Čech cohomology and covering dimension for topological spaces

by

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Abstract. For a topological space $X$ let us define the covering dimension of $X$ and the Čech cohomology groups of $X$ by using only normal open covers of $X$ instead of arbitrary open coverings. Then it will be shown that some of the basic theorems concerning the Čech cohomology groups and covering dimension of CW complexes or paracompact spaces, such as the Hopf classification theorem and the product theorem on dimension for the case of one factor being $κ$-compact, can be generalized to the case of arbitrary topological spaces.

In discussing the topological invariants for topological spaces, such as the Čech cohomology groups and the covering dimension, which are defined by using open coverings, it seems natural to make a modification by restricting open coverings to normal ones.

For the covering dimension of Tychonoff spaces (= completely regular Hausdorff spaces) such a modification was made by M. Katětov [8] and Yu. Smirnov [30]; a nice exposition of their results is given in Engelking [4]. Applying their modification to a general case, we shall define the covering dimension of a topological space $X$, denoted by $\dim X$, to be the least integer $n$ such that every finite normal open covering of $X$ admits a finite normal open covering of order $\leq n+1$ as its refinement.

In case $X$ is a normal space, $\dim X$ defined here coincides with the covering dimension of $X$ in the usual sense.

As for the $n$th Čech cohomology group $H^n(X; G)$ of a topological space $X$ with coefficients in an abelian group $G$, we shall define it by using only normal open coverings of $X$. In case $X$ is paracompact Hausdorff, $H^n(X; G)$ is the usual Čech cohomology group based on all open coverings.

The purpose of this paper is to show that with these definitions we can generalize some of the basic theorems concerning the Čech cohomology groups and dimension of paracompact Hausdorff spaces or CW complexes to the case of topological spaces.

Let $X$ be a topological space, $G$ an abelian group, and $Z$ the additive group of all integers. Let $|K(G, n)|$ be the geometric realization of the
Eilenberg-MacLane complex $K(G, n)$. Then the following results will be established.

I. $\dim (X \times Y) \leq \dim X + \dim Y$ if $X$ is a locally compact, paracompact Hausdorff space, and the equality holds if $Y$ is a locally finite polyhedron.

II. If $\dim X \leq n$, then the cohomotopy group $\pi^n(X, x_0)$, that is, the group of homotopy classes of continuous maps from $(X, x_0)$ to $(S^n, p_0)$, is isomorphic to $H^n(X; Z)$ for $n \geq 1$, where $S^n$ is an $n$-sphere and $x_0 \in X$, $p_0 \in S^n$. (Hopf's classification theorem.)

III. The $n$th homotopical cohomology group, that is, the group $[K(G, n)]$ of homotopy classes of continuous maps from $X$ to $K(G, n)$, is isomorphic to $H^n(X; G)$.

IV. If $Y$ is a compact Hausdorff space, then the Künneth formula

$$H^n(X \times Y; G) \cong \bigoplus_{p+q=n} H^p(X) \otimes H^q(Y; G)$$

holds.

These results have been proved hitherto only for the case of $X$ being a paracompact Hausdorff space; indeed, for this case, I was proved by K. Morita [9], II by G. H. Downer [3], III by P. J. Huber [7] (for the case where $X$ is a 3-space and $G$ is countable), T. Goto [5] and V. Bartik [1], and IV by V. Bartik [1]. Throughout this paper by a space we shall mean a non-empty topological space, and $I$ denotes the closed unit interval $[0, 1]$ in the real line and $N$ the set of all positive integers.

§ 1. The Tychonoff functor $\tau$. Throughout this section, let $X$ and $Y$ be spaces. Let $Y^X$ denote the set of all continuous maps $g: X \to Y$ with the compact-open topology and construct the product space $P(X) = H(T, \tau \times I^Y)$, where $T_\gamma = I$. By defining $\Phi_\tau(g)$ to be the point of $P(X)$ whose $\varphi$-coordinate is $g(x)$, we have a continuous map $\Phi_\tau: X \to P(X)$. For a continuous map $f: X \to Y$ we have a continuous map $P(f): P(X) \to P(Y)$ by defining $P(f)(\tau)$ to be the point of $P(Y)$ whose $\varphi$-coordinate is the constant $f$-coordinate of $\tau$ where $\tau \in P(X), \varphi \in I^Y$. Then the diagram

