References

[1] J. L. Cornette, *Connectivity functions and images on Peano continua*, Fund. Math. 58 (1966), pp. 183–192.
[2] K. R. Kellum and B. D. Garret, *Almost continuous real functions*, Proc. Amer. Math. Soc. 33 (1972), pp. 181–184.
[3] — *Almost continuous functions on I^n*, Fund. Math. 79 (1973), pp. 213–215.
[4] K. Kuratowski, *Topology*, vol. II, New York–London–Warszawa 1968.
[5] J. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. 47 (1959), pp. 249–263.

MILES COLLEGE

Accepté par la Rédaction le 17. 9. 1973

Contractive fixed points

by

Solomon Leader and **Stephen L. Hoyle** * (New Brunswick, N. J.)

Abstract. A contractive fixed point is defined to be a fixed point to which all orbits converge. Conditions giving contractive fixed points are studied for spaces equipped with a suitable equivalence relation on their sequences. The results of this study are then applied to uniform and metric spaces where they yield known as well as new generalizations of the Banach Contraction Principle.

1. Introduction. Let $f\colon X\to X$ where X is equipped with a suitable notion of sequential convergence. For each x in X we call the sequence $\langle x, fx, f^2x, \ldots\rangle$ the *orbit* of x. We call p a *contractive fixed point* if $fp = p$ and every orbit converges to p. Since convergent sequences in our spaces have unique limits, a contractive fixed point must be a unique fixed point.

We are interested here in existence theorems for contractive fixed points. The classical result of this type is the Banach contraction theorem [2] which has inspired the search for fixed point principles in metric and uniform spaces. In this search the contractive property of fixed points has sometimes been ignored by researchers.

Our main contribution here is to place the study of contractive fixed points in a more general setting than uniform spaces, but with sufficient structure to yield results. Specifically, we use the UL*-space of A. Goetz [5] which we call S-space (sequential structure space). Our results apply in particular to uniform spaces and thereby to metric spaces and topological groups.

2. S-spaces. Bold face capitals $\boldsymbol{K}$, $\boldsymbol{L}$, $\boldsymbol{M}$ will always denote infinite subsets of the set $\boldsymbol{N}$ of all natural numbers. For any sequence $\langle x_n\rangle$ let $\langle x_n\rangle_{\boldsymbol{M}}$ be the subsequence obtained by composing the unique order-preserving map of $\boldsymbol{N}$ onto $\boldsymbol{M}$ with the restriction of the sequence to $\boldsymbol{M}$.

An S-space is a nonempty set X equipped with an equivalence relation $\langle x_n\rangle \sim \langle y_n\rangle$ on sequences in X such that:

(S_1) For constant sequences, $\langle x\rangle \sim \langle y\rangle$ implies $x = y$.

(S_2) If $\langle x_n\rangle \sim \langle y_n\rangle$ then $\langle x_n\rangle_{\boldsymbol{M}} \sim \langle y_n\rangle_{\boldsymbol{M}}$ for all $\boldsymbol{M}$.

* Part of this work is derived from the latter author's 1971 Henry Rutgers Thesis.

(S_3) Given $\langle x_n\rangle$ and $\langle y_n\rangle$ such that every $\boldsymbol{M}$ contains some $\boldsymbol{K}$ with $\langle x_n\rangle_K \sim \langle y_n\rangle_K$, then $\langle x_n\rangle \sim \langle y_n\rangle$.

Convergence $x_n \to x$ is defined by $\langle x_n\rangle \sim \langle x\rangle$. A sequence is *Cauchy* if it is equivalent to each of its subsequences. X is *S-complete* if every Cauchy sequence in X converges to some point in X. A point x is an *S-limit point* of a sequence if some subsequence converges to x. X is *S-compact* if every sequence in X has an S-limit point in X. X is *S-bounded* if every sequence in X has a Cauchy subsequence. So X is S-compact if and only if it is S-complete and S-bounded.

Every (separated) uniform space $(X, \mathcal{U})$ becomes an S-space if we define $\langle x_n\rangle \sim \langle y_n\rangle$ to mean $(x_n, y_n) \in U$ ultimately for each U in $\mathcal{U}$. Sequential equivalence in uniform spaces is a proximity invariant. Explicitly, let δ be the proximity relation on 2^X induced by $\mathcal{U}$: $A\ \delta\ B$ means $A \times B$ meets every member of $\mathcal{U}$. Then, using braces to denote the range of a sequence, we have $\langle x_n\rangle \sim \langle y_n\rangle$ if and only if $\{x_n\}_M\, \delta\, \{y_n\}_M$ for all $\boldsymbol{M}$. (See section 18 in [11].)

In the special case of a metric space (X, d) sequential equivalence becomes $d(x_n, y_n) \to 0$.

For a compact Hausdorff space X equivalence is determined by the unique uniformity $\mathcal{U}$ compatible with the topology: $\mathcal{U}$ consists of all neighborhoods of the diagonal I in $X \times X$. (See Prop. 1 below.)

For U, V subsets of $X \times X$ we define $U \circ V$ by the composition law $(x, y) \circ (y, z) = (x, z)$. So $U[y]$ is the set of all x with (x, y) in U. For f a function on X the *graph* is the set of all (fx, x) with x in X.

In a topological space a point x is a *limit point* of a sequence $\langle x_n\rangle$ if every neighborhood of x contains some subsequence of $\langle x_n\rangle$. In an S-space equipped with a topology whose sequential convergence agrees with S-convergence every S-limit point of a sequence is a limit point, but the converse may fail if the First Axiom of Countability does not hold.

Our first proposition is of interest, but since it is not specifically used in the sequel, we omit the proof.

PROPOSITION 1. *In a uniform space $\langle x_n\rangle \sim \langle y_n\rangle$ implies that for all $\boldsymbol{M}$, $\langle x_n\rangle_M$ and $\langle y_n\rangle_M$ have the same set of limit points. The converse holds if X is compact.*

PROPOSITION 2. *For any S-space X each condition below implies the next:*

(a) $\langle x_n\rangle \sim \langle y_n\rangle$.

