On generalizations of Borsuk's homotopy extension theorem

by

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Abstract. As generalizations of Borsuk's theorem C. H. Dowker established two theorems concerning the homotopy extension property for closed subsets of a normal or collectionwise normal space X for which $X\times I$ is also normal. In this note it will be shown that his theorems are true without assuming the normality of $X\times I$.

1. Introduction. Let X be a topological space. A subspace A of X is said to have the *homotopy extension property* (abbreviated to HEP) in X with respect to a topological space Y if every partial homotopy

$$h_t\colon A\to Y \quad (0\leqslant t\leqslant 1)$$

of an arbitrary continuous map $f\colon X\to Y$ has an extension

$$f_t\colon X\to Y \quad (0\leqslant t\leqslant 1)$$

such that $f_0=f$.

The original homotopy extension theorem of Borsuk (cf. [2] and [5]) has been generalized as follows by C. H. Dowker [4].

THEOREM 1. *Let Y be an ANR for metric spaces which is separable and Čech complete. Then every closed subspace A of a countably paracompact normal space X has the HEP in X with respect to Y.*

THEOREM 2. *Let Y be an ANR for metric spaces. Then every closed subspace A of a countably paracompact, collectionwise normal space X has the HEP in X with respect to Y, if either A is a G_δ set or Y is Čech complete.*

The purpose of this note is to give a further generalization to each of these theorems.

Let m be an infinite cardinal number.

A subset A of a topological space X is said to be *P^m-embedded* (resp. *P-embedded*) if for every locally finite cozero-set cover $\mathfrak{V}$ of A of cardinality $\leqslant m$ (resp. of any cardinality) there exists a locally finite cozero-set cover $\mathfrak{U}$ of X of cardinality $\leqslant m$ (resp. of some cardinality) such that $\mathfrak{V}$ is refined by $\mathfrak{U}\cap A$, where $\mathfrak{U}\cap A=\{U\cap A\mid U\in\mathfrak{U}\}$.

Then the following results are known.

THEOREM 3 (Dowker [3]). *Every closed subset A of a collectionwise normal space is P-embedded.*

THEOREM 4 (Shapiro [9]). *If a subset A of a topological space X is C-embedded in X, then A is $P^{\aleph_0}$-embedded in X.*

The notion "P^m-embedded" in our sense is the same as "P^m-embedded" in the sense of H. L. Shapiro [9] which was introduced by R. Arens [1] under the name "m-normally embedded". This fact which is proved in [9], however, is not needed in the present paper.

Our main theorems are now stated as follows.

THEOREM 5. *Let Y be an ANR for metric spaces which is Čech complete and has weight $\leqslant m$. If a subspace A of a topological space X is P^m-embedded in X, then A has the HEP in X with respect to Y.*

THEOREM 6. *Let Y be an ANR for metric spaces which has weight $\leqslant m$. If A is a zero-set of a topological space X such that A is P^m-embedded in X, then A has the HEP in X with respect to Y.*

In view of Theorems 3 and 4, Theorems 1 and 2 are improved as follows by virtue of our Theorems 5 and 6. It is to be noted that there is a collectionwise normal Hausdorff space which is not countably paracompact (Rudin [8]).

THEOREM 7. *Let Y be a separable ANR for metric spaces. Then every closed subspace A of a normal space X has the HEP in X with respect to Y, if either A is a G_δ set or Y is Čech complete.*

THEOREM 8. *Let Y be an ANR for metric spaces. Then every closed set A of a collectionwise normal space X has the HEP in X with respect to Y, if either A is a G_δ set or Y is Čech complete.*

2. Proof of Theorem 5. Let N be the set of positive integers. By assumption on Y there is a normal sequence $\{\mathfrak{W}_i|\ i \in N\}$ of locally finite open covers of Y such that $\{\mathrm{St}(y, \mathfrak{W}_i)|\ i \in N\}$ is a local base at each point y of Y, the cardinality of $\mathfrak{W}_i$ does not exceed m, and such that Y is complete with respect to $\{\mathfrak{W}_i\}$.

Let us put

$$C = (X\times 0) \cup (A\times I)$$

where $I = [0, 1]$. Let

$$f\colon\ C \to Y$$

be a continuous map.