$$\begin{array}{ccc}
X & \to & Y \\
\phi \downarrow & & \downarrow \phi \\
P(X) & \to & P(Y)
\end{array}$$

is commutative. Let us put

$$\tau(X) = \text{Image of } \Phi_\tau,$$

$$\tau(f) = P(f)\tau(X): \tau(X) \to \tau(Y),$$

and denote by the same letter $\Phi_\tau$ the map from $X$ to $\tau(X)$ which coincides with $\Phi_\tau$ but has $\tau(X)$ as its range. Then we have a commutative diagram

$$\begin{array}{ccc}
X & \to & Y \\
\phi \downarrow & & \downarrow \phi \\
\tau(X) & \to & \tau(Y)
\end{array}$$

Thus $\tau$ is a covariant functor from the category of topological spaces and continuous maps into itself, and $(\Phi_\tau)^{-1}$ defines a natural transformation from the identity functor to $\tau$; $\tau(X)$ is a Tychonoff space and $\Phi_\tau: X \to \tau(X)$ is a homeomorphism if $X$ is itself a Tychonoff space (cf. Dugundji [3]). Hence we shall call $\tau$ the Tychonoff functor. The Tychonoff functor is the reflector from the category above to the full subcategory of Tychonoff spaces; that is,

**Lemma 1.1.** Any continuous map $f$ from $X$ into a Tychonoff space $B$ is factored through $\tau(f)$ such that $f = g \circ \Phi_\tau$ for some continuous map $g: \tau(f) \to B$; $g$ is determined uniquely by $f$.

For a normal open covering $\{G_\alpha | \alpha \in \Omega\}$ of a space $X$ there exist a continuous map $f$ from $X$ onto a metric space $T$ and a normal open covering $\{f^{-1}(G_\alpha) | \alpha \in \Omega\}$ of $T$ such that $f^{-1}(G_\alpha) \subset G_\alpha$ for each $\alpha \in \Omega$. By Lemma 1.1 the map $f$ is factored through $\tau(f)$. Therefore we have

**Lemma 1.2.** $\dim X \leq n$ if and only if $\dim \tau(f) \leq n$.

Moreover, we have

**Theorem 1.3.** $\dim X \leq n$ if and only if for every normal open covering $\mathcal{G}$ of $X$ there is a normal open covering $\mathcal{G}'$ of $X$ of order $\leq n + 1$ which is a refinement of $\mathcal{G}$.

**Proof.** In case $X$ is a Tychonoff space, this theorem is proved by Pasynkov [18]. For another proof, cf. Morita [15]. By virtue of Lemma 1.2 and the remark preceding it, the theorem for the general case follows from the theorem for the special case mentioned above.

The following lemma due to Pappier [19] is useful sometimes.

**Lemma 1.4.** If $X$ is a locally compact Hausdorff space, then $\tau(X \times Y) = \tau(X) \times \tau(Y)$.

**Proof.** If $T$ is a Tychonoff space, so is $T^X$ (cf. Engelking [4]). Since $Y$ is locally compact Hausdorff, there is a bijective map $\Phi: T^X \to (T, I^Y)$ such that $\Phi(f) = f(x, y)$ for $f \in T^X$. Hence there is a continuous map $g: \tau(X) \times Y \to \tau(X \times Y)$ such that $\Phi_\tau(g) = g \circ \Phi_\tau(y, x)$. On the other hand, since $\tau(X) \times Y$ is a Tychonoff space, by Lemma 1.1 there is a continuous map $h: \tau(X \times Y) \to \tau(X \times Y)$ such that $h \circ \tau(X \times Y)$ coincides with $\Phi_\tau(y, x)$ for $y, x \in T$. Therefore $h \circ \tau(X \times Y) = \tau(X \times Y)$.

