tion $h \in L$ such that $g \ll h \ll f$. Then $-h \in U$, $-g \in L$ and $-h \ll -g$. Again by (γ), there exists a function $-k \in L$ such that $-h \ll -k \ll -g$. Then $g \ll k \ll h \ll f$. If $h^*(x) = h(x)/(1+|h(x)|)$ when $h(x)$ is finite, $h^*(x) = 1$ when $h(x) = +\infty$, and $h^*(x) = -1$ when $h(x) = -\infty$, then h^* is lsc. If k^* is defined similarly in terms of k, k^* is usc and $k^* \leqslant h^*$. By Theorem 2 of [4] there is a continuous function b^* on X such that $k^* \leqslant b^* \leqslant h^*$. If $b(x) = b^*(x)/(1-|b^*(x)|)$ when $|b^*(x)| < 1$, $b(x) = +\infty$ when $b^*(x) = 1$, and $b(x) = -\infty$ when $b^*(x) = -1$, then b is an extended real-valued continuous function on X. Since $k \leqslant b \leqslant h$, it follows that $g \ll b \ll f$. Thus (δ) holds. It therefore remains to prove that (α) implies (γ).

Let $f \in L$, let $g \in U$, and assume that $g \ll f$. Then $A = \{x\colon f(x) = g(x) = +\infty\}$ and $B = \{x\colon f(x) = g(x) = -\infty\}$ are closed sets, and the sets $U(n) = \{x\colon f(x) > n\}$ and $V(n) = \{x\colon g(x) < -n\}$ are open for each natural number n. Since X is normal it follows that there exist closed sets $A(n)$ and $B(n)$, $n = 1, 2, \ldots$, such that $A(1) \cap B(1) = \emptyset$ and

$$U(n) \supset A(n) \supset A(n)^0 \supset A(n+1) \supset A\,,$$
$$V(n) \supset B(n) \supset B(n)^0 \supset B(n+1) \supset B\,.$$

Take $A' = \bigcap \{A(n)\colon n = 1, 2, \ldots\}$, $B' = \bigcap \{B(n)\colon n = 1, 2, \ldots\}$ and $W = X - (A' \cup B')$. For $x \in W$, put $u(x) = -\infty$ if $x \notin A(1)$ and $u(x) = n$ if $x \in A(n) - A(n+1)$. Then u is usc and $f(x) > u(x)$ for each x in W. Hence also $u \vee g$ is usc and for $x \in W$, $(u \vee g)(x) < f(x)$. Since W is an F_σ-subspace of X, the space W is countably paracompact and normal; hence there exists a continuous function v on W such that $(u \vee g)(x) < v(x) < f(x)$ for x in W. If $h(x) = -\infty$ when $x \in B'$, $h(x) = v(x)$ when $x \in W$ and $h(x) = +\infty$ when $x \in A'$, then h is lsc and $g \ll h \ll f$.

The authors are indebted to the referee for the significantly shorter and more elegant proof of the implication (α) implies (γ) which is given above.

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Čech cohomology and covering dimension for topological spaces

by

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Abstract. For a topological space X let us define the covering dimension of X and the Čech cohomology groups of X by using only normal open coverings of X instead of arbitrary open coverings. Then it will be shown that some of the basic theorems concerning the Čech cohomology groups and covering dimension of CW complexes or paracompact spaces, such as the Hopf classification theorem and the product theorem on dimension for the case of one factor being σ-compact, can be generalized to the case of arbitrary topological spaces.

In discussing the topological invariants for topological spaces, such as the Čech cohomology groups and the covering dimension, which are defined by using open coverings, it seems natural to make a modification by restricting open coverings to normal ones.

For the covering dimension of Tychonoff spaces (= completely regular Hausdorff spaces) such a modification was made by M. Katětov [8] and Yu. Smirnov [20]; a nice exposition of their results is given in Engelking [4]. Applying their modification to a general case, we shall define the covering dimension of a topological space X, denoted by $\dim X$, to be the least integer n such that every finite normal open covering of X admits a finite normal open covering of order $\leqslant n+1$ as its refinement. In case X is a normal space, $\dim X$ defined here coincides with the covering dimension of X in the usual sense.

As for the nth Čech cohomology group $H^n(X; G)$ of a topological space X with coefficients in an abelian group G, we shall define it by using only normal open coverings of X. In case X is paracompact Hausdorff, $H^n(X; G)$ is the usual Čech cohomology group based on all open coverings.

The purpose of this paper is to show that with these definitions we can generalize some of the basic theorems concerning the Čech cohomology groups and dimension of paracompact Hausdorff spaces or CW complexes to the case of topological spaces.

Let X be a topological space, G an abelian group, and Z the additive group of all integers. Let $|K(G, n)|$ be the geometric realization of the

Eilenberg–MacLane complex $K(G, n)$. Then the following results will be established.

I. $\dim(X\times Y) \leqslant \dim X + \dim Y$ if Y is a locally compact, paracompact Hausdorff space, and the equality holds if Y is a locally finite polyhedron.

II. If $\dim X \leqslant n$, then the cohomotopy group $\pi^n(X, x_0)$, that is, the group of homotopy classes of continuous maps from (X, x_0) to (S^n, p_0), is isomorphic to $H^n(X; \boldsymbol{Z})$ for $n \geqslant 1$, where S^n is an n-sphere and $x_0 \in X$, $p_0 \in S^n$. (Hopf's classification theorem.)

III. The nth homotopical cohomology group, that is, the group $[X; |K(G, n)|]$ of homotopy classes of continuous maps from X to $|K(G, n)|$, is isomorphic to $H^n(X; G)$.

IV. If Y is a compact Hausdorff space, then the Künneth formula

$$H^n(X\times Y; G) \cong \bigoplus_{p+q=n} H^p(X; H^q(Y; G))$$

holds.

These results have been proved hitherto only for the case of X being a paracompact Hausdorff space; indeed, for this case, I was proved by K. Morita [9], II by C. H. Dowker [2], III by P. J. Huber [7] (for the case where X is a k-space or G is countable), T. Goto [5] and V. Bartik [1], and IV by V. Bartik [1].

Throughout this paper by a space we shall mean a non-empty topological space, and I denotes the closed unit interval $[0, 1]$ in the real line and N the set of all positive integers.

§ 1. The Tychonoff functor τ. Throughout this section, let X and Y be spaces. Let Y^X denote the set of all continuous maps $\varphi\colon X\to Y$ with the compact-open topology and construct the product space $P(X) = \Pi\{I_\varphi \mid \varphi \in I^X\}$, where $I_\varphi = I$. By defining $\Phi_X(x)$ to be the point of $P(X)$ whose φ-coordinate is $\varphi(x)$, we have a continuous map $\Phi_X\colon X\to P(X)$. For a continuous map $f\colon X\to Y$ we have a continuous map $P(f)\colon P(X)\to P(Y)$ by defining $P(f)(t)$ to be the point of $P(Y)$ whose ψ-coordinate is the $\psi\circ f$-coordinate of t where $t\in P(X)$, $\psi\in I^Y$. Then the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Phi_X"'] & Y \arrow[d, "\Phi_Y"] \\ P(X) \arrow[r, "P(f)"'] & P(Y) \end{tikzcd}$$

is commutative. Let us put

$$\tau(X) = \text{Image of } \Phi_X\,,$$
$$\tau(f) = P(f)|\tau(X)\colon\ \tau(X)\to\tau(Y)\,,$$

and denote by the same letter Φ_X the map from X to $\tau(X)$ which coincides with Φ_X but has $\tau(X)$ as its range. Then we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Phi_X"'] & Y \arrow[d, "\Phi_Y"] \\ \tau(X) \arrow[r, "\tau(f)"'] & \tau(Y) \end{tikzcd}$$

Thus τ is a covariant functor from the category of topological spaces and continuous maps into itself and $\{\Phi_X\}$ defines a natural transformation from the identity functor to τ; $\tau(X)$ is a Tychonoff space and $\Phi_X\colon X\to\tau(X)$ is a homeomorphism if X is itself a Tychonoff space (cf. Dugundji [3]). Hence we shall call τ the *Tychonoff functor*. The Tychonoff functor is the reflector from the category above to the full subcategory of Tychonoff spaces; that is,

LEMMA 1.1. *Any continuous map f from X into a Tychonoff space R is factored through $\tau(X)$ such that $f = g\circ\Phi_X$ for some continuous map $g\colon \tau(X)\to R$; g is determined uniquely by f.*

For a normal open covering $\{G_\alpha \mid \alpha\in\Omega\}$ of a space X there exist a continuous map f from X onto a metric space T and a normal open covering $\{H_\alpha \mid \alpha\in\Omega\}$ of T such that $f^{-1}(H_\alpha)\subset G_\alpha$ for each $\alpha\in\Omega$. By Lemma 1.1 the map f is factored through $\tau(X)$. Therefore we have

LEMMA 1.2. *$\dim X\leqslant n$ if and only if $\dim\tau(X)\leqslant n$.*

Moreover, we have

THEOREM 1.3. *$\dim X\leqslant n$ if and only if for every normal open covering $\mathcal{G}$ of X there is a normal open covering $\mathcal{H}$ of X of order $\leqslant n+1$ which is a refinement of $\mathcal{G}$.*

Proof. In case X is a Tychonoff space, this theorem is proved by Pasynkov [18] (for another proof, cf. Morita [15]). By virtue of Lemma 1.2 and the remark preceding it, the theorem for the general case follows from the theorem for the special case mentioned above.

The following lemma due to Puppier [19] is useful sometimes.

LEMMA 1.4. *If Y is a locally compact Hausdorff space, then $\tau(X\times Y) = \tau(X)\times Y$.*

Proof. If T is a Tychonoff space, so is T^Y (cf. Engelking [4]). Since Y is locally compact Hausdorff, there is a bijective map $\varrho\colon T^{X\times Y}\to(T^Y)^X$ where $\varrho(f)(x)(y) = f(x, y)$ for $f\in T^{X\times Y}$. Hence there is a continuous map $g\colon \tau(X)\times Y\to\tau(X\times Y)$ such that $\Phi_{X\times Y} = g\circ(\Phi_X\times 1_Y)$.