(b) *For all $\boldsymbol{M}$, $\langle x_n\rangle_M$ and $\langle y_n\rangle_M$ have the same S-limit points.*

(c) *If for some $\boldsymbol{K}$ both $\langle x_n\rangle_K$ and $\langle y_n\rangle_K$ are convergent, then their limits are equal.*

If X is S-compact, then (a), (b), (c) *are all equivalent.*

Proof. The two implications follow easily from (S_2) and the definitions of convergence and S-limit.

To prove (c) $\Rightarrow$ (a) for X S-compact consider any $\boldsymbol{M}$. Using S-compactness choose $\boldsymbol{K}$ contained in $\boldsymbol{M}$ such that both $\langle x_n\rangle_K$ and $\langle y_n\rangle_K$ converge. By (c) these two subsequences are equivalent. So (a) follows from (S_3).

PROPOSITION 3. *In an S-compact space a sequence converges if and only if it has a unique S-limit point.*

Proof. Apply (a) $\Leftrightarrow$ (c) in Proposition 2 with y_n constant.

3. Contractive fixed points in S-spaces.

PROPOSITION 4. *Let X be an S-space and f: $X \to X$. Then each condition below implies the next:*

(0) *$x_n \to p$ implies $fx_n \to fp$ (S-continuity).*

(1) *$x_n \to p$ and $fx_n \to q$ imply $fp = q$ (S-closed graph).*

(2) *$x_n \to p$ and $fx_n \to p$ imply $fp = p$.*

(3) *$f^n x \to p$ implies $fp = p$.*

Proof. (0) $\Rightarrow$ (1) by uniqueness of limits under (S_1). (1) $\Rightarrow$ (2) by applying (1) with $q = p$. To prove (2) $\Rightarrow$ (3) let $f^n x \to p$. Then $f^{n+1}x \to p$ by (S_2), transitivity of equivalence, and the definition of S-convergence. So (2) with $x_n = f^n x$ gives (3).

PROPOSITION 5. *Let (X, d) be a metric space and f: $X \to X$ with the function $d(x, fx)$ lower semicontinuous on X. Then* (2) *holds under metric convergence.*

Proof. Under the hypothesis in (2), $d(x_n, fx_n) \to 0$. So by semicontinuity $d(p, fp) \leqslant \varliminf_{x\to p} d(x, fx) \leqslant \varliminf_{n\to\infty} d(x_n, fx_n) = 0$. So $fp = p$.

PROPOSITION 6. *Let X be an S-space and f: $X \to X$ such that* (2) *holds. Then the following are equivalent for p in X:*

(i) *p is an S-limit point of some sequence $\langle x_n\rangle$ which is equivalent to its image $\langle fx_n\rangle$.*

(ii) *There is some $x_n \to p$ with $\langle x_n\rangle \sim \langle fx_n\rangle$.*

(iii) *$fp = p$.*

Proof. Given (i) take a subsequence $\langle x_n\rangle_M$ converging to p. Then (ii) holds for this subsequence by (S_2). Given (ii) use transitivity of equivalence and the definition of S-convergence to conclude $fx_n \to p$. So (iii) follows from (2). Finally, given (iii) set $x_n = p$ to get (i).

PROPOSITION 7. *Let X be an S-compact space and f: $X \to X$ such that* (2) *holds. Then f has a fixed point if and only if some sequence $\langle x_n\rangle$ is equivalent to its image $\langle fx_n\rangle$. If p is a unique fixed point then $\langle x_n\rangle \sim \langle fx_n\rangle$ implies $x_n \to p$.*

Proof. To prove the first part use S-compactness and the equivalence of (i) and (iii) in Proposition 6. The implication (i) $\Rightarrow$ (iii) also gives the second part by Proposition 3.

The next result was proved for metric and uniform spaces by Wattel and de Groen [15].

Proposition 8. *Let X be an S-space and $f\colon X\to X$ such that* (2) *holds. Then f has a contractive fixed point if and only if*

(a) *all orbits are equivalent, and*

(b) *some orbit has an S-limit point.*

Proof. Necessity of (a), (b) is trivial. Conversely, let (a), (b) hold. By (b) there exists x whose orbit has an S-limit point p. Then by (a) the orbit of x is equivalent to the orbit of fx. So (i) of Proposition 6 holds, which gives (iii). All orbits converge to p by (a) since the orbit of p is $\langle p\rangle$.

Proposition 9. *Let X be an S-compact space and $f\colon X\to X$ such that* (2) *holds. Then f has a contractive fixed point iff all orbits are equivalent.*

Proof. (b) in Proposition 8 is redundant since X is S-compact.

Proposition 10. *Let X be an S-space and $f\colon X\to X$ such that*

(a) *every orbit is equivalent to its image, and*

(b) *all sequences which are equivalent to their images are equivalent.*

Then all orbits are equivalent Cauchy sequences. Hence, if some orbit has an S-limit point p, then all orbits converge to p. If, in addition, (3) *holds, then p is a contractive fixed point.*

Proof. (a), (b) trivially imply that all orbits are equivalent. For all x in X (a) gives $\langle f^nx\rangle\sim\langle f^{n+1}x\rangle$. So $\langle f^nx\rangle_M\sim\langle f^{n+1}x\rangle_M$ for all M by (S_2). Hence (b) gives $\langle f^nx\rangle\sim\langle f^nx\rangle_M$. So $\langle f^nx\rangle$ is Cauchy. The rest of the theorem is trivial.

We remark that (b) is a reasonable condition to impose. Under the hypothesis of Proposition 7 the existence of a contractive (hence, unique) fixed point gives (b) by the latter part of Proposition 7.

Proposition 11. *Let X be S-complete and $f\colon X\to X$ such that* (2) *holds. Then f has a contractive fixed point iff*

(a) *all orbits are equivalent, and*

(b) *some nonempty S-bounded subset A of X contains its image fA.*

Proof. Given (a), (b) we need only show that some orbit has an S-limit point. Then Proposition 8 will yield a contractive fixed point. Choose a in A. Then $f^na\in A$ for all n. So $\langle f^na\rangle$ has a Cauchy subsequence since A is S-bounded. Such a subsequence converges since X is S-complete. So $\langle f^na\rangle$ has an S-limit point.

