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On a question of Borsuk concerning non-continuous retracts I*

by

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Abstract. We are concerned with the question as to whether an acyclic plane continuum is an almost continuous retract of a 2-cell. It is shown that there exists a non-locally connected almost continuous retract of the unit square. On the other hand, it is shown that no pseudo-arc is an almost continuous retract of a Peano continuum.

In [5] Stallings states a question due to Borsuk which suggests the possibility of proving fixed point theorems using non-continuous retracts. The question is whether an acyclic plane continuum is an almost continuous retract of a 2-cell. An affirmative answer to this question would settle the long-standing question of whether such continua have the fixed point property for continuous functions, since an almost continuous retract of an n-cell has the fixed point property for continuous functions.

In the present paper we obtain partial solutions to Borsuk's question. We show the existence of a non-locally connected almost continuous retract of the unit square. On the other hand, we show that no pseudo-arc is an almost continuous retract of a Peano continuum. The first result contrasts sharply to a result of Cornette [1] that a connectivity retract of a unicoherent Peano continuum is a unicoherent Peano continuum.

Preliminaries. Suppose $f\colon A\to B$. We make no distinction between f and its graph. If each open set containing f also contains a continuous function with domain A, then f is *almost continuous*. If $f|C$ is a connected set whenever C is a connected subset of A then f is a *connectivity function*. Every connectivity function on an n-cube, with $n>1$, is almost continuous [3]. Now suppose $B\subset A$. We say that B is an *almost continuous* (*connectivity*) *retract* of A if there exists an almost continuous (connectivity) function $f\colon A\to B$ such that $f(b)=b$ for each b in B. The statement that the subset K of $A\times B$ is a *blocking set of f* means that K is closed, K contains no point of f and K intersects g whenever $g\colon A\to B$ is

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continuous. If no proper subset of K is a blocking set of f, we say that C is a *minimal blocking set* of f. Suppose $C \subset A \times B$. The projection of C into A will be denoted by $p(C)$. If $D \subset p(C)$, then $C|D$ will denote the part of C with A-projection D.

The main results. For each n, let L_n denote the line segment between the points $(1/n, -\frac{1}{2})$ and $(1/n, \frac{1}{2})$. Let K denote the line segment joining $(0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$. In the sequal M will denote the non-locally connected continuum $\mathrm{Cl}(K \cup L_2 \cup L_3 \cup \ldots)$. In Theorem 2 we show that M is an almost continuous retract of J^2 where $J = [-1, 1]$.

THEOREM 1. *Suppose $f\colon J^2 \to M$ is not almost continuous, then there exists a minimal blocking set K of f and $p(K)$ is a perfect subset of J^2.*

Proof. The proof that the set K exists is essentially that given in [2] for a more restricted case, and is ommitted. Clearly, $p(K)$ is non-degenerate.

To see that $p(K)$ is perfect, assume that Q is an isolated point of $p(K)$. Then $K - K|\{Q\}$ is a closed proper subset of K. By the minimality of K, there exists a continuous function $g\colon J^2 \to M$ such that g contains no point of $K - K|\{Q\}$. Let D be the interior of a circle C with center Q such that $D \cap p(K) = \{Q\}$. Define a function $h\colon J^2 \to M$ as follows. If P is in $J^2 - D$, let $h(P) = g(P)$. Let $h(Q) = f(Q)$. If P is in $(D \cap J^2) - \{Q\}$, let R denote the point at which the radial line segment from Q through P meets C. Let m be the length of the arc A in M with end-points $f(Q)$ and $g(R)$. Finally, let $h(P)$ be the point of A such that if n is the length of the subarc from $g(R)$ to $h(P)$ we have

$$n/m = \mathrm{dist}(C, Q)/\mathrm{dist}(R, Q) .$$

Then h is continuous and contains no point of K. This contradiction completes the proof.

THEOREM 2. *M is an almost continuous retract of J^2.*

Proof. Let $\mathfrak{B}$ be the collection to which the subset B of $J^2 \times M$ belongs if and only if B is closed and $p(B) - (p(B) \cap M)$ has cardinality c. We define a function f as follows. First, if P is in M, let $f(P) = P$. Next, there exists a well-ordering $B_1, B_2, \ldots, B_\omega, \ldots, B_a, \ldots$ of $\mathfrak{B}$ such that each element of $\mathfrak{B}$ is preceeded by fewer than c-many elements of $\mathfrak{B}$. Using transfinite induction, for each B_a in $\mathfrak{B}$ we may choose a unique point (P_a, Q_a) in B_a such that P_a is not in $M \cup \{F_\beta\colon \beta < a\}$ and let $f(P_a) = Q_a$. Finally if P is a point of $I^2 - M$ for which $f(P)$ is not yet defined, let $f(P) = (0, 0)$.