On the other hand, since $\tau(X)\times Y$ is a Tychonoff space, by Lemma 1.1 there is a continuous map $h\colon \tau(X\times Y)\to\tau(X)\times Y$ such that $\Phi_X\times 1_Y = h\circ\Phi_{X\times Y}$. Therefore $h\circ g = 1_{\tau(X)\times Y}$ and $g\circ h = 1_{\tau(X\times Y)}$. This proves Lemma 1.4.

§ 2. A factorization theorem for maps into metric spaces and a covering theorem for product spaces. Let X be a space and $\Phi = \{\mathfrak{U}_i \mid i \in N\}$ a normal sequence of open coverings of X. Let (X, Φ) be the space obtained from X by taking $\{\mathrm{St}(x, \mathfrak{U}_i) \mid i \in N\}$ as a local base (= a basis of neighborhoods) at each point x of X, and X/Φ the quotient space obtained from (X, Φ) by identifying such points x and y that $y \in \mathrm{St}(x, \mathfrak{U}_i)$ for each $i \in N$. Let us denote by φ the composite of the identity map from X onto (X, Φ) and the quotient map from (X, Φ) onto X/Φ. Then $\varphi\colon X \to X/\Phi$ is a continuous map. We shall call X/Φ the space associated with Φ, and φ the canonical map.

For any subset A of X let us put

$$\mathrm{Int}(A; \Phi) = \{x \in X \mid \mathrm{St}(x, \mathfrak{U}_i) \subset A \text{ for some } i \in N\}.$$

Then $\mathrm{Int}(A; \Phi)$ is open in (X, Φ) and, since $\varphi^{-1}\big(\varphi(\mathrm{Int}(A; \Phi))\big) = \mathrm{Int}(A; \Phi)$, $\varphi\big(\mathrm{Int}(A; \Phi)\big)$ is open in X/Φ. Let us put further

$$\mathfrak{U}'_i = \{\mathrm{Int}(U; \Phi) \mid U \in \mathfrak{U}_i\} \quad \text{for} \quad i \in N.$$

Then $\Phi' = \{\mathfrak{U}'_i \mid i \in N\}$ is also a normal sequence of open coverings of X and

$$\mathfrak{U}_i < \mathfrak{U}'_i, \quad \mathfrak{U}'_i < \mathfrak{U}_{i+1} \quad \text{for} \quad i \in N$$

where, for coverings $\mathfrak{U}$ and $\mathfrak{V}$, by $\mathfrak{U} < \mathfrak{V}$ we mean that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$. Hence $X/\Phi' = X/\Phi$. In particular, $\{\mathrm{St}(t, \varphi(\mathfrak{U}'_i)) \mid i \in N\}$ is a local base at each point t of X/Φ. Thus, X/Φ is metrizable. If $\mathfrak{U}_i$ consists of at most $\mathfrak{m}$ members, so does $\mathfrak{U}'_i$, and if the order of $\mathfrak{U}_i$ is not greater than $n+1$, so is the order of $\mathfrak{U}'_i$. These results were obtained in our previous paper [14] and the argument given there proves the following lemma. Here $w(X)$ denotes the weight of X (i.e. the minimal cardinal number $\mathfrak{m}$ such that X has an open basis of cardinality $\mathfrak{m}$).

LEMMA 2.1. *Let $\Phi = \{\mathfrak{U}_i \mid i \in N\}$ be a normal sequence of open coverings of X. Then X/Φ is metrizable, and if each $\mathfrak{U}_i$ consists of at most $\mathfrak{m}$ members where $\mathfrak{m} \geqslant \aleph_0$ (resp. has order $\leqslant n+1$), then $w(X/\Phi) \leqslant \mathfrak{m}$ (resp. $\dim X/\Phi \leqslant n$).*

As an immediate consequence we have

LEMMA 2.2. *Let $f\colon X \to T$ be a continuous map from a topological space X into a metric space T, and suppose that $w(T) \geqslant \aleph_0$. Then there exist a metric space S, a continuous onto map $\varphi\colon X \to S$ and a continuous map $g\colon S \to T$ such that $f = g \circ \varphi$, $w(S) \leqslant w(T)$ and $\dim S \leqslant \dim X$.*

Indeed, if $\{\mathfrak{W}_i \mid i \in N\}$ is a normal sequence of open coverings of T such that $\{\mathrm{St}(t, \mathfrak{W}_i) \mid i \in N\}$ is a local basis at each point t of T and each $\mathfrak{W}_i$ consists of at most $\mathfrak{m}$ members, where $\mathfrak{m} = w(T)$, then there is a normal sequence $\Phi = \{\mathfrak{U}_i \mid i \in N\}$ of open coverings of X such that $f^{-1}(\mathfrak{W}_i) < \mathfrak{U}_i$, order $\mathfrak{U}_i \leqslant \dim X + 1$ for $i \in N$ and each $\mathfrak{U}_i$ consists of at most $\mathfrak{m}$ members;

this is seen from Theorem 1.3. Hence by Lemma 2.1 it is easy to see that $S = X/\Phi$ has the desired property.

For the case of X being a Tychonoff space, Lemma 2.2 was stated without proof by Pasynkov in [17], and in [18] he gave a proof which, however, lacks a consideration about the weight of S; for the assertion that S can be chosen so that $w(S) \leqslant \aleph_0$ if $w(T) \leqslant \aleph_0$, he gave a separate proof in [18].

THEOREM 2.3. *Let X be a space, Y a locally compact, Lindelöf, Hausdorff space, and T a metric space. Then for any continuous map $f\colon X \times Y \to T$ there exist a metric space S, a continuous onto map $\varphi\colon X \to S$ and a continuous map $g\colon S \times Y \to T$ such that $f = g \circ (\varphi \times 1_Y)$ and $w(S) \leqslant \mathrm{Max}\big(w(T), w(Y)\big)$.*

Proof. Since Y is locally compact, there is a bijective map $\varrho\colon T^{X \times Y} \to (T^Y)^X$, where $\varrho(f)(x)(y) = f(x, y)$ for $f\colon X \times Y \to T$. Since Y is locally compact and Lindelöf, the function space T^Y is metrizable. Hence by Lemmas 1.1, 1.2 and 2.2 there are a metric space S, a continuous onto map $\varphi\colon X \to S$ and a continuous map $h\colon S \to T^Y$ such that $\varrho(f) = h \circ \varphi$, $w(S) \leqslant w(T^Y)$, and $\dim S \leqslant \dim X$. If we put $g = \varrho^{-1}(h)\colon S \times Y \to T$, then $f = g \circ (\varphi \times 1_Y)$. Since $w(T^Y) \leqslant w(Y)w(T)$, this proves Theorem 2.3.

LEMMA 2.4. *Let X and Y be the same as in Theorem 2.3. Then for any normal open covering $\mathfrak{G} = \{G_\alpha \mid \alpha \in \Omega\}$ of $X \times Y$ there is a metrizable space $S = X/\Phi$ associated with a normal sequence Φ of open coverings of X such that $(\varphi \times 1_Y)^{-1}(H_\alpha) \subset G_\alpha$, $\alpha \in \Omega$ for some normal open covering $\mathfrak{H} = \{H_\alpha \mid \alpha \in \Omega\}$ of $S \times Y$ where $\varphi\colon X \to X/\Phi = S$ is the canonical map.*

Proof. There are a metric space T, a continuous map $f\colon X \times Y \to T$ and an open covering $\mathfrak{K} = \{K_\alpha \mid \alpha \in \Omega\}$ of T such that $f^{-1}(K_\alpha) \subset G_\alpha$ for each α. By Theorem 2.3 the map f is factored through $S = X/\Phi$, which is associated with some normal sequence Φ of open coverings of X, such that $f = g \circ (\varphi \times 1_Y)$. The covering $\mathfrak{H} = g^{-1}(\mathfrak{K})$ has the desired property. This proves Lemma 2.4.

Remark. Lemma 2.4 can be proved similarly as in the proof of [15, Theorem 1] without appealing to Theorem 2.3. Thus, we have another proof of Theorem 2.3 by using Theorem 2.5 which relies upon Lemma 2.4.

THEOREM 2.5. *Let X be a space and Y a compact Hausdorff space. Let $\mathfrak{G} = \{G_\alpha \mid \alpha \in \Omega\}$ be an open covering of $X \times Y$. Then there exists an open covering $\mathfrak{U} = \{U_\lambda \mid \lambda \in \Lambda\}$ of X satisfying conditions* (a), (b) *and* (c) *below:*

(a) $\mathrm{card}\, \Lambda \leqslant \mathfrak{m}$ *or* $< \aleph_0$ *according as* $\mathfrak{m} \geqslant \aleph_0$ *or* $\mathfrak{m} < \aleph_0$ *where* $\mathfrak{m} = \mathrm{Max}\big(\mathrm{card}\, \Omega, w(Y)\big)$.

(b) *For a suitable collection $\{\mathfrak{V}_\lambda \mid \lambda \in \Lambda\}$ of finite open coverings of Y, the collection $\{U_\lambda \times V \mid V \in \mathfrak{V}_\lambda, \lambda \in \Lambda\}$ is an open covering of $X \times Y$ which refines $\mathfrak{G}$.*

(c) $\mathfrak{U}$ *is a normal open covering of* X *if and only if* $\mathcal{G}$ *is a normal open covering of* $X\times Y$.

Proof. There exists a collection $\{\mathcal{V}_\delta|\ \delta\in\Delta\}$ of finite open coverings of Y such that $\operatorname{card}\Delta\leqslant\mathfrak{m}$ or $<\aleph_0$ according as $\mathfrak{m}\geqslant\aleph_0$ or $\mathfrak{m}<\aleph_0$ and such that every finite open covering of Y is refined by some $\mathcal{V}_\delta$ with $\delta\in\Delta$. For each $\mathcal{V}_\delta$, let $\mathcal{V}_\delta=\{V_{\delta i}|\ i=1,\dots,r_\delta\}$ and let Γ_δ be the set of all maps $\varkappa\colon\{1,\dots,r_\delta\}\to\Omega$. Then $\operatorname{card}\Gamma_\delta\leqslant\mathfrak{m}$ or $<\aleph_0$ according as $\mathfrak{m}\geqslant\aleph_0$ or $\mathfrak{m}<\aleph_0$.