The converse is trivial: Let A in (b) consist of the unique fixed point.

Note that in Proposition 11 "S-bounded" can be replaced by "S-compact" in (b).

Proposition 12. *Let X be S-complete and $f\colon X\to X$ such that* (3) *holds. Then f has a contractive fixed point if and only if all suborbits are equivalent:*

$$\text{(i)}\qquad \langle f^nx\rangle\sim\langle f^ny\rangle_K \quad \textit{for all } x,y \textit{ and all } K.$$

Proof. Trivial.

For a uniform space $(X,\mathfrak{U})$ (i) says given x,y in X and W in $\mathfrak{U}$ there exists m such that $(f^jx,f^ky)\in W$ for all $j,k>m$. In particular, for a metric space (X,d) (i) says $\lim\limits_{j,k\to\infty} d(f^jx,f^ky)=0$ for all x,y.

Proposition 13. *Let X be S-complete and $f\colon X\to X$ such that* (2) *holds. Assume*

(a) *given x,y in X there exists M such that $\langle f^nx\rangle_M\sim\langle f^ny\rangle_M$, and*

(b) *if $\langle x_n\rangle\sim\langle fx_n\rangle$ and $\langle y_n\rangle$ is any sequence with y_n in the orbit of x_n for all n, then $\langle x_n\rangle\sim\langle y_n\rangle$.*

Then f has a contractive fixed point.

Proof. Consider any x in X. Set $y=fx$ in (a) to get M such that

$$\langle f^nx\rangle_M\sim\langle f^{n+1}x\rangle_M. \tag{4}$$

Apply (b) to (4) to conclude

$$\langle f^nx\rangle_M\sim\langle y_n\rangle_M \text{ if } y_m \text{ is in the orbit of } f^mx \text{ for all } m \text{ in } M. \tag{5}$$

We say K *dominates* M if for K and M the ranges of ascending sequences $\langle k_i\rangle$ and $\langle m_i\rangle$, $k_i\geqslant m_i$ for all i. Since f^kx is in the orbit of f^mx if $k\geqslant m$, (5) gives the following lemma:

If K dominates M then

$$\langle f^nx\rangle_M\sim\langle f^nx\rangle_K. \tag{6}$$

Since K dominates M if $K\subseteq M$ the lemma implies $\langle f^nx\rangle_M$ is Cauchy and, hence, converges to some p in X since X is S-complete. By (4) and (2), $fp=p$.

Given any L there exists $K\subseteq L$ such that K dominates M. Such a $K=\{k_1,k_2,\ldots\}$ can be defined inductively by letting k_i be the smallest number in L which exceeds both m_i and k_{i-1}, with the notational convention $k_0=0$. Therefore, by the lemma, every L contains some K for which (6) holds. By transitivity of equivalence, $\langle f^nx\rangle_K\sim\langle p\rangle$. So by ($S_3$), the orbit of x converges to p. Unicity of the fixed point p follows from (a) and (S_1).

4. Contractive fixed points in uniform spaces. For $f\colon X \to X$ we define the induced product mapping by

$$F(x, y) = (fx, fy). \tag{7}$$

A subset A of a uniform space $(X, \mathfrak{U})$ has *small after-images* under $f\colon X \to X$ if given V in $\mathfrak{U}$ there exist m in N and z in X such that $f^m A \subseteq V[z]$. Equivalently, given W in $\mathfrak{U}$ there exists m such that $F^m(A \times A) \subseteq W$.

PROPOSITION 14. *Let $(X, \mathfrak{U})$ be an S-complete uniform space and $f\colon X \to X$ such that* (3) *holds. Then f has a contractive fixed point if and only if the orbital range of every two-point set has small after-images.*

Proof. The direct implication is trivial. Conversely, given x, y in X let A be the range of the orbits of x, y. By hypothesis A has small after-images. So for arbitrary W in $\mathfrak{U}$ there exists m such that $(f^j x, f^k y) \in W$ for all $j, k > m$. So (i) of Proposition 12 holds. Therefore Proposition 12 gives a contractive fixed point.

For compact uniform spaces we have the following analogue of Proposition 9.

PROPOSITION 15. *Let $(X, \mathfrak{U})$ be compact and $f\colon X \to X$ be continuous. Then f has a contractive fixed point if and only if all orbits are equivalent.*

Proof. The direct implication is trivial. To prove the converse let all orbits be equivalent. In particular $\langle f^n x\rangle \sim \langle f^{n+1}x\rangle$ for all x. So $(f^n x, f^{n+1}x)$ is ultimately in any given neighborhood of the diagonal I. Hence, the graph of f meets every neighborhood of I. Therefore, since the graph is closed because f is continuous, the graph meets I. That is, f has a fixed point p. Since all orbits are equivalent, p is contractive.

Proposition 15 has the following extension.

PROPOSITION 16. *Let $(X, \mathfrak{U})$ be a complete uniform space and $f\colon X \to X$ with closed graph. Then f has a contractive fixed point if and only if*

(a) $\varliminf_{n\to\infty} F^{-n} U = X \times X$ *for all U in $\mathfrak{U}$ and*

(b) *there is some nonempty, totally bounded subset A of X such that $fA \subseteq A$.*

Proof. (a) says in terms of (7) that all orbits are equivalent. So (a) is necessary for a contractive fixed point. (b) is necessary with A consisting of the fixed point.

Conversely, given (a) and (b) we need only show that $f\bar{A} \subseteq \bar{A}$ to get a contractive fixed point from Proposition 15 applied to the space $\bar{A}$. $\bar{A}$ is compact since A is totally bounded and X is complete. So if $f\bar{A} \subseteq \bar{A}$, then f must be continuous since its graph is closed.

Given $x \in \bar{A}$ we contend $fx \in \bar{A}$. Choose a net $\langle a_\alpha\rangle$ in A with $a_\alpha \to x$. We may assume $\langle fa_\alpha\rangle$ is convergent since it lies in the compact set $\bar{A}$ and hence has a convergent subnet. So $fa_\alpha \to y$ in $\bar{A}$. Since the graph is closed, $y = fx$. So $fx \in \bar{A}$.