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§ 2. A factorization theorem for maps into metric spaces and a covering theorem for product spaces. Let $X$ be a space and $\Phi = \{U_i : i \in N\}$ a normal sequence of open coverings of $X$. Let $(X, \Phi)$ be the space obtained from $X$ by taking $(\text{St}(s, U_i) : i \in N)$ as a local base (as a basis of neighborhoods) at each point $x \in X$, and $X/\Phi$ the quotient space obtained from $(X, \Phi)$ by identifying each point $x$ and $y$ in $\text{St}(s, U_i)$ for each $i \in X$. Let us denote by $\varphi$ the composite of the identity map from $X$ onto $(X, \Phi)$ and the quotient map from $(X, \Phi)$ onto $X/\Phi$. Then $\varphi : X \to X/\Phi$ is a continuous map. We shall call $X/\Phi$ the space associated with $\Phi$, and $\varphi$ the canonical map.

For any subset $A$ of $X$ let us put

$$\text{Int}(A; \Phi) = \{x \in X | \text{St}(s, U_i) \subset A \text{ for some } i \in N\}.$$  

Then $\text{Int}(A; \Phi)$ is open in $(X, \Phi)$ and, since $\varphi^{-1}[\text{Int}(A; \Phi)] = \text{Int}(A; \Phi)$, $\varphi[\text{Int}(A; \Phi)]$ is open in $X/\Phi$. Let us put further

$$U_i = \text{Int}(U_i; \Phi) = U \in U_i \text{ for } i \in N.$$  

Then $\Phi' = \{U_i : i \in N\}$ is also a normal sequence of open coverings of $X$ and

$$(U_i : i \in N), (U_i : i \in N) \text{ for } i \in N$$

where, for coverings $U$ and $\Phi$, by $U_i \subset U$ we mean that $U$ is a refinement of $U_i$. Hence $X/\Phi' = X/\Phi$. In particular, $(\text{St}(s, U_i) : i \in N)$ is a local base at each point $x$ of $X$. Thus, $X/\Phi$ is metrizable. If $U_i$ consists of at most $m$ members, so does $U_i$, and if the order of $U_i$ is not greater than $m+1$, so is the order of $U_i$. These results were obtained in our previous paper [14] and the argument given there proves the following lemma. Here $w(X)$ denotes the weight of $X$ (i.e. the minimal cardinal number $m$ such that $X$ has an open base of cardinality $m$).

**Lemma 2.1.** Let $\Phi = \{U_i : i \in N\}$ be a normal sequence of open coverings of $X$. Then $X/\Phi$ is metrizable, and if each $U_i$ consists of at most $m$ members where $m > n_0$ (resp. has order $< n+1$), then $w(X/\Phi) < m$ (resp. $\dim X/\Phi < n$).

As an immediate consequence we have

**Lemma 2.2.** Let $f : X \to T$ be a continuous map from a topological space $X$ into a metric space $T$, and suppose that $T(\Phi) > n_0$. Then there exists a metric space $S$, a continuous onto map $g : S \to T$ and a continuous map $h : S \to T$ such that $f = g \circ h$, $w(S) < w(T)$ and $\dim S < \dim T$.

Indeed, if $\{U_i : i \in N\}$ is a normal sequence of open coverings of $X$ such that $(\text{St}(i, U_i) : i \in N)$ is a local basis at each point $x$ of $T$ and each $U_i$ consists of at most $m$ members, where $m = w(T)$, then there is a normal sequence $\Phi = \{U_i : i \in N\}$ of open coverings of $X$ such that $f^{-1}(U_i) < U_i$, order $U_i < \dim X$ for $i \in N$ and each $U_i$ consists of at most $m$ members;

this is seen from Theorem 1.3. Hence by Lemma 2.1 it is easy to see that $S = X/\Phi$ has the desired property.

For the case of $X$ being a Tychonoff space, Lemma 2.2 was stated without proof by Hausdorff in [17], and in [18] he gave a proof which, however, lacks a consideration about the weight of $S$; for the assertion that $S$ can be chosen so that $w(S) < n_0$ if $w(T) < n_0$, he gave a separate proof in [18].

**Theorem 2.3.** Let $X$ be a space, $Y$ a locally compact, Lindelöf, Hausdorff space, and $T$ a metric space. Then for any continuous map $f : X \times X \to T$ there exist a metric space $S$, a continuous onto map $\varphi : X \to S$ and a continuous map $g : S \times S \to T$ such that $f = (g \circ (p \times q))$ and $w(S) < \max\{w(T), w(Y)\}$.