For $\delta\in\Delta$, $\varkappa\in\Gamma_\delta$ let us put

$$U(\delta,\varkappa)=\{x\in X|\ \{x\}\times\operatorname{Cl}V_{\delta i}\subset G_{\varkappa(i)}\quad\text{for each } i\leqslant r_\delta\}.$$

Since $\operatorname{Cl}V_{\delta i}$ is compact, $U(\delta,\varkappa)$ is an open set of X. For any point x of X there is a finite subset γ of Ω such that $\{x\}\times Y\subset\bigcup\{G_\alpha|\ \alpha\in\gamma\}$. Hence there is $\delta\in\Delta$ such that $\{\{x\}\times\operatorname{Cl}V|\ V\in\mathcal{V}_\delta\}$ is a refinement of $\{G_\alpha|\ \alpha\in\gamma\}$. Hence $\mathfrak{U}=\{U(\delta,\varkappa)|\ \varkappa\in\Gamma_\delta,\delta\in\Delta\}$ is an open covering of X and

$$\{U(\delta,\varkappa)\times V|\ V\in\mathcal{V}_\delta,\ \varkappa\in\Gamma_\delta,\ \delta\in\Delta\}$$

is an open covering of $X\times Y$ which refines $\mathcal{G}$.

Let us put $\Lambda=\{(\delta,\varkappa)|\ \ \varkappa\in\Gamma_\delta,\ \delta\in\Delta\}$, and $\mathcal{V}_{(\delta,\varkappa)}=\mathcal{V}_\delta$. Then $\operatorname{card}\Lambda\leqslant\mathfrak{m}$ or $<\aleph_0$ according as $\mathfrak{m}\geqslant\aleph_0$ or $\mathfrak{m}<\aleph_0$. Thus, conditions (a) and (b) are satisfied.

Next, let $\mathcal{G}=\{G_\alpha|\ \alpha\in\Omega\}$ be a normal open covering of $X\times Y$. Then by Lemma 2.4 there exist a metric space S, a continuous map $\varphi\colon X\to S$ and a normal open covering $\mathcal{H}=\{H_\alpha|\ \alpha\in\Omega\}$ of $S\times Y$ such that $(\varphi\times 1_Y)^{-1}(H_\alpha)\subset G_\alpha$ for each α. Let us put

$$M(\delta,\varkappa)=\{s\in S|\ \{s\}\times\operatorname{Cl}V_{\delta i}\subset H_{\varkappa(i)}\ \text{for } i\leqslant r_\delta\}.$$

By applying the result proved in the preceding paragraph to $\{H_\alpha\}$ we see that $\{M(\delta,\varkappa)\}$ is an open covering of S. Since S is paracompact and $\varphi^{-1}(M(\delta,\varkappa))\subset U(\delta,\varkappa)$ for each $(\delta,\varkappa)$, $\mathfrak{U}=\{U(\delta,\varkappa)\}$ is a normal covering of X.

Conversely, suppose that $\mathfrak{U}$ is a normal covering of X. Then $\mathfrak{U}\times Y=\{U_\lambda\times Y|\ \lambda\in\Lambda\}$ is a normal covering of $X\times Y$ and $\{U_\lambda\times V|\ V\in\mathcal{V}_\lambda\}$ is a normal covering of $U_\lambda\times Y$ by Morita [13, Lemma 1.4]. Hence by [11, Theorem 1.2] $\{U_\lambda\times V|\ V\in\mathcal{V}_\lambda,\ \lambda\in\Lambda\}$ is a normal covering of $X\times Y$, and so $\mathcal{G}$ is a normal covering of $X\times Y$. This proves that condition (c) is satisfied, and the proof of Theorem 2.5 is completed.

As an application of Theorem 2.5 we have a direct proof of the following theorem of ours (cf. Morita [11, 12]).

THEOREM 2.6. *Let X be a normal space, $\mathfrak{m}$ an infinite cardinal number, and D the discrete space consisting of exactly two points* 0 *and* 1. *Then the following statements are equivalent.*

(a) X *is* $\mathfrak{m}$-*paracompact.*

(b) $X\times Y$ *is normal for any compact Hausdorff space* Y *of weight* $\leqslant\mathfrak{m}$.

(c) $X\times I^{\mathfrak{m}}$ *is normal.*

(d) $X\times D^{\mathfrak{m}}$ *is normal.*

Proof. (a) $\Rightarrow$ (b). Let $\mathcal{G}=\{G_\alpha|\ \alpha\in\Omega\}$ be an open covering of $X\times Y$ such that $\operatorname{card}\Omega\leqslant\mathfrak{m}$. Since $w(Y)\leqslant\mathfrak{m}$, by Theorem 2.5 there is an open covering $\mathfrak{U}=\{U_\lambda|\ \lambda\in\Lambda\}$ such that $\operatorname{card}\Lambda\leqslant\mathfrak{m}$ and for a suitable collection $\{\mathcal{V}_\lambda|\ \lambda\in\Lambda\}$ of finite open coverings of Y the collection $\mathcal{W}=\{U_\lambda\times V|\ V\in\mathcal{V}_\lambda,\ \lambda\in\Lambda\}$ is an open covering of $X\times Y$ which refines $\mathcal{G}$. Let X be $\mathfrak{m}$-paracompact. Then $\mathfrak{U}$ is normal and hence by Theorem 2.5 $\mathcal{G}$ is normal. This proves the implication (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are obvious.

(d) $\Rightarrow$ (a). Suppose that $X\times D^{\mathfrak{m}}$ is normal. Let Ω be a set with $\operatorname{card}\Omega=\mathfrak{m}$. For each $\alpha\in\Omega$, let us put $Y_\alpha=D$, and construct the product space $Y=\Pi\{Y_\alpha|\ \alpha\in\Omega\}$, which is homeomorphic to $D^{\mathfrak{m}}$; we denote by ϱ_α the projection from Y onto Y_α. Let $\mathcal{G}=\{G_\alpha|\ \alpha\in\Omega\}$ be any open covering of X and put

$$H_i=\bigcup\{G_\alpha\times\varrho_\alpha^{-1}(i)|\ \alpha\in\Omega\}\quad\text{for}\quad i=0,1.$$

Then $\mathcal{H}=\{H_0,H_1\}$ is a normal open covering of $X\times Y$. Hence by Theorem 2.5 there is a normal open covering $\{U_\lambda|\ \lambda\in\Lambda\}$ of X such that for a suitable collection $\{\mathcal{V}_\lambda|\ \lambda\in\Lambda\}$ of finite open coverings of Y the collection $\{U_\lambda\times V|\ V\in\mathcal{V}_\lambda;\ \lambda\in\Lambda\}$ is an open covering of $X\times Y$ which refines $\mathcal{H}$. Here $\mathcal{V}_\lambda$ can be chosen so that for some finite subset $\gamma=\{\alpha_1,\dots,\alpha_n\}$ of Ω we have

$$\mathcal{V}_\lambda=\{\bigcap_{i=1}^{n}\varrho_{\alpha_i}^{-1}(k_i)|\ k_i=0\text{ or }1\text{ for } i\leqslant n\}.$$

Suppose that

$$U_\lambda\times\bigcap_{i=1}^{n}\varrho_{\alpha_i}^{-1}(k_i)\subset H_0.$$

Pick a point y of $\bigcap_{i=1}^{n}\varrho_{\alpha_i}^{-1}(k_i)$ such that $y_\alpha=1$ for $\alpha\notin\{\alpha_1,\dots,\alpha_n\}$. If $x\notin\bigcup\{G_{\alpha_i}|\ i=1,\dots,n\}$, then $(x,y)\notin H_0$. Hence we have

$$U_\lambda\subset\bigcup\{G_{\alpha_i}|\ i=1,\dots,n\}.$$

This shows by Morita [11, Corollary 1.3] that $\mathcal{G}$ is a normal covering of X.

Thus, the proof of Theorem 2.6 is completed.

§ 3. The Čech cohomology groups. Let (X,A) be a pair of a space X and its subspace A and G an abelian group. Let $\{\mathfrak{U}_\lambda|\ \lambda\in\Lambda\}$ be the family of all locally finite normal open coverings of X. Let us denote by K_λ and L_λ the nerves of $\mathfrak{U}_\lambda$ and $\mathfrak{U}_\lambda|\ A=\{U\cap A|\ U\in\mathfrak{U}_\lambda\}$ respectively. For coverings $\mathfrak{U}_\lambda$ and $\mathfrak{U}_\nu$ with $\mathfrak{U}_\nu<\mathfrak{U}_\lambda$ we define a canonical projection $\varphi_\nu^\lambda\colon(K_\lambda,L_\lambda)$

$\to (K_\nu, L_\nu)$ such that $\varphi_\nu^\lambda(u) = v$ implies $U \subset V$ where U and V are members of $\mathfrak{U}_\lambda$ and $\mathfrak{U}_\nu$ corresponding to vertices u of K_λ and v of K_ν respectively. As is well-known, any two canonical projections are contiguous and hence induce the same homomorphism from $H^n(K_\nu, L_\nu; G)$ to $H^n(K_\lambda, L_\lambda; G)$. Thus, we have a direct system $\{H^n(K_\lambda, L_\lambda; G), (\varphi_\nu^\lambda)^*\}$ of abelian groups; we denote its direct limit by $H^n(X, A; G)$ and call it the *n-th Čech cohomology group* of (X, A) with coefficients in G. Since for an open set U of X, $U \cap A \neq \emptyset$ if and only if $U \cap \mathrm{Cl} A \neq \emptyset$, we have

$$H^n(X, A; G) \cong H^n(X, \mathrm{Cl} A; G)\,.$$

Any continuous map $f\colon (X, A) \to (Y, B)$ induces a homomorphism $f^*\colon H^n(Y, B; G) \to H^n(X, A; G)$.