The remaining results (Propositions 17–23) of this section give sufficient conditions for a contractive fixed point in a uniform space. These conditions all hold for Banach contractions. Condition (iii) in Proposition 17 was used by C. S. Wong [16] to prove a non-contractive fixed point theorem.

PROPOSITION 17. *Let $(X, \mathfrak{U})$ be an S-complete uniform space and $f\colon X \to X$ such that*

(i) *for each U in $\mathfrak{U}$ the graph of f is contained in $\varliminf_{n=\infty} F^{-n} U$,*

(ii) (3) *holds, and*

(iii) *given U in $\mathfrak{U}$ there exists V in $\mathfrak{U}$ such that (x, fx) and (y, fy) in V imply $(fx, fy) \in U$.*

Then f has a contractive fixed point.

Proof. Given x in X and U in $\mathfrak{U}$ apply (i) to the graph point (fx, x) to get n such that $(f^{k+1}x, f^k x) \in U$ for all $k \geqslant n$. So (a) of Proposition 10 holds. Therefore, (ii) and (iii) with Proposition 10 yield a contractive fixed point.

PROPOSITION 18. *Let $(X, \mathfrak{U})$ be S-complete and $f\colon X \to X$. Let $\mathfrak{B}$ be a symmetric base for $\mathfrak{U}$ satisfying*

(a) *given x in X there exist U in $\mathfrak{B}$, y in X, and n in $\mathbf{N}$ such that $f^k x \in U[y]$ for all $k \geqslant n$,*

(b) *given U in $\mathfrak{B}$ and y in X, $U[y]$ has small after-images.*

Then every orbit converges. If, moreover, (3) *holds and*

(c) *$\mathfrak{B}$ covers $X \times X$*

then there is a contractive fixed point.

Proof. By (a) and (b) the orbital range of any point has small after-images. So every orbit is Cauchy, hence convergent since X is S-complete. If (3) holds, then the limit of an orbit is a fixed point.

Given (c) let p and q be fixed points. By (c) there exists U in $\mathfrak{B}$ with (p, q) in U. So both p and q are in $U[q]$. Hence, by (b), given W in $\mathfrak{U}$ there exists m such that $(p, q) = F^m(p, q) \in W$. So, since $(X, \mathfrak{U})$ is separated, $p = q$.

PROPOSITION 19. *Let $(X, \mathfrak{U})$ be S-complete and $f\colon X \to X$ such that* (3) *holds. Let $\mathcal{A}$ be a symmetric base satisfying*

(A) *given x in X there exist U in $\mathcal{A}$, y in X, and n in $\mathbf{N}$ such that $f^k x \in U[y]$ for all $k \geqslant n$ and*

(B) *given U, W in $\mathcal{A}$ there exists m with $F^m U \subseteq W$.*

Then every orbit converges to a fixed point. If, moreover,

(C) $X\times X=\bigcup_{U\in\mathcal{A}} U^\infty$ *where* $U^\infty=\bigcup_{k=1}^{\infty} U^k$,

then there is a contractive fixed point.

Proof. Let $\mathfrak{B}$ consist of all U^k with U in $\mathcal{A}$ and k in $\boldsymbol{N}$. Then (C) gives (c) of Proposition 18. Since $\mathfrak{B}$ contains $\mathcal{A}$, (A) gives (a). We need only show (B) gives (b) to get Proposition 19 from Proposition 18.

Consider y in X and U^k in $\mathfrak{B}$. We contend $U^k[y]$ has small after-images. Given V in $\mathfrak{U}$ choose W in the base with $W^k\subseteq V$. Then get m from (B) which implies $F^mU^k\subseteq(F^mU)^k\subseteq W^k\subseteq V$. So

$$f^m(U^k[y])\subseteq(F^mU^k)[f^my]\subseteq V[f^my]\,.$$

PROPOSITION 20. *Let* $(X,\mathfrak{U})$ *be S-complete,* $\mathfrak{B}$ *a symmetric base for* $\mathfrak{U}$, *and* $f\colon X\to X$ *with* (3) *satisfied such that*

(i) $\mathfrak{B}$ *covers* $X\times X$ *and*

(ii) *given* U, V *in* $\mathfrak{B}$ *there exists* n *in* $\boldsymbol{N}$ *such that*

$$F^nU\circ F^{n+1}U\circ\ldots\circ F^{n+j}U\subseteq V\qquad\textit{for all } j\geqslant 0\,.$$

Then f has contractive fixed point.

Proof. We contend Proposition 20 is subsumed by Proposition 19. Indeed, (i) trivially implies (C), and (ii) with $j=0$ is (B). To verify (A) let x be given in X. Using (i) choose U in $\mathfrak{B}$ with (x,fx) in U. Apply (ii) with $V=U$ to get n. Let $y=f^nx$. Then for all j in $\boldsymbol{N}$,

$$\begin{aligned}(y,f^{n+j}x)&=(f^nx,f^{n+1}x)\circ(f^{n+1}x,f^{n+2}x)\circ\ldots\circ(f^{n+j-1}x,f^{n+j}x)\\&\in F^nU\circ F^{n+1}U\circ\ldots\circ F^{n+j-1}U\subseteq U\end{aligned}$$

by (ii). Therefore, since $U=U^{-1}$, $f^{n+j}x\in U[y]$ for all $j\geqslant 0$ which gives (A).

Although the next result is subsumed by Proposition 13, an independent proof is shorter.

PROPOSITION 21. *Let* $(X,\mathfrak{U})$ *be S-complete and* $f\colon X\to X$ *such that*

(i) *every two-point subset of X has small after-images and*

(ii) *given U in $\mathfrak{U}$ there exists W in $\mathfrak{U}$ such that* $(z,f^kz)\in U$ *for all k in $\boldsymbol{N}$ and all z with* (z,fz) *in W.*

Then every orbit converges. If (3) *also holds, then f has a contractive fixed point.*

Proof. Given x in X and U in $\mathfrak{U}$ get W from (ii). Then apply (i) to x, fx to get n such that $(f^nx,f^{n+1}x)\in W$. Apply (ii) with $z=f^nx$ to conclude that $(f^nx,f^{n+k}x)\in U$ for all k in $\boldsymbol{N}$. So the orbit of x is Cauchy, hence converges to some p by S-completeness. Given (3), $fp=p$.