We will now complete the proof by showing that f is almost continuous. Assume the contrary. Then there exists a minimal blocking set K of f. Since K is closed and $p(K)$ is perfect, by the construction of f, we have that $p(K) \subset M$. For each integer $n \geqslant 2$, let $g_n\colon J^2 \to (M \cap [1/n, 1] \times J)$ be a continuous function such that if $0 \leqslant x \leqslant 1/n$ and $-\frac{1}{2} \leqslant y \leqslant \frac{1}{2}$, then $g_n(x, y) = (1/n, y)$ and such that $g_n(P) = P$ for each P in $M \cap ([1/n, 1] \times J)$. Then each g_n intersects K in a point of the form $P_n = \big((1/(n+m), y), (1/n, y)\big)$, where m is a positive integer. Some subsequence of $P_2, P_3, P_4, \ldots$ converges to a point of the form $\big((0, y), (0, y)\big)$ in M. But since $f\big((0, y)\big) = (0, y)$ and K is closed, we have that K contains a point of f, a contradiction.

In the next theorem we prove that no-pseudo-arc is an almost continuous retract of a Peano continuum. First, we need some definitions. An ε-chain is a finite collection of open sets $G_1, \ldots, G_n$ such that $\mathrm{diam}\, G_i \leqslant \varepsilon$, for $i = 1, \ldots, n$ and $G_i \cap G_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$. A continuum is called *snake-like* if for each positive number ε it can be covered by an ε-chain. A *pseudo-arc* is an hereditarilly indecomposable snake-like continuum. For definitions of terms not defined here and for information and references concerning pseudo-arcs, see [4, pp. 224–226]. The property of pseudo-arcs needed in the next proof is that a pseudo-arc contains no arc and hence no non-degenerate Peano continuum.

THEOREM 3. *Suppose S is a pseudo-arc and N is a Peano continuum. Then S is not an almost continuous retract of N.*

Proof. Assume that $S \subset N$ and that $f\colon N \to S$ is almost continuous and leaves points of S fixed. Let P and Q be two points of S and let $A_1, \ldots, A_n$ be an ε-chain covering S such that P is in A_i and Q is in A_j where $i < j-1$. Let

$$D = (N \times S) - [(\{P\} \times C_1) \cup (\{Q\} \times C_2)] ,$$

where

$$C_1 = S - (A_{i+1} \cup \ldots \cup A_n) \quad \text{and} \quad C_2 = S - (A_1 \cup \ldots \cup A_{j-1}) .$$

Then D is an open set containing f. Since f is almost continuous, D contains a continuous function $g\colon N \to S$. But $g(P)$ is in $K_1 = A_1 \cup \ldots \cup A_i$ and $g(Q)$ is in $K_2 = A_j \cup \ldots \cup A_n$. Since $K_1 \cap K_2 = \emptyset$, we have that $g(N) \subset S$ is a non-degenerate Peano continuum, a contradiction.

Remarks. 1. I conjecture that Theorem 2 is true whenever M is an arc-wise connected acyclic plane continuum.

2. In [3] Kellum shows that almost continuous functions can behave rather wildly under functional composition. In light of this fact and the results of the present paper, it might be worth-while to study almost continuous retracts of almost continuous retracts. It is conceivable that a fixed point theorem might be found in this way.

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Contractive fixed points

by

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Abstract. A contractive fixed point is defined to be a fixed point to which all orbits converge. Conditions giving contractive fixed points are studied for spaces equipped with a suitable equivalence relation on their sequences. The results of this study are then applied to uniform and metric spaces where they yield known as well as new generalizations of the Banach Contraction Principle.

1. Introduction. Let $f\colon X \to X$ where X is equipped with a suitable notion of sequential convergence. For each x in X we call the sequence $\langle x, fx, f^2x, \ldots\rangle$ the *orbit* of x. We call p a *contractive fixed point* if $fp = p$ and every orbit converges to p. Since convergent sequences in our spaces have unique limits, a contractive fixed point must be a unique fixed point.

We are interested here in existence theorems for contractive fixed points. The classical result of this type is the Banach contraction theorem [2] which has inspired the search for fixed point principles in metric and uniform spaces. In this search the contractive property of fixed points has sometimes been ignored by researchers.

Our main contribution here is to place the study of contractive fixed points in a more general setting than uniform spaces, but with sufficient structure to yield results. Specifically, we use the UL*-space of A. Goetz [5] which we call S-space (sequential structure space). Our results apply in particular to uniform spaces and thereby to metric spaces and topological groups.

2. S-spaces. Bold face capitals $\boldsymbol{K}$, $\boldsymbol{L}$, $\boldsymbol{M}$ will always denote infinite subsets of the set $\boldsymbol{N}$ of all natural numbers. For any sequence $\langle x_n\rangle$ let $\langle x_n\rangle_{\boldsymbol{M}}$ be the subsequence obtained by composing the unique order-preserving map of $\boldsymbol{N}$ onto $\boldsymbol{M}$ with the restriction of the sequence to $\boldsymbol{M}$.

An S-space is a nonempty set X equipped with an equivalence relation $\langle x_n\rangle \sim \langle y_n\rangle$ on sequences in X such that:

(S_1) For constant sequences, $\langle x\rangle \sim \langle y\rangle$ implies $x = y$.

(S_2) If $\langle x_n\rangle \sim \langle y_n\rangle$ then $\langle x_n\rangle_{\boldsymbol{M}} \sim \langle y_n\rangle_{\boldsymbol{M}}$ for all $\boldsymbol{M}$.

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