If $\mathfrak{U}_\lambda$ is locally finite there is a continuous map $\varphi_\lambda\colon (X, A) \to (K_\lambda, L_\lambda)$ such that $\varphi_\lambda^{-1}(\mathrm{St}(u; K_\lambda)) \subset U$ where u is a vertex of K_λ corresponding to the set U of $\mathfrak{U}_\lambda$; φ_λ is called a canonical map. Here by the same letter K_λ we denote the polyhedron $|K_\lambda|$ with the weak topology. Then two canonical maps from (X, A) to (K_λ, L_λ) are homotopic. If we denote by $\{\mathfrak{U}_\lambda |\ \lambda \in \Lambda'\}$ the set of all locally finite normal open coverings of X, we have

$$H^n(X, A; G) \cong \varinjlim \{H^n(K_\lambda, L_\lambda; G), (\varphi_\nu^\lambda)^* |\ \lambda, \nu \in \Lambda'\}$$

and $(\varphi_\lambda)^*\colon H^n(K_\lambda, L_\lambda; G) \to H^n(X, A; G)$ coincides with the projection.

LEMMA 3.1. *If* $\dim X \leqslant n$, *then* $H^r(X, A; G) = 0$ *for* $r > n$.

Proof. This is obvious by Theorem 1.3.

For normal sequences $\Phi = \{\mathfrak{U}_i |\ i \in N\}$ and $\Psi = \{\mathfrak{V}_i |\ i \in N\}$ of open coverings of X we write $\Phi < \Psi$ if each $\mathfrak{U}_i$ is refined by some $\mathfrak{V}_j$. Now, let $\{\Phi_\alpha |\ \alpha \in \Omega\}$ be the totality of normal sequences Φ_α of open coverings of X. In § 2 we defined the space X/Φ_α and the canonical map $\varphi_\alpha\colon X \to X/\Phi_\alpha$. If $\Phi_\beta < \Phi_\alpha$, there is a canonical map $\varphi_\beta^\alpha\colon X/\Phi_\alpha \to X/\Phi_\beta$ which is continuous and $\varphi_\beta^\alpha \circ \varphi_\alpha = \varphi_\beta$. If $\Phi_\gamma < \Phi_\beta < \Phi_\alpha$, then $\varphi_\gamma^\alpha = \varphi_\gamma^\beta \circ \varphi_\beta^\alpha$. Since $\{\Phi_\alpha\}$ forms a directed set with respect to the partial order defined above, $\{X/\Phi_\alpha; \varphi_\beta^\alpha\}$ defines an inverse system of metrizable spaces. Its limit will be denoted by $\mu(X)$. In case X is a Tychonoff space $\mu(X)$ is the completion of X with respect to its finest uniformity (cf. [14]). In general, $\mu(X) = \mu(\tau(X))$ where τ is the Tychonoff functor defined in § 1.

For two pairs (X, A) and (Z, C) of spaces let us denote by $[X, A; Z, C]$ the set of all the homotopy classes of continuous maps $f\colon (X, A) \to (Z, C)$. Then by Theorem 2.3 we have at once the following theorem. Here $\cong$ means the existence of an equivalence in the category of sets or abelian groups as the case may be.

THEOREM 3.2. *Let* (X, A), (Y, B) *and* (Z, C) *be pairs of spaces such that* Y *is locally compact, Hausdorff and Lindelöf and* Z *is metrizable. Then*

$$[(X, A) \times (Y, B); Z, C] \cong \varprojlim \{[(X/\Phi_\alpha, \varphi_\alpha(A)) \times (Y, B); Z, C], [\varphi_\beta^\alpha \times 1]\}\,.$$

In case $\mathfrak{m} = \mathrm{Max}(w(Y), w(Z)) \geqslant \aleph_0$ *and* $\dim X \leqslant n$, *the formula holds even if we restrict* Φ_α *at the right hand side to normal sequences of open coverings of cardinality* $\leqslant \mathfrak{m}$ *and of order* $\leqslant n+1$.

By virtue of Lemma 2.4 we obtain a theorem on cohomology groups corresponding to Theorem 3.2.

THEOREM 3.3. *Under the same assumption on* (X, A) *and* (Y, B) *as in Theorem* 3.2, *we have*

$$H^n((X, A) \times (Y, B); G) \cong \varinjlim \{H^n((X/\Phi_\alpha, \varphi_\alpha(A)) \times (Y, B); G), (\varphi_\beta^\alpha \times 1_Y)^*\},$$

with the same supplements as in Theorem 3.2 *where* $\mathfrak{m} = w(X) \geqslant \aleph_0$.

Theorem 3.2 shows that the problems concerning the homotopy sets of continuous maps from a space X to a metric space can be reduced to those for the case of X being metrizable.

§ 4. An expansion theorem for sets of homotopy classes of maps into polyhedra. Let (X, A) be a pair of spaces, and for an infinite cardinal number $\mathfrak{m}$ let $\{\mathfrak{U}_\lambda |\ \lambda \in \Lambda(\mathfrak{m})\}$ be a cofinal subset of the directed set of all locally finite normal open coverings of X consisting of at most $\mathfrak{m}$ sets; K_λ and L_λ are respectively the nerves of $\mathfrak{U}_\lambda$ and $\mathfrak{U}_\lambda | A$ with the weak topology, $\varphi_\lambda\colon (X, A) \to (K_\lambda, L_\lambda)$ are canonical maps, and $\varphi_\nu^\lambda\colon (K_\lambda, L_\lambda) \to (K_\nu, L_\nu)$ are canonical projections.

Let (Q, Q_0) be a simplicial pair (that is, a pair of a simplicial complex and its subcomplex with the weak or metric topology) such that Q has at most $\mathfrak{m}$ vertices.

For a continuous map $f\colon (X, A) \to (Q, Q_0)$ we call a covering $\mathfrak{U}_\lambda$ a *bridge* for f if there exists a continuous map $f_\lambda\colon (K_\lambda, L_\lambda) \to (Q, Q_0)$ such that $f \simeq f_\lambda \circ \varphi_\lambda$ where $\simeq$ means "is homotopic to" as usual; each continuous map f_λ with this property is called a *bridge map* for f. These notions were first introduced by S. T. Hu in 1948 for the absolute case.

LEMMA 4.1. *For any continuous map* $f\colon (X, A) \to (Q, Q_0)$ *there exists a bridge* $\mathfrak{U}_\lambda$ *with* $\lambda \in \Lambda(\mathfrak{m})$ *and any refinement* $\mathfrak{U}_\nu$ *of* $\mathfrak{U}_\lambda$ *is also a bridge for* f.

Proof. Let us put

$$\mathfrak{W} = \{\mathrm{St}(q; Q) |\ q \text{ ranging over vertices of } Q\}\,.$$

Then $\mathfrak{W}$ is a normal open covering of Q since Q is paracompact. Then there is $\mathfrak{U}_\lambda$ with $\lambda \in \Lambda(\mathfrak{m})$ such that $f^{-1}(\mathfrak{W}) < \mathfrak{U}_\lambda$. Let $\mathfrak{U}_\lambda = \{U_\alpha^\lambda |\ \alpha \in \Omega_\lambda\}$. Then for each vertex u_α^λ of K_λ we can choose a vertex q of Q so that

$$U_\alpha^\lambda \subset f^{-1}(\mathrm{St}(q; Q))\,.$$

Let us put $f_\lambda(u_\alpha^\lambda) = q$. If q is not in Q_0, then $\mathrm{St}(q; Q) \cap Q_0 = \emptyset$. Hence if $U_\alpha^\lambda \cap A \neq \emptyset$, then $f_\lambda(u_\alpha^\lambda)$ is a vertex of Q_0. Thus f_λ defines a simplicial map

$$f_\lambda\colon (K_\lambda, L_\lambda) \to (Q, Q_0)\,.$$

For any point x of X let $\{\alpha_0, \ldots, \alpha_n\}$ be the totality of $\alpha \in \Omega_\lambda$ such that $x \in U_\alpha^\lambda$. Then $f(x)$ and $(f_\lambda \circ \varphi_\lambda)(x)$ lie in the simplex of Q which is spanned by $f_\lambda(u_{\alpha_i}^\lambda)$, $i = 0, 1, \ldots, n$ and q_j, $j = 0, 1, \ldots, s$ where $[q_0, q_1 \ldots q_s]$ is the carrier of $f(x)$ in Q. Hence we have $f \simeq f_\lambda \circ \varphi_\lambda$: $(X, A) \to (Q, Q_0)$.

LEMMA 4.2. *Let $\mathfrak{U}_\nu$ be a locally finite normal open covering of X with $\nu \in \Lambda(\mathfrak{m})$, and let f_ν, g_ν: $(K_\nu, L_\nu) \to (Q, Q_0)$ be two continuous maps such that*

$$f_\nu \circ \varphi_\nu \simeq g_\nu \circ \varphi_\nu \colon (X, A) \to (Q, Q_0) .$$

Then there exists a locally finite normal open covering $\mathfrak{U}_\lambda$ of X with $\lambda \in \Lambda(\mathfrak{m})$ such that

(a) $\mathfrak{U}_\nu < \mathfrak{U}_\lambda$,

(b) $f_\nu \circ \varphi_\nu^\lambda \simeq g_\nu \circ \varphi_\nu^\lambda$: $(K_\lambda, L_\lambda) \to (Q, Q_0)$.

Proof. Let $(K_{\nu'}, L_{\nu'})$ be a subdivision of (K_ν, L_ν) such that $\{u_\beta^{\nu'} |\ \beta \in \Omega_{\nu'}\}$ is the set of all vertices of $K_{\nu'}$ and

(1) $$f_\nu^{-1}(\mathfrak{W}),\ g_\nu^{-1}(\mathfrak{W}) < \{\mathrm{St}(u_\beta^{\nu'}; K_{\nu'}) |\ \beta \in \Omega_{\nu'}\} .$$

Here $\mathfrak{W}$ is the same covering of Q as in the proof of Lemma 4.1.