Then for each i there is a locally finite cozero-set cover $\mathfrak{L}_i = \{L_{i\lambda}|\ \lambda \in \Lambda_i\}$ of A with card $\Lambda_i \leqslant m$ such that $f^{-1}(\mathfrak{W}_i) \cap (A\times I)$ is refined by

$$\{L_{i\lambda}\times B|\ B \in \mathfrak{K}_{i\lambda},\ \lambda \in \Lambda_i\}$$

for suitable finite open covers $\mathfrak{K}_{i\lambda}$ of I. This is proved in our previous paper [7].

By assumption on A there is a locally finite cozero-set cover $\mathcal{M}_i$ of X of cardinality $\leqslant m$ such that $\mathcal{M}_i \cap A$ is a refinement of $\mathfrak{L}_i$. Here we may assume that $\mathcal{M}_i = \{M_{i\lambda}|\ \lambda \in \Lambda_i\}$ and $M_{i\lambda} \cap A \subset L_{i\lambda}$ for each λ.

Now, by induction on i we shall construct locally finite cozero-set covers $\mathfrak{U}_i = \{U_{i\alpha}|\ \alpha \in \Omega_i\}$ of X with card $\Omega_i \leqslant m$, $i \in N$, and families $\{\mathfrak{H}_{i\alpha}|\ \alpha \in \Omega_i\}$, $i \in N$, of finite open covers of I such that the following conditions (1) to (3) are satisfied where $\mathfrak{V}_i = \{U_{i\alpha}\times H|\ \alpha \in \Omega_i,\ H \in \mathfrak{H}_{i\alpha}\}$.

(1) $\mathfrak{H}_{i\alpha}$ consists of sets of diameter $< 1/i$.

(2) $\mathfrak{V}_i$ is a star-refinement of $\mathfrak{V}_{i-1}$.

(3) $\mathfrak{V}_i \cap (A\times I)$ and $\mathfrak{V}_i \cap (X\times 0)$ refine $f^{-1}(\mathfrak{W}_i) \cap (A\times I)$ and $f^{-1}(\mathfrak{W}_i) \cap (X\times 0)$ respectively.

Indeed, suppose that $\mathfrak{U}_{i-1}$ and $\mathfrak{H}_{i-1,\alpha}$, $\alpha \in \Omega_{i-1}$, are constructed. Then $\mathfrak{V}_{i-1}$ is a normal open cover of $X\times I$ of cardinality $\leqslant m$. Hence by [7, Theorem 2.5] there exist a locally finite cozero-set cover $\mathfrak{P} = \{P_\gamma|\ \gamma \in \Gamma\}$ of X with card $\Gamma \leqslant m$ and a family $\{\mathfrak{Q}_\gamma|\ \gamma \in \Gamma\}$ of finite open covers of I such that the cover

$$\{P_\gamma\times Q|\ Q \in \mathfrak{Q}_\gamma,\ \gamma \in \Gamma\}$$

of $X\times I$ is a star-refinement of $\mathfrak{V}_{i-1}$. Let $\mathfrak{U}_i = \{U_{i\alpha}|\ \alpha \in \Omega_i\}$ with card $\Omega_i \leqslant m$ be a locally finite cozero-set cover of X which refines $\mathfrak{P}$, $\mathcal{M}_i$ and $\varkappa^{-1}f^{-1}(\mathfrak{W}_i)$, where $\varkappa$ is a map from X to $X\times I$ defined by $\varkappa(x) = (x, 0)$ for $x \in X$.

For each $\alpha \in \Omega_i$ let us choose $\lambda \in \Lambda_i$ and $\gamma \in \Gamma$ so that $U_{i\alpha} \subset M_{i\lambda} \cap P_\gamma$, and define $\mathfrak{H}_{i\alpha}$ to be a finite cover of I by open sets of diameter $< 1/i$ which is a refinement of $\mathfrak{K}_{i\lambda}$ and $\mathfrak{Q}_\gamma$. Then it is easy to see that conditions (1) to (3) are satisfied.