**Proof.** Since $Y$ is locally compact, there is a bijective map $g : X \to T$ such that $g(f)(y) = f(s, y)$ for $f : X \times X \to T$. Since $Y$ is locally compact and Lindelöf, the function space $T^Y$ is metrizable. Hence by Lemmas 3.1, 2.1 and 2.2 there is a metric space $S$, a continuous onto map $\varphi : X \to S$ and a continuous map $h : S \to T^Y$ such that $f = (g \circ h)$. If $w(T) < \infty$, then $w(S) < \dim X$. If we put $g = g^{-1}(h) : S \times S \to T$, and $w(T) < \infty$, then $w(T) < \infty$. Since $w(T) < \infty$, this proves Theorem 2.3.

**Lemma 2.4.** Let $X$ and $Y$ be the same as in Theorem 2.3. Then for any normal open covering $\mathcal{U} = \{U_i : i \in N\}$ of $X \times Y$ there is a metrizable space $S = X \times Y$ associated with a normal sequence $\Phi$ of open coverings of $X$ such that $p(\pi_1)^{-1}(U_i) \subset G_i$ for some normal open covering $\mathcal{G} = \{G_i : i \in N\}$ of $X \times Y$, where $p : X \times Y \to X$.

**Proof.** There are a metric space $T$, a continuous $f : X \times Y \to T$ and an open covering $\mathcal{G} = \{G_i : i \in N\}$ of $T$ such that $f^{-1}(G_i) \subset U_i$ for each $i$. By Theorem 2.3 the map $f$ is factored through $S = X \times Y$, which is associated with some normal sequence $\Phi$ of open coverings of $X$, such that $f = (g \circ (p \times q))$. The covering $\mathcal{G}$ has the desired property.

This proves Lemma 2.4.

**Remark.** Lemma 2.4 can be proved similarly as in the proof of [15, Theorem 1] without appealing to Theorem 2.3. Thus, we have another proof of Theorem 2.3 by using Theorem 2.3 which relies upon Lemma 2.4.

**Theorem 2.5.** Let $X$ be a space and $Y$ a compact Hausdorff space. Then there exists an open covering $\mathcal{U} = \{U_i : i \in N\}$ of $X \times Y$ such that $\text{card}(U_i) < \infty$ and $\text{card}(U_i) < \infty$ according as $m > n_0$ or $m < n_0$ where $m = \max\{\text{card}(U_i), w(T)\}$.

(6) For a suitable collection $\{U_i : i \in N\}$ of finite open coverings of $X \times Y$, the collection $\{U_i \times V : V \in U_i, \lambda \in A\}$ is an open covering of $X \times Y$ which refines $\mathcal{U}$.  

(a) $\mathcal{U}$ is a normal open covering of $X$ if and only if $\mathcal{V}$ is a normal open covering of $X \times Y$.

Proof: There exists a collection $\{\mathcal{U}_i \mid \delta \in A\}$ of finite open coverings of $Y$ such that $\operatorname{card} \delta \leq m$ or $\operatorname{card} \delta < \kappa_0$ according as $m > \kappa_0$ or $m < \kappa_0$, and such that every finite open covering of $Y$ is refined by some $\mathcal{U}_i$ with $\delta \in A$. For each $\delta \in A$, let $\mathcal{U}_i = \{F_{\delta i} \mid i = 1, \ldots, r_i\}$ and let $F_{\delta i}$ be the set of all maps $x: \{1, \ldots, r_i\} \to Y$. Then Card $F_{\delta i} < m$ or $\operatorname{card} \delta < \kappa_0$ according as $m > \kappa_0$ or $m < \kappa_0$.

For $\delta \in A$, $x \in T_i$, let us put

$$U(\delta, x) = \{x \times X \mid \{x \times \mathcal{V}_{\delta x} \subset T_{\delta x} \} \text{ for each } i \leq r_i\}.$$

Since $\mathcal{V}_{\delta x}$ is compact, $U(\delta, x)$ is an open set of $X$. For any point $x$ of $X$ there is a finite subset $\mathcal{V} \subset \mathcal{U}$ such that $x \times X \subset \bigcup \{\mathcal{V}_{\delta x} \mid \delta \in \mathcal{A}\}$. Hence there is $\delta \in \mathcal{A}$ such that $x \times X \subset \bigcup \{\mathcal{V}_{\delta x} \mid \delta \in \mathcal{A}\}$ is a refinement of $\{x \mid x \in T_{\delta x}, x \in T_{\delta x}, \delta \in \mathcal{A}\}$. Hence $U = \{U(\delta, x) \times \mathcal{V}_{\delta x} \mid \delta \in \mathcal{A}\}$ is an open covering of $X \times Y$.