By assumption there is a continuous map Φ: $(X, A) \times I \to (Q, Q_0)$ such that

(2) $$\Phi(x, 0) = (f_\nu \circ \varphi_\nu)(x) \quad \text{for} \quad x \in X ,$$

(3) $$\Phi(x, 1) = (g_\nu \circ \varphi_\nu)(x) \quad \text{for} \quad x \in X .$$

Then by Theorem 2.5 there exists a locally finite normal open covering $\mathfrak{U}_\lambda$ of X with $\lambda \in \Lambda(\mathfrak{m})$ satisfying the following conditions:

(4) $$\mathfrak{U}_\lambda = \{U_\alpha^\lambda |\ \alpha \in \Omega_\lambda\} ,$$

(5) $$\mathfrak{U}_\lambda > \{\varphi_\nu^{-1} \mathrm{St}(u_\beta^{\nu'};\ K_{\nu'}) |\ \beta \in \Omega_{\nu'}\} ,$$

(6) there exists a collection $\{\mathcal{M}_\alpha |\ \alpha \in \Omega_\lambda\}$ of finite open coverings of $I = [0, 1]$ such that $\{U_\alpha^\lambda \times \mathrm{Cl}\, M |\ M \in \mathcal{M}_\alpha, \alpha \in \Omega_\lambda\}$ is a covering of $X \times I$ which refines $\Phi^{-1}(\mathfrak{W})$.

For each covering $\mathcal{M}_\alpha$ there is a refinement of the form:

$$\{(t_{\alpha,j-1}, t_{\alpha,j+1}) |\ j = 0, 1, \ldots, r_\alpha\}, \text{ where } 0 = t_{\alpha,0} < t_{\alpha,1} < \ldots < t_{\alpha,r_\alpha} = 1$$

and $(t_{\alpha,-1}, t_{\alpha,1})$ and $(t_{\alpha,r_\alpha-1}, t_{\alpha,r_\alpha+1})$ mean respectively $[0, t_{\alpha,1})$ and $(t_{\alpha,r_\alpha-1}, 1]$.

Let u_α^λ be the vertex of K_λ corresponding to U_α^λ. By (6), for each u_α^λ and $t_{\alpha,j} \in I$ there is a vertex $q_{\alpha,j}$ of Q such that

(7) $$U_\alpha^\lambda \times (t_{\alpha,j-1}, t_{\alpha,j+1}) \subset \Phi^{-1}\big(\mathrm{St}(q_{\alpha,j}; Q)\big) .$$

If $U_\alpha^\lambda \cap A \neq \emptyset$, then we have $q_{\alpha,j} \in Q_0$. Let us put

$$\psi(u_\alpha^\lambda, t_{\alpha,j}) = q_{\alpha,j} .$$

Then $q_{\alpha,j}$ and $q_{\alpha,j+1}$ are vertices of a simplex of Q.

For $t_{\alpha,j} < t < t_{\alpha,j+1}$ let $\psi(u_\alpha^\lambda, t)$ be the point of Q which divides the segment from $q_{\alpha,j}$ to $q_{\alpha,j+1}$ in the ratio $(t - t_{\alpha,j})$: $(t_{\alpha,j+1} - t)$.

Let $[u_{\alpha_0}^\lambda, u_{\alpha_1}^\lambda, \ldots, u_{\alpha_n}^\lambda]$ be a simplex of K_λ (resp. L_λ). Then we have

$$U_{\alpha_i}^\lambda \times t \subset \Phi^{-1}\big(\mathrm{St}(q_{\alpha_i,j}; Q)\big) \quad \text{or} \quad \Phi^{-1}\big(\mathrm{St}(q_{\alpha_i,j}; Q)\big) \cap \Phi^{-1}\big(\mathrm{St}(q_{\alpha_i,j+1}; Q)\big)$$

according as $t = t_{\alpha_i,j}$ or $t_{\alpha_i,j} < t < t_{\alpha_i,j+1}$. Hence $\psi(u_{\alpha_i}^\lambda, t)$ for $i = 0, 1, \ldots, n$ are vertices or points on the 1-faces of a simplex of Q (resp. Q_0). Therefore the map ψ can be extended linearly over the simplex $[u_{\alpha_0}^\lambda, \ldots, u_{\alpha_n}^\lambda]$ with t fixed, and we have a map

$$\psi\colon (K_\lambda \times I, L_\lambda \times I) \to (Q, Q_0)$$

by using the same letter ψ. The map ψ is clearly continuous over $[u_{\alpha_0}^\lambda, u_{\alpha_1}^\lambda, \ldots, u_{\alpha_n}^\lambda] \times I$ and hence over $K_\lambda \times I$ since $K_\lambda \times I$ has the weak topology.

For each $\alpha \in \Omega_\lambda$ there is $\beta \in \Omega_{\nu'}$ such that

(8) $$U_\alpha^\lambda \subset \varphi_\nu^{-1}\big(\mathrm{St}(u_\beta^{\nu'}; K_{\nu'})\big) .$$

Let us define a simplicial map

$$\tilde{\varphi}_\nu^\lambda\colon (K_\lambda, L_\lambda) \to (K_{\nu'}, L_{\nu'})$$

by $\tilde{\varphi}_\nu^\lambda(u_\alpha^\lambda) = u_\beta^{\nu'}$.

For each $\beta \in \Omega_{\nu'}$ there is a vertex $f_{\nu'}(u_\beta^{\nu'})$ of Q such that

(9) $$\mathrm{St}(u_\beta^{\nu'}; K_{\nu'}) \subset f_\nu^{-1}\big(\mathrm{St}(f_{\nu'}(u_\beta^{\nu'}); Q)\big) .$$

Hence we have a simplicial map $f_{\nu'}$: $(K_{\nu'}, L_{\nu'}) \to (Q, Q_0)$ which is a simplicial approximation of f_ν. Therefore we have by (8) and (9)

$$U_\alpha^\lambda \subset (f_\nu \circ \varphi_\nu)^{-1}\big(\mathrm{St}(f_{\nu'} \tilde{\varphi}_\nu^\lambda(u_\alpha^\lambda); Q)\big) .$$

On the other hand, by (7) we have

$$U_\alpha^\lambda \times 0 \subset \Phi^{-1}\big(\mathrm{St}(\psi(u_\alpha^\lambda, 0); Q)\big)$$

and hence we get from (2)

$$U_\alpha^\lambda \subset (f_\nu \circ \varphi_\nu)_\nu^{-1}\big(\mathrm{St}(\psi(u_\alpha^\lambda, 0); Q)\big) .$$

Now let us put

$$\psi_0(y) = \psi(y, 0) \quad \text{for} \quad y \in K_\lambda .$$

Then ψ_0: $(K_\lambda, L_\lambda) \to (Q, Q_0)$ is a simplicial map and we have

$$\psi_0 \simeq f_{\nu'} \circ \tilde{\varphi}_\nu^\lambda \simeq f_\nu \circ \tilde{\varphi}_\nu^\lambda .$$

Similarly, we have

$$\psi_1 \simeq g_\nu \circ \tilde{\varphi}_\nu^\lambda ,$$

where $\psi_1\colon (K_\lambda, L_\lambda)\to(Q, Q_0)$ is defined by $\psi_1(y) = \psi(y, 1)$ for $y \in K_\lambda$. Since $\psi_0 \simeq \psi_1$, we have

$$f_\nu \circ \tilde{\varphi}^\lambda_\nu \simeq g_\nu \circ \tilde{\varphi}^\lambda_\nu\colon (K_\lambda, L_\lambda)\to(Q, Q_0)\,.$$

Let $\varphi_0\colon (K_{\nu'}, L_{\nu'})\to(K_\nu, L_\nu)$ be a simplicial map such that $\mathrm{St}(u^{\nu'}_\beta; K_{\nu'}) \subset \mathrm{St}(\varphi_0(u^{\nu'}_\beta); K_\nu)$. Then we have

$$U^\lambda_\alpha \subset \varphi_\nu^{-1}\big(\mathrm{St}(\tilde{\varphi}^\lambda_\nu(u^\lambda_\alpha); K_\nu)\big) \subset \varphi_\nu^{-1}\big(\mathrm{St}(\varphi_0\tilde{\varphi}^\lambda_\nu(u^\lambda_\alpha); K_\nu)\big) \subset U^\nu_\gamma\,,$$

where U^ν_γ is the set of $\mathfrak{U}_\nu$ corresponding to $\varphi_0\tilde{\varphi}^\lambda_\nu(u^\lambda_\alpha)$. Thus $\varphi_0\tilde{\varphi}^\lambda_\nu\colon K_\lambda\to K_\nu$ is a canonical projection which will be denoted by φ^λ_ν. Since φ_0 is homotopic to the identity map, we have

$$f_\nu \circ \varphi^\lambda_\nu \simeq g_\nu \circ \varphi^\lambda_\nu\colon (K_\lambda, L_\lambda)\to(Q, Q_0)\,.$$

This completes the proof of Lemma 4.2.

As an immediate consequence of Lemmas 4.1 and 4.2 we obtain the following theorem.

THEOREM 4.3. *Let (X, A) be a pair of spaces and (Q, Q_0) a simplicial pair such that Q has at most $\mathfrak{m}$ vertices, where $\mathfrak{m}$ is an infinite cardinal number. Let $\{\mathfrak{U}_\lambda|\ \lambda \in \Lambda(\mathfrak{m})\}$ be a cofinal subset of the directed set of all locally finite normal open coverings of X of cardinality $\leqslant \mathfrak{m}$. Let K_λ and L_λ be the nerves of $\mathfrak{U}_\lambda$ and $\mathfrak{U}_\lambda|A$ respectively, $\varphi_\lambda\colon (X, A)\to(K_\lambda, L_\lambda)$ canonical maps and $\varphi^\lambda_\nu\colon (K_\lambda, L_\lambda)\to(K_\nu, L_\nu)$ canonical projections, where $\lambda, \nu \in \Lambda(\mathfrak{m})$. Then $\{[K_\lambda, L_\lambda; Q, Q_0],\ (\varphi^\lambda_\nu)^{\#}|\ \lambda, \nu \in \Lambda(m)\}$ is a direct system and $\{(\varphi_\lambda)^{\#}|\ \lambda \in \Lambda(\mathfrak{m})\}$ defines a bijective map*:

$$[X, A; Q, Q_0] \cong \varinjlim\{[K_\lambda, L_\lambda; Q, Q_0],\ \lambda \in \Lambda(\mathfrak{m})\}\,.$$

Let (Y, B) be a compact pair. Then by a theorem of Milnor the function space $(Q, Q_0)^{(Y,B)}$ has the homotopy type of a simplicial pair. Hence by Theorem 4.3 we have

THEOREM 4.4. *Let (X, A) be a pair of spaces, (Y, B) a pair of compact Hausdorff spaces and (Q, Q_0) a pair of spaces dominated by a simplicial pair* [1]. *Let $\{\mathfrak{U}_\lambda|\ \lambda \in \Lambda\}$ be a cofinal subset of the directed set of all locally finite normal open coverings of X and let $K_\lambda, L_\lambda, \varphi_\lambda$ and φ^λ_ν be the same as in Theorem 4.3 except that $\lambda, \nu \in \Lambda(\mathfrak{m})$. Then $\{[(K_\lambda, L_\lambda)\times(Y, B); Q, Q_0],\ (\varphi^\lambda_\nu\times 1)^{\#}|\ \lambda, \nu \in \Lambda\}$ is a direct system and $\{(\varphi_\lambda\times 1)^{\#}|\ \lambda \in \Lambda\}$ defines a bijective map*:

$$[(X, A)\times(Y, B); Q, Q_0] \cong \varinjlim\{[(K_\lambda, L_\lambda)\times(Y, B); Q, Q_0],\ \lambda \in \Lambda\}\,.$$

Remark. It follows from Theorem 4.3 and [10] that $[X, A; S^n, s_0]$ forms a group called the *n-th cohomotopy group* of (X, A), if $\dim X/A < 2n-1$ and $A \neq \emptyset$ (where S^n is an n-sphere). By virtue of Theorem 4.3, we can prove that the cohomotopy sequence of (X, A) is exact if X is a normal space of dimension $<2n-1$ and A a closed subset of X, since any countable normal open covering of A is refined by restriction of a countable normal open covering of X. This settles a question stated in Morita [10, footnote 3) on p. 254].