Since $\mathrm{St}(U_{i\alpha}\times H, \mathfrak{V}_i) \supset \mathrm{St}(U_{i\alpha}\times t, \mathfrak{V}_i) \supset \mathrm{St}(U_{i\alpha}, \mathfrak{U}_i)\times t$ for $t \in H \in \mathfrak{H}_{i\alpha}$, the following holds by (3).

(4) $\mathfrak{U}_i$ is a star-refinement of $\mathfrak{U}_{i-1}$.

Therefore $\Phi = \{\mathfrak{U}_i|\ i \in N\}$ is a normal sequence of open covers of X.

Let (X, Φ) be a topological space obtained from X by taking $\{\mathrm{St}(x, \mathfrak{U}_i)|\ i \in N\}$ as a local base at each point x of X, and X/Φ the quotient space obtained from (X, Φ) by identifying two points x and y such that $y \in \mathrm{St}(x, \mathfrak{U}_i)$ for each $i \in N$. Let us denote by φ the composite of the identity map from X onto (X, Φ) and the quotient map from (X, Φ) onto X/Φ. Then $\varphi\colon X \to X/\Phi$ is a continuous map and the space X/Φ is metrizable. This fact is proved in [6]. As is seen from the arguments in [6], by replacing each $\mathfrak{U}_i$ by a suitable refinement we can assume that

(5) $\varphi^{-1}\varphi(U_{i\alpha}) = U_{i\alpha}$ is open in (X, Φ) for $\alpha \in \Omega_i$, $i \in N$.

Let us put $\Psi = \{\mathfrak{V}_i|\ i \in N\}$. Since Ψ is also a normal sequence of open covers of $X \times I$, similarly as above we can construct the spaces $(X \times I, \Psi)$, $(X \times I)/\Psi$ and the map $\psi\colon X \times I \to (X \times I)/\Psi$.

Let (x, t) be any point of $X \times I$. For $\delta > 0$, let us put

$$N(t;\ \delta) = \{t' \in I|\ |t - t'| < \delta\}.$$

Then by (1)

$$\mathrm{St}\big((x, t), \mathfrak{V}_i\big) \subset \mathrm{St}(x, \mathfrak{U}_i) \times N(t;\ 1/i).$$

Let $\{\alpha_1, \ldots, \alpha_n\}$ be the set of all $\alpha \in \Omega_i$ such that $x \in U_{i\alpha}$. Then there is a positive number δ such that if $t \in H$ for $H \in \mathcal{H}_{i\alpha_j}$ with $1 \leqslant j \leqslant n$ then $N(t;\ \delta) \subset H$. Hence we have

$$\mathrm{St}(x, \mathfrak{U}_i) \times N(t;\ \delta) \subset \mathrm{St}\big((x, t), \mathfrak{V}_i\big).$$

Therefore, we can conclude that

$$(X \times I)/\Psi = (X/\Phi) \times I, \quad \psi = \varphi \times 1.$$

Let us put $S = X/\Phi$. Then by the construction above there is a single-valued map

$$g\colon (S \times 0) \cup \big(\varphi(A) \times I\big) \to Y$$

such that f is the composite of $\varphi \times 1|C$ and g.

By (3) each of the maps

$$g_0 = g|S \times 0, \qquad g_1 = g|\varphi(A) \times I$$

is uniformly continuous when we regard $S \times 0$ and $\varphi(A) \times I$ as uniform subspaces of the space $S \times I$ with the uniformity $\{\psi(\mathfrak{V}_i)|\ i \in N\}$ and Y as a uniform space with the uniformity $\{\mathfrak{W}_i|\ i \in N\}$. Since Y is complete with respect to $\{\mathfrak{W}_i\}$, g_1 is extended uniquely to a continuous map

$$\tilde{g}_1\colon \mathrm{Cl}_S\varphi(A) \times I \to Y.$$

Since

$$(S \times 0) \cap \big(\mathrm{Cl}_S\varphi(A) \times I\big) = \mathrm{Cl}_{S \times I}\big(\varphi(A) \times 0\big),$$

a map $\tilde{g}\colon (S \times 0) \cup \big(\mathrm{Cl}_S\varphi(A) \times I\big) \to Y$ defined by

$$\tilde{g}|S \times 0 = g_0, \qquad g|\mathrm{Cl}_S\varphi(A) \times I = \tilde{g}_1$$

is single-valued and hence continuous.