(b) $X \times Y$ is normal for any compact Hausdorff space $Y$ of weight $\leq m$.

(c) $X \times \mathbb{R}$ is normal.

(d) $X \times \mathbb{D}$ is normal.

Proof: (a) $\Rightarrow$ (b). Let $\mathcal{U} = \{U_{\delta x} \mid \delta \in \mathcal{A}\}$ be an open covering of $X \times Y$ such that $\operatorname{card} \mathcal{U} \leq m$. Since $\operatorname{card} \mathcal{V} < m$, by Theorem 2.5 there is an open covering $\mathcal{V} = \{U_{\delta x} \mid \delta \in \mathcal{A}\}$ of $X \times Y$ such that $\operatorname{card} \mathcal{V} < m$ and for a suitable collection $\{\mathcal{V}_{\delta x} \mid \delta \in \mathcal{A}\}$ of finite open coverings of $Y$ the collection $\mathcal{U} = \{U_{\delta x} \times \mathcal{V}_{\delta x} \mid \delta \in \mathcal{A}\}$ is an open covering of $X \times Y$ which refines $\mathcal{U}$. Let $X$ be $\mathfrak{m}$-paracompact. Then $\mathcal{U}$ is normal and hence by Theorem 2.5 $\mathcal{V}$ is normal. This proves the implication (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are obvious.

(d) $\Rightarrow$ (a). Suppose that $X \times \mathbb{D}$ is normal. Let $\mathcal{U}$ be a set with card $\mathcal{U} = m$. For each $\delta \in \mathcal{U}$ let us put $Y_{\delta} = D$, and construct the product space $X = \prod \{Y_{\delta} \mid \delta \in \mathcal{U}\}$ which is homeomorphic to $\mathbb{D}$; denote by $\pi_\delta$ the projection from $Y$ onto $Y_{\delta}$. Let $\mathcal{U} = \{\mathcal{U}_{\delta} \mid \delta \in \mathcal{U}\}$ be any open covering of $X$ and put

$$H_{\delta} = \bigcup \{\mathcal{U}_{\delta} \times \mathcal{U}_{\delta} \mid \delta \in \mathcal{U}\}$$

is an open covering of $X \times Y$ which refines $\mathcal{U}$.

Then $X = \{H_{\delta} \mid \delta \in \mathcal{U}\}$ is a normal open covering of $X \times Y$. Hence by Theorem 2.5 there is a normal open covering $\{U_{\delta x} \mid \delta \in \mathcal{A}\}$ of $X \times Y$ such that for a suitable collection $\{\mathcal{V}_{\delta x} \mid \delta \in \mathcal{A}\}$ of finite open coverings of $Y$ the collection $\{U_{\delta x} \times \mathcal{V}_{\delta x} \mid \delta \in \mathcal{A}\}$ is an open covering of $X \times Y$ which refines $\mathcal{U}$. Here $\mathcal{V}_{\delta x}$ can be chosen so that for some finite subset $\gamma = \{\gamma_1, ..., \gamma_n\}$ of $\delta$ we have

$$\forall \delta \in \gamma \exists \delta \in \mathcal{A} \leq \mathcal{U}$$

Suppose that

$$U_{\delta x} \times \mathcal{V}_{\delta x} \subset H_{\delta}.$$

Pick a point $y$ of $\bigcup \mathcal{U}_{\delta x}(k_i)$ such that $y = 1$ for $i \in \gamma\{\delta_1, ..., \delta_n\}$. If $x \not\in \bigcup \mathcal{U}_{\delta x}(k_i)$ then $(x, y) \not\in H_{\delta}$. Hence we have

$$U_{\delta x} \subset \bigcup \mathcal{U}_{\delta x}(k_i) \subset \bigcup \mathcal{V}_{\delta x}(k_i).$$

This shows by Morita [11, Corollary 1.3] that $\mathcal{U}$ is a normal covering of $X$. Thus, the proof of Theorem 2.6 is completed.