[1] A pair of an ANR (for metric spaces) and its closed ANR subspace is dominated by a simplicial pair (Hu [6, p. 211]).

The following theorem will be used in § 5.

THEOREM 4.5. *Let S^n be an n-sphere $(n \geqslant 1)$ with base-point s_0. Let X be a space of dimension $\leqslant n$ and x_0 its base-point. Then the suspension map*

$$S\colon [X, x_0; S^n, s_0]\to[SX, x_0; SS^n, s_0]$$

defined by $S[f] = [Sf]$ is bijective.

Here S denotes the operation of taking the reduced suspension; that is, SX is the quotient space of $X\times I$ by identifying all the points of $X\times \dot{I} \cup x_0\times I$ to a single point which we consider as the base-point of SX and denote by the same letter x_0 as the base-point of X.

Proof. With the notations of Theorem 4.3, the diagram

$$\begin{array}{ccc}
[X, x_0; S^n, s_0] & \xleftarrow{[\varphi_\lambda]} & [K_\lambda, k_\lambda; S^n, s_0] \\
\Big\downarrow{\scriptstyle S} & & \Big\downarrow{\scriptstyle S} \\
[SX, x_0; SS^n, s_0] & \xleftarrow[S[\varphi_\lambda]]{} & [SK_\lambda, k_\lambda; SS^n, s_0]
\end{array}$$

is commutative where without loss of cofinality we may restrict $\mathfrak{U}_\lambda$ to those coverings which have exactly one member containing x_0 and k_λ is a vertex of K_λ corresponding to such a member. We can assume here that $\dim K_\lambda \leqslant n$ for each λ.

In case $n \geqslant 2$, it follows from the suspension theorem in Spanier [21, p. 458] that

$$S\colon [K_\lambda, k_\lambda; S^n, s_0]\to[SK_\lambda, k_\lambda; SS^n, s_0]$$

is bijective. In case $n = 1$, this fact can be verified directly.

Therefore, by virtue of Theorem 4.4 we have Theorem 4.5.

§ 5. The covering dimension.

Let I^n be the n-dimensional cube and $\dot{I}^n$ its boundary $(n \geqslant 1)$.

A continuous map $f\colon X\to I^n$ is called *essential* if any continuous map $g\colon X\to I^n$ such that $g|f^{-1}(\dot{I}^n) = f|f^{-1}(\dot{I}^n)$ satisfies $g(X) = I^n$; otherwise f is called *inessential*. Thus $f\colon X\to I^n$ is essential if and only if there is no continuous map $g\colon X\to I^n$ which is homotopic to f relative to $f^{-1}(\dot{I}^n)$ and $g(X) \neq I^n$. Theorem 5.1 below is given by Smirnov [20] for the case of X being Tychonoff. In view of Lemmas 1.1 and 1.2 we have Theorem 5.1 for the general case.

THEOREM 5.1. *For a space X, $\dim X \leqslant n$ if and only if every continuous map $f\colon X \to I^{n+1}$ is inessential.*

Hence, if $\dim X = n$, there is an essential map $f\colon X \to \dot{I}^n$.

THEOREM 5.2. *Let X be a space such that $\dim X = n$. Let $f\colon X \to \dot{I}^n$ be an essential map. Then the continuous map $f \times 1\colon X \times I \to \dot{I}^n \times I$ is essential.*

Proof. Let us put

$$A = f^{-1}(\dot{I}^n)\,, \qquad Y = X \times I\,, \qquad B = (f \times 1)^{-1}(\dot{I}^{n+1})\,,$$

where $\dot{I}^{n+1} = (\dot{I}^n \times I) \cup (I^n \times \dot{I})$. Then $B = (A \times I) \cup (X \times \dot{I})$. Let us construct quotient spaces and quotient maps as follows:

$$\varphi\colon X \to X/A = X_0\,, \qquad \psi\colon Y \to Y/B = Y_0\,,$$

$$\alpha\colon I^n \to I^n/\dot{I}^n = S^n\,, \qquad \beta\colon I^n \times I \to (I^n \times I)/\dot{I}^{n+1} = S^{n+1}\,,$$

and put

$$x_0 = \varphi(A)\,, \qquad y_0 = \psi(B)\,,$$
$$s_0 = \alpha(\dot{I}^n)\,, \qquad t_0 = \beta(\dot{I}^{n+1})\,.$$

Then we have two continuous maps

$$f_0\colon (X_0, x_0) \to (S^n, s_0)\,, \qquad g_0\colon (Y_0, y_0) \to (S^{n+1}, t_0)$$

so that the diagrams

$$\begin{tikzcd}
(X, A) \arrow[r, "\varphi"] \arrow[d, "f"'] & (X_0, x_0) \arrow[d, "f_0"] \\
(I^n, \dot{I}) \arrow[r, "\alpha"'] & (S^n, s_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
(Y, B) \arrow[r, "\psi"] \arrow[d, "f\times 1"'] & (Y_0, y_0) \arrow[d, "g_0"] \\
(I^n \times I, \dot{I}^{n+1}) \arrow[r, "\beta"'] & (S^{n+1}, t_0)
\end{tikzcd}$$

are commutative.

As usual, let S be the operation of taking the reduced suspension. Then we have

$$Y_0 = SX_0\,, \qquad S^{n+1} = SS^n, \qquad g_0 = Sf_0\,.$$

LEMMA 5.3. *$f^*\colon H^n(I^n, \dot{I}^n; Z) \to H^n(X, A; Z)$ is not zero.*

If we have proved Lemma 5.3, then f_0 is not null homotopic and hence by Theorem 4.5 g_0 is not null homotopic, and consequently

$$(f \times 1)^*\colon H^{n+1}(I^n \times I, \dot{I}^{n+1}) \to H^{n+1}(Y, B)$$

is not zero. Therefore $f \times 1$ is essential. This proves Theorem 5.2.

Proof of Lemma 5.3. If $\psi\colon X \to Y$ is a continuous map from X onto a metric space Y such that $f = g \circ \psi$ with some continuous map $g\colon Y \to \dot{I}^n$, then g is essential. Hence by Theorems 3.2 and 3.3, it is sufficient to prove Lemma 5.3 for the case of X being a metric space. In this case, in the commutative diagram

$$\begin{tikzcd}
 & H^{n-1}(A; Z) \arrow[r, "\delta"] & H^n(X, A; Z) & \\
0 \arrow[r] & H^{n-1}(\dot{I}^n; Z) \arrow[r, "\delta"'] \arrow[u, "(f|A)^*"] & H^n(I^n, \dot{I}^n; Z) \arrow[r] \arrow[u, "f^*"'] & 0
\end{tikzcd}$$

the lower sequence is exact and $\delta(f|A)^*(\iota) \neq 0$ for a generator ι of $H^{n-1}(\dot{I}^n; Z)$; this is seen by Hopf's extension theorem (cf. Dowker [2, Theorem 5.2]) since $f|A$ cannot be extended to a continuous map from X to $\dot{I}^n$. Thus, $f^*\delta(\iota) \neq 0$. This proves Lemma 5.3.

COROLLARY 5.4. $\dim(X \times I) \geqslant \dim X + 1$.

If we use the special case of the Künneth formula:

$$H^{n+1}\big((X, A) \times (I, \dot{I}); Z\big) \cong H^n(X, A; Z)\,,$$

we have another proof of Corollary 5.4 by Lemma 5.3.

In a previous paper [15] we have proved the following theorem for the case of X being a Tychonoff space, from which the theorem for the general case follows immediately in view of Lemmas 1.2 and 1.4.

THEOREM 5.5. *If X is a space and Y a locally compact, paracompact Hausdorff space, then* $\dim(X \times Y) \leqslant \dim X + \dim Y$.

LEMMA 5.6. *Let X be a space and Y a normal, Hausdorff P-space in the sense of Morita* [13]. *If a subset B of Y is locally compact, σ-compact and closed, then $X \times B$ is C-embedded in $X \times Y$.*

Proof. Let f be any real-valued continuous function defined over $X \times B$. Then by Theorem 2.3 there is a continuous map $\varphi\colon X \to T$ for some metric space T such that $f = g \circ (\varphi \times 1_B)$ for some real-valued continuous function g defined over $T \times B$. Since $T \times Y$ is normal by Morita [13] and $T \times B$ is closed in $T \times Y$, there is a real-valued continuous function h defined over $T \times Y$ such that $h|T \times B = g$. Then $h \circ (\varphi \times 1_Y)$ is an extension of f.

THEOREM 5.7. *Let X be a space and Y a σ-compact regular Hausdorff space. Then* $\dim(X \times Y) \leqslant \dim X + \dim Y$.