Unicity of p follows from (i) since our spaces are separated.

PROPOSITION 22. *Let* $(X,\mathfrak{U})$ *be S-complete and* $f\colon X\to X$ *such that*

(a) *every two-point subset of X has small after-images and*

(b) *given V in $\mathfrak{U}$ there exists W in $\mathfrak{U}$ such that*

$$FW\circ F^2W\circ\ldots\circ F^kW\subseteq V\qquad\textit{for all } k \textit{ in } \boldsymbol{N}.$$

Then f has a contractive fixed point.

Proof. We contend Proposition 22 follows from Proposition 21. (3) holds by Proposition 4 since f is (uniformly) continuous by (b) for $k=1$. We need only show (b) implies (ii).

Given U in $\mathfrak{U}$ choose V in $\mathfrak{U}$ with $V^2\subseteq U$. Choose W satisfying (b) with $W\subseteq V$. If $(z,fz)\in W$, then

$$\begin{aligned}(z,f^kz)&=(z,fz)\circ(fz,f^2z)\circ\ldots\circ(f^{k-1}z,f^kz)\\&\in W\circ FW\circ F^2W\circ\ldots\circ F^{k-1}W\subseteq W\circ V\subseteq V^2\subseteq U\end{aligned}$$

for all k. So (ii) holds.

PROPOSITION 23. *Let* $(X,\mathfrak{U})$ *be S-complete and* $f\colon X\to X$ *such that*

(a) *every two-point subset of X has small after-images and*

(b) *there exists a base $\mathfrak{B}$ for $\mathfrak{U}$ such that given U in $\mathfrak{B}$ there exists W in $\mathfrak{U}$ with* $(FU)\circ(W\cap\mathcal{G})\subseteq U$, *where $\mathcal{G}$ is the graph of f.*

Then f has a contractive fixed point p and f is continuous at p.

Proof. We contend the existence of a contractive fixed point follows from Proposition 21. We first show (b) implies (ii). We may assume that $W\subseteq U$ in (b). Then $(fz,z)\in W$ implies by induction that $(f^kz,z)\in U$ for all k. The latter condition holds for $k=1$ because $W\subseteq U$. Given $(f^kz,z)\in U$ then $(f^{k+1}z,fz)\in FU$. So

$$(f^{k+1}z,z)=(f^{k+1}z,fz)\circ(fz,z)\in FU\circ W\cap G\subseteq U$$

by (b). Hence, (ii) holds.

So by Proposition 21 every orbit converges. For $f^nx\to p$ and arbitrary U in $\mathfrak{B}$ we have $(p,f^nx)\in U$ ultimately. So $(fp,f^{n+1}x)\in FU$ ultimately. For W from (b) we have, since $\langle f^nx\rangle$ is Cauchy, $(f^{n+1}x,f^nx)\in W\cap\mathcal{G}$ ultimately. So ultimately

$$(fp,f^nx)=(fp,f^{n+1}x)\circ(f^{n+1}x,f^nx)\in FU\circ W\cap\mathcal{G}\subseteq U$$

by (b). Therefore, $f^nx\to fp$. So $fp=p$ by (S_1) which gives (3). So f has a contractive fixed point p by Proposition 21.

To show f is continuous at p we contend that $f(U[p])\subseteq U[p]$ for all U in $\mathfrak{B}$. Let $x\in U[p]$. Then $(x,p)\in U$, so $(fx,p)=(fx,fp)\in FU$.

Hence
$$(fx, p) = (fx, p) \circ (p, p) \in FU \circ W \cap \mathcal{G} \subseteq U.$$
So $fx \in U[p]$.

Condition (b) in Proposition 23 is a weakening of a condition introduced by R. J. Knill [8] who defined a *uniform contraction* to be a self-mapping f on a uniform space with a symmetric base $\mathcal{B}$ such that given U in $\mathcal{B}$ there exists W in $\mathcal{B}$ with

$$FU \circ W \subseteq U. \tag{8}$$

We may always assume that $W \subseteq U$ in (8). Moreover, since $I \subseteq W$, (8) implies $FU \subseteq U$. So a uniform contraction is uniformly continuous. By induction (8) implies for $W \subseteq U$,

$$F^k(U \circ W^k) \subseteq U \quad \text{for all } k \text{ in } N. \tag{9}$$

(See [8].) Knill proved in [8] that a uniform contraction on an S-complete uniform space has a contractive fixed point if the space is well-chained [10]. Knill's theorem follows from Proposition 23 because every uniform contraction satisfies (a) in a well-chained space. To show this we have $W^k \subseteq U \circ W^k \subseteq F^{-k}U$ for all k by (9). Hence, since X is well-chained, $X \times X = W^\infty \subseteq \bigcup_{k=1}^{\infty} F^{-k}U$ which gives (a).

5. Contractive fixed points in metric spaces.

PROPOSITION 24. *Let (X, d) be a complete metric space and $f\colon X \to X$ such that*

(a) *$d(x, fx)$ is lower semicontinuous on X,*

(b) *$d(f^n x, f^{n+1}x) \to 0$ for all x (asymptotic regularity) and*

(c) *given $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \varepsilon$ for all x, y with $d(x, fx) + d(y, fy) < \delta$.*

Then f has a contractive fixed point.

Proof. Apply Proposition 10 with Propositions 4 and 5. Or use Proposition 17 using the base

$$\mathcal{B} = \{U_\varepsilon\colon \varepsilon > 0\} \quad \text{where} \quad U_\varepsilon = d^{-1}[0, \varepsilon) \tag{10}$$

to get (iii) from (c).

For a Banach contraction

$$d(fx, fy) \leqslant M d(x, y) \quad \text{for all } x, y \tag{11}$$

where $0 < M < 1$, (a) and (b) are trivially satisfied. (c) holds for $\delta = (1-M)\varepsilon$ since (11) and the triangle inequality imply $(1-M)d(x, y) \leqslant d(x, fx) + d(y, fy)$. So Proposition 24 subsumes the Banach Contraction Theorem.

