k_0$. As in Case I, one shows that $\mathcal{T}|_M$ is induced by a non-Archimedean norm and that $\mathrm{Aut}(\overline{M}) = G(\overline{M})$. Let $U(\overline{M})$ be the group of units in the ring of the integers of $\bar{M}$ for the norm inducing $\mathcal{C}$ in $\bar{M}$. Let $A \subset U(\bar{M})$ be an infinite subset containing no convergent subsequences. (Such a subset must exist since $U(\bar{M})$ is not compact in a topology induced from $\bar{M}$). Then putting for any $f(y) \in \bar{M}$, $\varphi_a(f(y)) = f(ay)$, we should obtain a net $\{\varphi_a\}$ in a compact $G(\bar{M})$, having no convergent subnets. A contradiction. Finally, $K$ is an algebraic extension of the complete normed field $k_0$. But $K$, being a pseudonormed, complete algebraic extension of $k_0$, is of bounded degree by [5] (Theorem 9), i.e. the degrees of all elements of $K$ over $k_0$ are bounded in common.

In order to finish the proof in Case II we shall need the following

LEMMA ([1]). *Let $\widetilde{F}$ be an algebraic closure of a non-Archimedean valued field $F$. Denote by $\widetilde{F}^s$ a separable algebraic closure of $F$ in $\widetilde{F}$ and extend the norm from $F$ to $\widetilde{F}$ in any way. Then $\widetilde{F}^s$ lies dense in $\widetilde{F}$.*

Let us put $F = k_0$, and note that $\mathcal{C}$ is equivalent to a topology induced by a norm extending the norm of $k_0$, since they are equivalent on every finite extension of $k_0$ (and then the topology is the product topology). If $K_s = K \cap \widetilde{k}_0^s$, then by the Lemma $K_s$ lies dense in $K$. Since the extension $K_s$ over $k_0$ is separable algebraic with the elements of bounded degrees, then by Abel's theorem (see [15]) on the primitive element, $[K_s : k_0]$ is finite. But, since $k_0$ is complete, the topology of $K_s$ is the product topology induced from $k_0$; hence $K_s$ is a closed subfield of $K$. Since, moreover, $K_s$ lies dense in $K$, we must have $K = K_s$, and, $[K : k_0]$ is finite.

This proves the theorem.

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