Before proceeding to the proof of Theorem 5.7 we state Lemmas 5.8 and 5.9 below, which were proved for the case of Tychonoff spaces by Katětov and Smirnov (cf. [4, p. 268]) and by Katětov (cf. [4, p. 285]) respectively.

LEMMA 5.8. *Let A be a subspace of a space X which is C^*-embedded in X. If $\dim X \leqslant n$, then $\dim A \leqslant n$.*

LEMMA 5.9. *Let* $\{A_i|\ i \in N\}$ *be a countable covering of a space* X *such that each* A_i *is* C^**-embedded in* X. *If* $\dim A_i \leqslant n$ *for each* i, *then* $\dim X \leqslant n$.

These lemmas can be reduced to the case of Tychonoff spaces by Lemma 1.2 and the following lemma.

LEMMA 5.10. *Let* A *be a subspace of a space* X *which is* C^**-embedded in* X. *Then* $\Phi_X(A)$ *is* C^**-embedded in* $\tau(X)$, *and* $\dim \Phi_X(A) = \dim A$, *where* Φ_X: $X \to \tau(X)$ *is the natural map defined in* § 1.

Proof. The first assertion is obvious. Let f: $X \to I$ be a continuous map. Then by Lemma 1.1 f is factored through $\tau(X)$ such that $f = g \circ \Phi_X$ for a continuous map g: $\tau(X) \to I$, and, for $a \in A$, $\Phi_X(a) \in g^{-1}((0,1])$ if and only if $a \in f^{-1}((0,1])$. It follows immediately from this fact that $\dim \Phi_X(A) = \dim A$.

Now we shall return to the proof of Theorem 5.7.

Let $\{B_i|\ i \in N\}$ be a countable closed covering of Y by compact subsets. Since Y is a normal P-space, by Lemma 5.6 each subspace $X \times B_i$ is C-embedded in $X \times Y$. Hence by Lemma 5.9 and Theorem 5.5 we have Theorem 5.7.

THEOREM 5.11. *If* X *is a space and* Y *a countable* CW *complex, then* $\dim(X \times Y) = \dim X + \dim Y$.

Proof. Suppose that $\dim Y = n$. Then there is a compact subset B of Y which is homeomorphic to the n-cube I^n. By Lemma 5.6, $X \times B$ is C-embedded in $X \times Y$. Hence by Lemma 5.8 we have $\dim(X \times Y) \geqslant \dim(X \times B)$.

On the other hand, by repeated application of Corollary 5.4 we have $\dim(X \times B) \geqslant \dim X + n$.

These two inequalities, together with Theorem 5.7, imply the desired equality of the theorem.

THEOREM 5.12. *If* X *is a* σ*-compact regular Hausdorff space and* Y *a space with* $\dim Y = 1$, *then* $\dim(X \times Y) = \dim X + \dim Y$.

Proof. In view of Lemmas 5.6, 5.8 and Theorem 5.7, it suffices to prove that $\dim(X \times Y) \geqslant \dim X + 1$ in case X is itself compact. Since $\dim Y = 1$ there exists a continuous map h: $Y \to I$ such that $\mathrm{Cl}\, V - V \neq \emptyset$ for any open set V satisfying $h^{-1}(0) \subset V \subset Y - h^{-1}(1)$.

Suppose that $\dim X = n$. Then by Theorem 5.1 there is an essential map f: $X \to I^n$. Let X_0 be the space which is constructed in the proof of Theorem 5.2. Then we have an essential map f_0: $(X_0, x_0) \to (S^n, s_0)$. Let us define a continuous map Φ: $X_0 \times Y \to I^{n+1}$ by

$$\Phi(x, y) = \big(1 - h(y)\big) f_0(x) + h(y) s_0 \quad \text{for} \quad x \in X_0,\ y \in Y,$$

where we regard S^n as the boundary $\dot{I}^{n+1}$ of I^{n+1}.

Since the set $A = f^{-1}(\dot{I}^n)$ is a G_δ-set, there exist a countable number of compact sets A_i with $i \in N$ such that

$$X - A = \bigcup \{A_i|\ i \in N\}.$$

Hence we have

$$X_0 - x_0 = \bigcup \{\varphi(A_i)|\ i \in N\}.$$

Here φ: $X \to X_0$ is the map defined in the proof of Theorem 5.2; we note that $\varphi(A_i)$ is homeomorphic to A_i for $i \in N$.

Suppose that $\dim(X \times Y) \leqslant n$ and $n \geqslant 1$; the theorem is obvious in case $n = 0$. Since

$$X_0 \times Y = x_0 \times Y \cup \big(\bigcup \{\varphi(A_i) \times Y|\ i \in N\}\big),$$

it follows from Lemmas 5.6, 5.8 and Theorem 5.7 that $\dim(X_0 \times Y) \leqslant n$. Hence Φ is inessential by Theorem 5.1. Thus, there exists a continuous map Ψ: $(X_0, x_0) \times Y \to (S^n, s_0)$ such that

$$\Psi(x, y) = f_0(x) \quad \text{for} \quad y \in h^{-1}(0),$$
$$\Psi(x, y) = s_0 \quad \text{for} \quad y \in h^{-1}(1).$$

Let us put $\Psi_y(x) = \Psi(x, y)$ and denote by V the set of points y of Y such that Ψ_y: $(X_0, x_0) \to (S^n, s_0)$ is not null homotopic. Then we have $h^{-1}(0) \subset V \subset Y - h^{-1}(1)$. For each point y of Y there is an open neighborhood $U(y)$ such that Ψ_y and $\Psi_{y'}$ are homotopic for any point y' of $U(y)$, since X_0 is compact. Therefore V and $Y - V$ are open. This contradicts the assumption that $\mathrm{Cl}\, V - V \neq \emptyset$. Thus, the proof of the theorem is completed.

The following lemma is sometimes useful.

LEMMA 5.13. *Let* (X, x_0) *be a pointed space of dimension* n. *Then* $\dim SX = n+1$.

Proof. For any point q_0 of I^n, there is a homotopy F: $I^n \times I \to I^n$ such that

$$F(q, 0) = q \quad \text{for} \quad q \in I^n,$$
$$F(q, t) = q \quad \text{for} \quad q \in \dot{I}^n,$$
$$F(q_0, 1) \in \dot{I}^n.$$

Hence there exists an essential map f: $X \to I^n$ such that $f(x_0) \in \dot{I}^n$. Let us apply the arguments in the proof of Theorem 5.2 to this map f. Then we have the commutative diagram

$$\begin{tikzcd} (Y, B) \arrow[rr, "\psi"] \arrow[dr, "\psi_0"'] & & (Y_0, y_0) \\ & (SX, SA) \arrow[ur, "\psi_1"'] & \end{tikzcd}$$

where ψ_0 and ψ_1 are quotient maps. Since $\psi^*\colon H^{n+1}(Y_0, y_0; Z) \to H^{n+1}(Y, B; Z)$ is an isomorphism by Corollary 6.3 below,

$$\psi_1^*\colon H^{n+1}(Y_0, y_0; Z) \to H^{n+1}(SX, SA; Z)$$

is one-to-one. Since g_0 is not null homotopic, $H^{n+1}(Y_0, y_0; Z) \neq 0$ by Lemma 5.14 and Theorem 6.7 below. Hence $H^{n+1}(SX, SA; Z) \neq 0$. This shows that $\dim SX \geqslant n+1$.

On the other hand, since $\dim(X \times I) \leqslant n+1$, we have $\dim SX \leqslant n+1$ by Lemma 5.14 below. Therefore, $\dim SX = n+1$.

LEMMA 5.14. *If A is a non-empty subset of a space X of finite dimension, then* $\dim X/A \leqslant \dim X$.

Proof. Let $f\colon X \to X/A$ be the quotient map and let us put $Y = X/A$ and $y_0 = f(A)$. Let $\mathcal{H} = \{H_i|\ i = 1, \dots, m\}$ be a finite covering of Y by cozero-sets. Let Γ be a subset of $\{1, 2, \dots, m\}$ such that $\Gamma = \{i|\ y_0 \in H_i\}$. Then $\bigcap \{H_i|\ i \in \Gamma\}$ is also a cozero-set and hence there is a continuous map $\varphi\colon Y \to I$ such that $\bigcap \{H_i|\ i \in \Gamma\} = \{y \in Y|\ \varphi(y) > 0\}$. Let us put

$$C = \{y \in Y|\ \varphi(y) \geqslant a\},$$
$$V_0 = \{y \in Y|\ \varphi(y) > \tfrac{1}{2}a\},$$
$$V_i = H_i - C \quad \text{for} \quad i = 1, 2, \dots, m,$$

where $a = \varphi(y_0) > 0$. Then $\{V_i|\ i = 0, 1, \dots, m\}$ is a covering of Y by cozero-sets and a refinement of $\mathcal{H}$, and $y_0 \notin V_i$ for $i > 0$. Suppose that $\dim X \leqslant n$. Since $\{f^{-1}(V_i)|\ i = 0, 1, \dots, m\}$ is a finite covering of X by cozero-sets, there exists a finite covering $\mathcal{G} = \{G_0, G_1, \dots, G_m\}$ of X by cozero-sets such that

$$G_i \subset f^{-1}(V_i) \quad \text{for} \quad i = 0, 1, \dots, m,$$

and the order of $\mathcal{G}$ does not exceed $n+1$. Let $\mathcal{L}$ be a normal open covering of X which is a star-refinement of $\mathcal{G}$ and let us put

$$M_i = \bigcup \{L \in \mathcal{L}|\ \operatorname{St}(L, \mathcal{L}) \subset G_i\} \quad \text{for} \quad i = 0, 1, \dots, m.$$

Then we have

$$\operatorname{St}(M_i, \mathcal{L}) \subset G_i \quad \text{for} \quad i = 0, 1, \dots, m,$$
$$X = \bigcup \{M_i|\ i = 0, 1, \dots, m\}.$$

Since $\mathcal{L}$ is normal, for each i there is a continuous map $\varphi_i\colon X \to I$ such that

$$\varphi_i(x) = \begin{cases} 1 & \text{for} \quad x \in M_i, \\ 0 & \text{for} \quad x \in X - \operatorname{St}(M_i, \mathcal{L}). \end{cases}$$

Since $G_i \cap A = \emptyset$ for $i > 0$, we have $\operatorname{St}(M_i, \mathcal{L}) \cap A = \emptyset$ for $i > 0$ and $A \subset M_0$. Let us put

$$U_i = \{x \in X|\ \varphi_i(x) > 0\} \quad \text{for} \quad i = 0, 1, \dots, m,$$
$$\psi_i(y) = \varphi_i(f^{-1}(y)) \quad \text{for} \quad y \in Y.$$

Then $\psi_i\colon Y \to I$ is single-valued for each $i \geqslant 0$, and $\varphi_i = \psi_i f$. Hence ψ_i is continuous. Since

$$U_i = f^{-1}f(U_i) \quad \text{and} \quad f(U_i) = \{y \in Y|\ \psi_i(y) > 0\},$$

$\{f(U_i)|\ i = 0, 1, \dots, m\}$ is a finite covering of Y by cozero-sets which is a refinement of $\mathcal{H}$ and has order $\leqslant n+1$. This proves Lemma 5.14.