PROPOSITION 25. *Let (X, d) be complete and $f\colon X \to X$ such that*

(i) *every orbit is bounded and*

(ii) *every bounded set has small after-images.*

Then every orbit converges. Hence, f has a contractive fixed point if (3) holds.

Proof. Apply Proposition 18 with $\mathcal{B}$ defined by (10). Then (i) gives (a) with arbitrary y and $n = 1$. (ii) immediately gives (b). (c) is trivial since $d(x, y) < \infty$ for all x, y.

Proposition 25 subsumes the Banach Contraction Theorem. We shall apply it to prove a stronger result of V. M. Sehgal [14]. We have dropped Sehgal's assumption of continuity.

PROPOSITION 26. *Let (X, d) be complete and $f\colon X \to X$ with a constant $c < 1$ such that for each x in X there exists n with*

$$d(f^n x, f^n y) \leqslant c\, d(x, y) \quad \text{for all } y. \tag{12}$$

Then f has a contractive fixed point.

Proof. We first show that (12) implies (i) and (ii) in Proposition 25, and then that (3) follows from (12) and the convergence of all orbits.

To get (i) consider any x in X. Choose n so that (12) holds. Define

$$R = \operatorname*{Max}_{i=1,\dots,n} d(x, f^i x). \tag{13}$$

Then (i) follows from Sehgal's lemma [14],

$$d(x, f^k x) \leqslant \frac{R}{1-c} \quad \text{for all } k. \tag{14}$$

To get (14) one can prove by induction on m that

$$d(x, f^{mn+i}x) \leqslant (1 + c + c^2 + \dots + c^m)R \quad \text{for} \quad i = 1, \dots, n. \tag{15}$$

For $m = 0$ (15) follows from (13). Given (15) for a particular m apply (12) to get

$$d(f^n x, f^{(m+1)n+i}x) \leqslant c\, d(x, f^{mn+i}x). \tag{16}$$

Now (13) implies

$$d(x, f^n x) \leqslant R. \tag{17}$$

Adding (16) and (17), applying the triangle inequality, and then applying (15) we get (15) with m replaced by $m+1$.

To prove (ii) consider a ball B of radius γ about x. By (12) B has some after-image $f^n B$ contained in a ball of radius $c\gamma$ about $f^n x$. By induction there exists arbitrarily large m such that B has an after-image in a ball of radius $c^m\gamma$. So (ii) holds since $c<1$.

So by Proposition 25 every orbit converges. Let p be the limit of the orbit of some point z. Apply (12) with $x=p$ and $y=f^kz$ for arbitrary k to get $d(f^np, f^{n+k}z)\leqslant cd(p, f^kz)$. Hence, letting $k\to\infty$, we get $f^np=p$. That is, p is a periodic point. But the orbit of p converges, so $fp=p$. Hence (3) holds.

PROPOSITION 27. *Let (X,d) be complete and $f\colon X\to X$ such that*

(a) $\inf_n d(f^nx, f^ny)=0$ *for all x, y and*

(b) *given $\varepsilon>0$ there exists $\delta>0$ such that $d(z, f^kz)<\varepsilon$ for all k in N and all z for which $d(z, fz)<\delta$.*

Then every orbit converges. So f has a contractive fixed point if (3) holds.

Proof. Apply Proposition 21 noting that $\mathcal{U}$ in (ii) can be replaced by any base $\mathcal{B}$. Use $\mathcal{B}$ defined by (10). Then (a) is exactly (i) and (b) is (ii).

Proposition 27 subsumes the Banach Contraction Theorem. Indeed, (a) follows from the equivalence of all orbits, a consequence of the inductive extension of (11) to

$$d(f^nx, f^ny)\leqslant M^nd(x,y)\,. \tag{18}$$

(b) follows from the triangle inequality and (18) with $y=fx$ which yield

$$d(x, f^kx)\leqslant(1-M)^{-1}d(x, fx)\,. \tag{19}$$

So under (11), (b) holds with $\delta=(1-M)\varepsilon$.

The next generalization of the Banach Contraction Theorem is due to S. Reich [12] who extended some results of R. Kannan [6], [7]. We have added the conclusion that f must be continuous at p.

PROPOSITION 28. *Let (X,d) be a metric space and $f\colon X\to X$. Let there exist non-negative constants a, b, c such that $a+b+c<1$ and for all x, y in X*

$$d(fx, fy)\leqslant ad(x, fx)+bd(y, fy)+cd(x,y)\,. \tag{20}$$

Then all orbits are equivalent Cauchy sequences. If they have a limit p, then p is a contractive fixed point and f is continuous at p.

Proof. We shall apply Proposition 10 to get the first conclusion.

To get (a) of Proposition 10 set $y=fx$ in (20) to conclude that for $M=\dfrac{a+c}{1-b}$

$$d(fx, f^2x)\leqslant Md(x, fx)\,. \tag{21}$$

By induction on n extend (21) to

$$d(f^nx, f^{n+1}x)\leqslant M^nd(x, fx)\,. \tag{22}$$

Since $M<1$, (22) implies (a).

To verify (b) let $d(x_n, fx_n)\to 0$ and $d(y_n, fy_n)\to 0$. Then (20) yields

$$\overline{\lim}\, d(fx_n, fy_n)\leqslant c\,\overline{\lim}\, d(x_n, y_n)\,. \tag{23}$$

But, since $d(x_n, y_n)\leqslant d(x_n, fx_n)+d(fx_n, fy_n)+d(fy_n, y_n)$, and

$$\overline{\lim}\, d(x_n, y_n)\leqslant\overline{\lim}\, d(fx_n, fy_n)\,. \tag{24}$$

Since $c<1$, (23) and (24) imply $d(x_n, y_n)\to 0$. So (b) holds. Hence, by Proposition 10, all orbits are equivalent Cauchy sequences.