Since $S \times I$ is metrizable and $(S \times 0) \cup \big(\mathrm{Cl}_S\varphi(A) \times I\big)$ is closed in $S \times I$, by the original homotopy extension theorem of Borsuk (cf. [2, p. 94]) there is a continuous map

$$h\colon S \times I \to Y$$

which coincides with $\tilde{g}$ over $(S \times 0) \cup \big(\mathrm{Cl}_S\varphi(A) \times I\big)$. Since $h \circ (\varphi \times 1)\colon X \times I \to Y$ is an extension of $f\colon C \to Y$, the proof of Theorem 5 is completed.

3. Proof of Theorem 6. Let A be a zero-set of X. Then there is a continuous map $\alpha\colon X \to I$ such that $A = \{x \in X|\ \alpha(x) = 0\}$. Let us put $\mathcal{G}_i = \{G_0, G_i\}$, where $G_0 = \{x \in X|\ \alpha(x) > 0\}$, $G_i = \{x \in X|\ \alpha(x) < 1/i\}$ for $i \in N$. For a normal sequence $\{\mathfrak{U}_i|\ i \in N\}$ of locally finite cozero-set covers of X which was constructed in the proof of Theorem 5, we further require that $\mathfrak{U}_i$ is a refinement of $\mathcal{G}_i$ for each i. Then it is easy to see that $\varphi^{-1}\varphi(A) = A$ and A is closed in (X, Φ). This shows that $\varphi(A)$ is closed also in S.

Therefore, by pursuing the proof of Theorem 5, we have Theorem 6.

4. Remarks. A slight modification of our proof described above yields the following theorem.

THEOREM 9. *Let Y be an AR for metric spaces which has weight $\leqslant m$. Let A be a subset of a topological space X which is P^m-embedded in X and C a closed subset of a compact metric space Z. Then every continuous map $f\colon (X \times C) \cup (A \times Z) \to Y$ is extended to a continuous map $g\colon X \times Z \to Y$ if either A is a zero-set or Y is Čech complete.*

COROLLARY 10. *Let A be a subset of a topological space X. Then A is P^m-embedded in X iff for an AR Y with weight $\leqslant m$ every continuous map $f\colon A \to Y$ is extended to a continuous map $g\colon X \to Y$, where Y is further assumed to be Čech complete unless A is a zero-set.*

Proof. The "only if" part is a direct consequence of Theorem 9. To prove the "if" part, assume that the condition of the theorem is satisfied. Let $\mathfrak{U}$ be any locally finite cozero-set cover of A of cardinality $\leqslant m$. Then there exist a complete bounded metric space T of weight $\leqslant m$, a continuous map $f\colon A \to T$ and a locally finite cozero-set cover $\mathfrak{W}$ of T of cardinality $\leqslant m$ such that $f^{-1}(\mathfrak{W})$ refines $\mathfrak{U}$. By a theorem of Kuratowski–Wojdyslawsky [5, p. 81] there is a Banach space L containing T such that the convex hull Z_0 of T has weight $\leqslant m$ and T is closed in L. Then the closure Z of Z_0 in L is convex (and hence an AR) and has weight $\leqslant m$. Hence by assumption the map $f\colon A \to T$ is extended to a continuous map $g\colon X \to Z$. Since T is closed in Z, there is a locally finite cozero-set cover $\mathfrak{V}$ of Z of cardinality $\leqslant m$ such that $\mathfrak{V} \cap T$ refines $\mathfrak{W}$. Then $g^{-1}(\mathfrak{V}) \cap A$ refines $\mathfrak{U}$. Thus, A is P^m-embedded in X.

As another application of our method we have the following theorem.

THEOREM 11. *Let A and B be zero-sets of a topological space X. If A and B are P^m-embedded in X, so is $A \cup B$.*

Proof. Let Y be an AR which has weight $\leqslant m$. Let $f\colon A \cup B \to Y$ be any continuous map. Then by the method described in § 3 we can find a metric space S, a continuous map $\varphi\colon X \to S$ and a map $g\colon \varphi(A \cup B) \to Y$ such that

(6) $\quad \varphi(A)$ and $\varphi(B)$ are closed in S,

(7) $f = g \circ (\varphi | A \cup B)$,

(8) $g|\varphi(A)$ and $g|\varphi(B)$ are continuous.