§ 6. Homotopical cohomology groups and the Künneth formula. Let G be an abelian group and $K = |K(G, n)|$ the geometric realization of the Eilenberg–MacLane complex $K(G, n)$; let k_0 be the identity element of the weak abelian group K. For any pair (X, A) of spaces let us define a map

$$\Psi_{X,A}\colon [X, A; K, k_0] \to H^n(X, A; G)$$

by $\Psi_{X,A}([f]) = f^*(\zeta)$ where ζ is an n-characteristic element of $H^n(K, k_0; G)$. It is known that $[X, A; K, k_0]$ is an abelian group and that $\Psi_{X,A}$ is a homomorphism.

As is known, $\Psi_{X,A}$ is an isomorphism if (X, A) is a simplicial pair. Since $\Psi_{X,A}$ is natural with respect to a pair (X, A), by applying Theorem 4.3 we have the following theorem.

THEOREM 6.1. *$\Psi_{X,A}$ is an isomorphism for any pair (X, A).*

In case $A = \emptyset$, we can define a natural homomorphism

$$\Psi_X\colon [X; K] \to H^n(X; G)$$

similarly and we have

THEOREM 6.2. *Ψ_X is an isomorphism for any space X.*

For a pair (X, A) with A non-empty, let us denote by X/A the space obtained from X by contracting A to a point q_A, and $\varphi\colon (X, A) \to (X/A, q_A)$ the quotient map. Then we have a commutative diagram:

$$\begin{tikzcd}
{[X, A; K, k_0]} \arrow[r, "\Psi"] & H^n(X, A; G) \\
{[X/A, q_A; K, k_0]} \arrow[r, "\Psi"'] \arrow[u, "\varphi^\#"] & H^n(X/A, q_A; G). \arrow[u, "\varphi^*"']
\end{tikzcd}$$

Since $\varphi^\#$ is bijective, φ^* is an isomorphism. Thus, we have

COROLLARY 6.3. $H^n(X, A; G) \cong H^n(X/A, q_A; G)$. *In particular, $H^n(X, A; G) \cong H^n(X/A; G)$ for $n \geqslant 1$.*

COROLLARY 6.4. *If $f \simeq g\colon (X, A) \to (Y, B)$, then $f^* = g^*\colon H^n(Y, B; G) \to H^n(X, A; G)$.*

Proof is obvious from the fact that in this case we have

$$f^{\#} = g^{\#}\colon [Y, B; K, k_0] \to [X, A; K, k_0].$$

LEMMA 6.5. *If (Y, B) is a compact pair, then $(K, k_0)^{(Y,B)}$ has the same homotopy type as $\bigtimes_{q=0}^{n} |K(H^{n-q}(Y, B; G), q)|$.*

Proof. Let Y/B be the quotient space of Y obtained by identifying all the points of B to a single point q_B. Then $(K, k_0)^{(Y,B)} \cong (K, k_0)^{(Y/B, q_B)}$. Hence we have only to deal with pointed spaces; we do so without making explicit mention of base-points.

Let $L = |K(G, n+1)|$ be the geometric realization of the Eilenberg–MacLane complex $K(G, n+1)$. Then L^Y is a weak abelian group. Let M be the path-component of the identity element of the total singular complex $S(L^Y)$ of L^Y. Then M is a connected group complex. Since we have

$$\pi_{q+1}(M) = \pi_{q+1}(L^Y) \cong [S^{q+1}; L^Y] \cong [Y; \Omega^{q+1}L]$$
$$\cong [Y; |K(G, n-q)|] \cong H^{n-q}(Y; G)$$

for $0 \leqslant q \leqslant n$ and $\pi_{q+1}(M) = 0$ for $q > n$, by Moore [16, Theorem 3.29] we have

$$M \simeq |\bigtimes_{q=0}^{n} K(H^{n-q}(Y; G), q+1)|.$$

Hence by taking the geometric realization we have

$$|M| \simeq |\bigtimes_{q=0}^{n} K(H^{n-q}(Y; G), q+1)| \simeq \bigtimes_{q=0}^{n} |K(H^{n-q}(Y; G), q+1)|.$$

On the other hand, since L^Y has the homotopy type of a CW complex, by taking loop spaces we have

$$\Omega|M| = \Omega|S(L^Y)| \simeq \Omega(L^Y) \simeq (\Omega L)^Y \simeq K^Y.$$

Thus, Lemma 6.5 is proved.

We are now in a position to prove the following Künneth formula.

THEOREM 6.6. *If (Y, B) is a compact pair, then*

$$H^n((X, A) \times (Y, B); G) \cong \bigoplus_{q=0}^{n} H^q(X, A; H^{n-q}(Y, B))$$

for any pair (X, A).

Proof. By Lemma 6.5 we have

$$[(X, A) \times (Y, B); K, k_0] \cong [X, A; (K, k_0)^{(Y,B)}, k_0{}^Y]$$
$$\cong \bigoplus_{q=0}^{n} [X, A; |K(H^{n-q}(Y, B; G), q)|, k_0],$$

and hence Theorem 6.6 follows from Theorem 6.1.

Next, let S^n be an n-sphere and p_0 a point of S^n. For a pair (X, A) of spaces with A closed we define a map

$$\theta_{X,A}\colon [X, A; S^n, p_0] \to H^n(X, A; \boldsymbol{Z})$$

by $\theta_{X,A}([f]) = f^*(\xi)$, where ξ is a generator of $H^n(S^n, \boldsymbol{Z})$; $\theta_{X,A}$ is natural with respect to a pair (X, A). Then we have the following generalization of the Hopf classification theorem.

THEOREM 6.7. *If (X, A) is a pair of spaces with A non-empty and $\dim X/A \leqslant n$, then $\theta_{X,A}$ is an isomorphism for $n \geqslant 1$.*

Proof. Let $\varphi\colon (X, A) \to (X/A, q_A)$ be the quotient map. Then we have a commutative diagram

$$\begin{tikzcd}
{[X, A; S^n, p_0]} \arrow[r, "\theta"] & H^n(X, A; \boldsymbol{Z}) \\
{[X/A, q_A; S^n, p_0]} \arrow[r, "\theta"'] \arrow[u, "\varphi^{\#}"] & H^n(X/A, q_A; \boldsymbol{Z}) \arrow[u, "\varphi^*"']
\end{tikzcd}$$

and $\varphi^{\#}$ and φ^* are bijective. Hence it is sufficient to deal with the case of pointed spaces. It is known that if X is a simplicial complex, $x_0 \in X$ and $\dim X \leqslant n$, then θ_{X,x_0} is an isomorphism for $n \geqslant 1$. Hence we have the theorem by Theorems 1.3 and 4.3.

Finally, we shall prove the following theorem.

THEOREM 6.8. *Let (X, A) be a pair of spaces and G an abelian group which is generated by at most $\mathfrak{m}$ elements ($\mathfrak{m} \geqslant \aleph_0$). Then we have*

$$H^n(X, A; G) \cong \varinjlim \{H^n(K_\lambda, L_\lambda; G), (\varphi_\nu^\lambda)^* |\ \lambda, \nu \in \Lambda(\mathfrak{m})\},$$

where $\{\mathfrak{U}_\lambda |\ \lambda \in \Lambda(\mathfrak{m})\}$, K_λ, L_λ and φ_ν^λ are the same as in Theorem 4.3.

Proof. The theorem follows from Theorems 4.3 and 6.1, since $|K(G; n)|$ is chosen to be a simplicial complex with at most $\mathfrak{m}$ vertices.

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Measurable relations

by

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Abstract. The measurability properties of relations (= set valued functions) are developed. First the logical relations among the various definitions of measurability are worked out and used to determine sufficient conditions for the intersection of measurable relations to be measurable. These results are then used to generalize the selection theorems of Kuratowski and Ryll-Nardzewski, Castaing, and Aumann, to generalize Filippov's implicit function theorem, and to prove the existence of a measurable selector extending a given measurable partial selector. The paper concludes with some applications to relations with values in a locally convex space.

1. Introduction. Measurable relations, i.e., set valued functions which assign to each element t of a measurable space T a subset of a topological space X in a manner satisfying any one of several possible definitions of measurability, have been studied extensively in recent years by numerous authors (Aumann [A-1, 2], Castaing [C], Debreu [D], Jacobs [J], Kuratowski and Ryll–Nardzewski [KR], McShane and Warfield [MW], Rockafellar [R], Van Vleck and the author [HV-1, 2, 3] and many others.) Much of this work either assumes that the measurable structure on T is that of a Radon measure on a locally compact space or that X is a very special kind of space, say compact metric or Euclidean. The purpose of this paper is to develop the properties of measurable relations in the general situation where T is an abstract measurable space and X is separable metric. It turns out that to work with T this general we must usually (but not always) introduce compactness somewhere, either in X or in the values of a multifunction with values in X. Alternatively, we obtain a similar body of results assuming that X is a Souslin space and that a σ-finite measure is defined on the measurable subsets of T.

In the main, we will confine our attention to the general properties of measurable relations, and to selection, extension, and implicit function theorems.

In Section 2 we give most of the necessary definitions and terminology, and state without proof some trivial but often used properties of measurable relations. Section 3 is an account of the logical relationships among the various definitions of measurability. Section 4 is concerned with measurability of the intersection, complement, and boundary of