If all orbits converge to p we contend $p=fp$. By (20) applied to p and $f^{n-1}x$ we get

$$d(fp, f^nx)\leqslant ad(p, fp)+bd(f^{n-1}x, f^nx)+cd(p, f^{n-1}x)\,. \tag{25}$$

As $n\to\infty$ (25) yields $d(fp, p)\leqslant ad(p, fp)$ which implies $d(p, fp)=0$ since $a<1$. So $p=fp$.

To show f is continuous at p set $y=p=fp$ in (20) to get

$$d(fx, p)\leqslant ad(x, fx)+cd(x, p)\leqslant a[d(x, p)+d(p, fx)]+cd(x, p)\,.$$

So

$$0\leqslant(1-a)d(fx, p)\leqslant(a+c)d(x, p)$$

which gives the continuity of f at $p=fp$.

Kannan [7] gave a special case of Proposition 28 with $a=b$ and $c=0$. He demanded the hypothesis that f be continuous at p which our result shows is redundant.

In [13] Reich relaxes the condition $a+b+c<1$ by demanding only that $c<1$. But he adds the hypothesis that $d(x, fx)$ be lower semicontinuous (which gives (2) by Proposition 5) and that some sequence in X be equivalent to its image. Then all such sequences converge to a unique fixed point p. However, p need not be contractive and f need not be continuous at p. In Reich's version [12] of Proposition 28 he omits the conclusion that the fixed point is contractive, which his proof yields.

Contractive fixed points in compact metric spaces have been investigated by D. F. Bailey [1]. Edelstein [4] proved that every contraction on a compact metric space has a contractive fixed point. Actually one can conclude that all orbits converge uniformly to the fixed point. This is our next result. We denote the diameter of a set A by $d[A]$.

Proposition 29. *Let (X, d) be a compact metric space and $f\colon X \to X$ such that*

(26) $$d(fx, fy) < d(x, y) \quad \text{for all } x \neq y\,.$$

Then

(27) $$d[f^n X] \downarrow 0\,.$$

So all orbits converge uniformly to a unique fixed point.

Proof. We need two lemmas.

Lemma A. *Given a nested sequence $Y_n \supseteq Y_{n+1}$ of nonempty, compact subsets of (X, d), then for $Y = \bigcap_{n=1}^{\infty} Y_n$, $d[Y] = \lim_{n\to\infty} d[Y_n]$.*

Proof. Clearly $d[Y] \leqslant d[Y_{n+1}] \leqslant d[Y_n]$. So $d[Y] \leqslant \lim d[Y_n]$. To reverse the inequality choose a_n, b_n in Y_n by compactness so that

(28) $$d(a_n, b_n) = d[Y_n]\,.$$

Choosing appropriate subsequences we may assume under compactness that $a_n \to a$ and $b_n \to b$. So

(29) $$d(a_n, b_n) \to d(a, b)\,.$$

Now Y_n is closed and $a_m, b_m \in Y_n$ for all $m \geqslant n$. So $a, b \in Y_n$ for all n. That is, $a, b \in Y$. So

(30) $$d(a, b) \leqslant d[Y]\,.$$

By (28), (29), (30) $\lim d[Y_n] \leqslant d[Y]$.

Lemma B. *If f is a contraction* (26) *on a compact metric space (Y, d), then either f is constant or $d[fY] < d[Y]$.*

Proof. If f is not constant, then $d[fY] > 0$. We can choose a, b in Y with $d(fa, fb) = d[fY]$. So $a \neq b$ since $d[fY] > 0$. Hence (26) gives $d[fY] = d(fa, fb) < d(a, b) \leqslant d[Y]$ which proves Lemma B.

To prove Proposition 29 first apply Lemma A with $Y_n = f^n X$. Then to get (27) we must show $d[Y] = 0$. From the definition of Y, $fY = Y$ and Y is compact and nonempty. So Lemma B implies f is constant on Y. So Y consists of a single point, hence $d[Y] = 0$.

Contractions on compact metric spaces and Banach contractions on metric spaces are special cases of the contractions introduced by Meier and Keeler [9] who proved the following result.

Proposition 30. *Let (X, d) be a complete metric space and $f\colon X \to X$. Suppose that given $\varepsilon > 0$ there exists $\delta > 0$ such that*

(31) $$d(fx, fy) < \varepsilon \quad \text{for all } x, y \text{ with } d(x, y) < \varepsilon + \delta\,.$$

Then f has a contractive fixed point.

Proof. We contend Proposition 30 follows from Proposition 23.

To verify (a) one can easily show that (31) implies $d(f^n x, f^n y) \downarrow 0$ for all x, y as is done in [9].

To verify (b) express (31) in the form $FU_{\varepsilon+\delta} \subseteq U_\varepsilon$ in terms of (7) and (10). So $(FU_{\varepsilon+\delta}) \circ U_\delta \subseteq U_\varepsilon \circ U_\delta \subseteq U_{\varepsilon+\delta}$ by the triangle inequality. Thus (b) holds for $U = U_{\varepsilon+\delta}$ with $W = U_\delta$. Such U form a base since ε and δ can be arbitrarily small.

Mappings with contractive modulus of continuity turn out to be special cases of Meier-Keeler contractions (31). However, the greater strength of contractive modulus conditions can yield important conclusions which do not hold for all Meier-Keeler contractions. Consider, for example, the contractive modulus condition introduced by Browder [3].

Browder has a nondecreasing, right-continuous function $\emptyset(\gamma)$ defined for $\gamma \geqslant 0$ such that $\emptyset(\gamma) < \gamma$ whenever $\gamma > 0$ and

$$d(fx, fy) \leqslant \emptyset\big(d(x, y)\big) \quad \text{for all } x, y\,.$$

For such a $\emptyset$ one can choose $\delta > 0$ given $\varepsilon > 0$, so that $\emptyset(\varepsilon + \delta) < \varepsilon$. Then if $d(x, y) < \varepsilon + \delta$,

$$d(fx, fy) \leqslant \emptyset\big(d(x, y)\big) \leqslant \emptyset(\varepsilon + \delta) < \varepsilon\,.$$

So (31) holds for a Browder contraction. Hence, by Proposition 30, every Browder contraction on a complete metric space has a contractive fixed point.