Thus, g is continuous and hence it is extended to a continuous map $h\colon S \to Y$. Then $h \circ \varphi\colon X \to \mathbf{Y}$ is an extension of f.

Theorem 11 does not hold in general if we drop the assumption that A and B are zero-sets of X.

EXAMPLE. Let us put

$$X = W(\omega_1+1)\times W(\omega_1+1) - (\omega_1, \omega_1),$$

$$A = W(\omega_1)\times \omega_1, \qquad B = \omega_1 \times W(\omega_1).$$

Then A and B are C-embedded in X but $A \cup B$ is not.

Added in proof. In case $Y = [0, 1]$ or $(0, 1)$ Theorem 7 has been proved also by M. Starbird (*The normality of products with a compact or a metric factor*, Ph. D. Thesis Univ. of Wisconsin, 1974).

The converse of Theorem 5 holds. Cf. K. Morita and T. Hoshina, *C-embedding and the homotopy extension property*, to appear in Gen. Topology and Appl.

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Conservative extensions and the two cardinal theorem for stable theories

by

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Abstract. DEFINITION. $\mathcal{B}$ is a *conservative extension* of $\mathcal{A}$ if $\mathcal{B} \succeq \mathcal{A}$ and if for every set X definable in $\mathcal{B}$ $X \cap |\mathcal{A}|$ is definable in $\mathcal{A}$. We use this definition to give a short proof of Lachlan's result: If $\mathcal{A}$ is a proper elementary submodel of $\mathcal{B}$ and $\mathcal{A}, \mathcal{B}$ are models of a stable theory T but $D(\mathcal{A}) = D(\mathcal{B})$ then there exists a proper elementary extension $\mathcal{C}$ of $\mathcal{B}$ with $D(\mathcal{C}) = D(\mathcal{A})$.

Let L be a countable first order language and D a unary L-formula. In [3] Lachlan proved that if T is stable, $\mathcal{A}$ and $\mathcal{B}$ are models of T with $|\mathcal{A}| \neq |\mathcal{B}|$, $\mathcal{A} \preceq \mathcal{B}$ and $D(\mathcal{A}) = D(\mathcal{B})$ then there exists a proper elementary extension $\mathcal{C}$ of $\mathcal{B}$ with $D(\mathcal{C}) = D(\mathcal{A})$. We give a simpler proof of this result. In addition, if $\mathcal{B}$ is countable we can weaken the hypothesis "T is stable" to "$\mathcal{B}$ is a conservative extension of $\mathcal{A}$". We define the notion of conservative extension below and remark that T is stable if and only if every elementary extension of every model of T is conservative. Our proof of the main theorem is evidence for taking this as the definition of a stable theory.

In general our notation follows [9]. We vary however by letting $F_n(L)$ denote the set of L-formulas with n free variables and $F_n(X)$ the formulas with n free variables in the expansion of L which names each member of X. We write $|\mathcal{A}|$ for the universe of the structure $\mathcal{A}$ and $|\ |$ for cardinality. Thus $||\mathcal{A}||$ denotes the cardinality of the universe of $\mathcal{A}$. If $\mathcal{A}$ is an L-structure and $X \subseteq \mathcal{A}$, $A \in F_n(X)$ then

$$A(\mathcal{A}) = \{\langle a_0, \ldots, a_{n-1}\rangle | \mathcal{A} \models A(a_0, \ldots, a_{n-1})\}.$$

We particularly want to acknowledge our debt to Andreas Blass for many stimulating conversations on the notion of conservative extension and for suggesting some simplifications in the proofs here.

The following definition came to our attention because of its application to Peano arithmetic in [6, 7].

DEFINITION. $\mathcal{B}$ is a *conservative extension* of $\mathcal{A}$ ($\mathcal{B} \succeq_c \mathcal{A}$) if $\mathcal{B} \succeq \mathcal{A}$ and for every formula $B \in F_n(\mathcal{B})$ there is an $A \in F_n(\mathcal{A})$ such that $B(\mathcal{B}) \cap |\mathcal{A}|^n = A(\mathcal{A})$.