For a Browder contraction f on (X, d) one has $d[f^k B] \leqslant \emptyset^k(d[B])$ for all k and all bounded subsets B of X. Hence, since $\emptyset^k(\gamma) \downarrow 0$ for all $\gamma \geqslant 0$, $d[f^k B] \downarrow 0$. So all orbits originating in a given bounded set B converge uniformly.

The latter condition does not hold for Meier-Keeler contractions. This is shown by the following example which we owe essentially to Richard Bumby.

Let $X = \boldsymbol{N}$ with $d(m, n) = 2 - 1/m - 1/n$ for $m \neq n$. The triangle inequality follows from the identity

$$d(m, k) + d(n, k) = d(m, n) + 2\,\frac{k-1}{k} \quad \text{for distinct } m, n, k\,.$$

(X is isometric to a subspace of l_1.) Let $f1 = 1$ and $fn = n-1$ for $n > 1$. Then $fX = X$ since $n = f(n+1)$ for all n in X. So $d[f^k X] = d[X] = 2$.

Now f is a contraction (26) and the range of d is well-ordered: Every nonempty subset has a minimum. The latter condition implies that for $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \varepsilon + \delta$ implies $d(x, y) \leqslant \varepsilon$, hence $d(fx, fy) < \varepsilon$ by (26). So (31) holds.

As a final remark we note that under (31) the monotone sequence $\langle d[f^n B]\rangle$ for B a bounded subset of X either converges to 0 or is ultimately constant.

References

[1] D. F. Bailey, *Some theorems on contractive mappings*, J. London Math. Soc. 41 (1966), pp. 101–106.

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur applications aux intégrales*, Fund. Math. 3 (1922), pp. 133–181.

[3] F. E. Browder, *On the convergence of successive approximations for nonlinear functional equations*, Proc. Kon. Nedrl. Akad. Wetensch. Amsterdam A71 (1968), pp. 27–35.

[4] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. 37 (1962), pp. 74–79.

[5] A. Goetz, *On a notion of uniformity for L-spaces of Fréchet*, Colloq. Math. 9 (1962), pp. 223–231.

[6] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. 60 (1968), pp. 71–76.

[7] — *Some results on fixed points II*, Amer. Math. Monthly 76 (1969), pp. 405–408.

[8] R. J. Knill, *Fixed points of uniform contractions*, J. Math. Anal. Appl. 12 (1965), pp. 449–455.

[9] A. Meir and E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. 28 (1969), pp. 326–329.

[10] S. G. Mrówka and W. J. Pervin, *On uniform connectedness*, Proc. Amer. Math. Soc. 15 (1964), pp. 446–449.

[11] S. A. Naimpally and B. D. Warrack, *Proximity Spaces*, Cambridge 1970.

[12] S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull. 14 (1971), pp. 121–124.

[13] — Problem 5775, Amer. Math. Monthly 78 (1971), p. 84.

[14] V. M. Sehgal, *A fixed point theorem for mappings with a contractive iterate*, Proc. Amer. Math. Soc. 23 (1969), pp. 631–634.

[15] E. Wattel and P. P. N. de Groen, *A general fixed point theorem*, Math. Cent. Amsterdam Afd. Zuiv. Wisk. 9 (1969), p. 5.

[16] C. S. Wong, *A fixed point theorem for certain functions on a complete Hausdorff uniform space*, Notices Amer. Math. Soc. 18 (1971), p. 191.

RUTGERS UNIVERSITY
New Brunswick, N. J.

Accepté par la Rédaction le 17. 9. 1973

Quasi-nonexpansive multi-valued maps and selections

by

Chyi Shiau, Kok-Keong Tan (Halifax, Nova Scotia)
and **Chi Song Wong** * (Windsor, Ontario)

Abstract. Two classes of quasi-nonexpansive multi-valued maps are investigated. (1) Let f be a map of a metrically convex complete metric space X into the family of all non-empty compact subsets of X. Then f has a fixed point if there exists a self map φ on $[0,\infty)$ such that $\varphi(t)<t$ for all $t>0$ and f is φ-contractive. (2) Let X be a weakly compact convex subset of a Banach space B and f be a continuous map of X into the family of all non-empty closed convex subset of X. Then f has a fixed point and has a Kannan selection; the selection so chosen is continuous if further B is strictly convex and X is compact. The relation between selections and fixed points are investigated. As an example, it is proved that every Kannan map of the unit interval into its subintervals has a Kannan selection and therefore a fixed point; all such maps can be explicitly illustrated.

1. Introduction. Let (X,d) be a (non-empty) metric space. Let $\mathrm{bc}(X)$ be the family of all non-empty bounded closed subsets of X endowed with the Hausdorff metric D induced by d [9]. Let f be a map of X into $\mathrm{bc}(X)$. f is *contractive* (*nonexpansive*) *at a point* x in X if $D(f(x),f(y))<d(x,y)$ $(\leqslant d(x,y))$ for all y in X other than x. Let φ be a map of $[0,\infty)$ into itself. φ is *contractive* if $\varphi(0)=0$ and $\varphi(t)<t$ for all $t>0$. Let φ be a contractive self map on $[0,\infty)$. f is *φ-contractive at a point* x in X if $D(f(x),f(y))\leqslant\varphi(d(x,y))$ for all y in X. f is *nonexpansive* (*contractive*, *φ-contractive*) on X if f is nonexpansive (resp. contractive, φ-contractive) at each point in X. f is *quasi-nonexpansive* (*quasi-contractive*, *quasi-φ-contractive*) if the fixed point set $F_f=\{x\in X\colon x\in f(x)\}$ of f is non-empty and f is nonexpansive (resp. contractive, φ-contractive) at each point in F_f. For convenience, we shall identify a singleton with the point it contains. Thus if f is single-valued, our notion of quasi-nonexpansiveness coincides with the one introduced by W. G. Dotson [6].

* The second and third authors are partially supported by the Canadian Mathematical Congress for participating in the Summer Research Institute and by the National Research Council of Canada under the grant numbers, respectively, A8096 and A8518.