§ 3. The Čech cohomology groups. Let $(X, A)$ be a pair of a space $X$ and its subspace $A$ and $G$ an abelian group. Let $\{U_{\delta x} \mid \delta \in \mathcal{A}\}$ be the family of all locally finite normal open coverings of $X$. Let us denote by $E_1, L_1$ the nerves of $U_{\delta x}$ and $U_{\delta x} \cap A = (U \cap A) / U \times U_{\delta x}$ respectively. For coverings $U_{\delta x}$ and $U_{\delta x}$, with $U_{\delta x} < U_{\delta x}$ we define a canonical projection $\pi_1: (X_1, L_1)$
In case $m = \text{Max}(w(X), w(Z)) \geq r_0$ and $\dim X \leq n$, the formula holds even if we restrict $\Theta_\gamma$ at the right hand side to normal sequences of open coverings of cardinality $\leq m$ and of order $\leq n+1$.

By virtue of Lemma 2.4 we obtain a theorem on ochronology groups corresponding to Theorem 3.2.

**Theorem 3.3.** Under the same assumption on $(X, A)$ and $(Y, B)$ as in Theorem 3.2, we have

$$H^m(X, A; \mathbb{Z}) \cong \lim \left( H^m(X, A; \mathbb{Z}) \times \lim \left( H^m(Y, B; \mathbb{Z}) \right) \right),$$

with the same supplements as in Theorem 3.2 where $m = w(X) \geq r_0$.

Theorem 3.2 shows that the problems concerning the homotopy classes of maps from a space $X$ to a metric space can be reduced to those for the case of $X$ being metrizable.

§ 4. An expansion theorem for sets of homotopy classes of maps into polyhedra. Let $(X, A)$ be a pair of spaces, and for an infinite cardinal number $m$ let $\{U_\lambda \mid \lambda \in \Lambda(m)\}$ be a cofinal subset of the directed set of all locally finite normal open coverings of $X$ consisting of at most $m$ sets; $K_0$ and $K_1$ respectively the nerves of $U_\lambda$ and $U_\lambda \cap A$ with the weak topology, $\varphi_\lambda: (X, A) \to (K_0, K_1)$ are canonical maps, and $\varphi_0: (K_0, K_1) \to (K_0, K_1)$ are canonical projections.

Let $(Q, Q_0)$ be a simplicial pair (that is, a pair of a simplicial complex and its subcomplex with the weak or metric topology) such that $Q$ has at most $m$ vertices.

For a continuous map $f: (X, A) \to (Q, Q_0)$ we call a covering $U_\lambda$ a bridge for $f$ if there exists a continuous map $f_\lambda: (K_0, K_1) \to (Q, Q_0)$ such that $f \cong f_\lambda \circ \varphi_0$, where $\cong$ means "is homotopic to" as usual; each continuous map $f_\lambda$ with this property is called a bridge map for $f$. These notions were first introduced by S. T. Hu in 1948 for the absolute case.

**Lemma 4.3.** For any continuous map $f: (X, A) \to (Q, Q_0)$ there exists a bridge $U_\lambda$ with $\lambda \in \Lambda(m)$ and every refinement $U_\mu$ of $U_\lambda$ is also a bridge for $f$.

**Proof.** Let us put

$$\mathcal{U} = \{\text{St}(q; Q) \mid q \text{ ranging over vertices of } Q\}.$$

Then $\mathcal{U}$ is a normal open covering of $Q$ since $Q$ is paracompact. Then there is $U_\lambda$ with $\lambda \in \Lambda(m)$ such that $f^{-1}(U_\lambda) \subset U_\lambda$. Let $U_\lambda = \{U_\lambda^i \mid i \in \Lambda\}$. Then for each vertex $u_\lambda^i$ of $K_0$ we can choose a vertex $q_i$ of $Q$ so that

$$U_\lambda^i \subset f^{-1}(\text{St}(q_i; Q)).$$

Let us put $f_\lambda(u_\lambda^i) = q_i$. If $q_i$ is not in $Q_0$, then $\text{St}(q_i; Q) \cap Q_0 = \emptyset$. Hence if $u_\lambda^i \cap A \neq \emptyset$, then $f_\lambda(u_\lambda^i)$ is a vertex of $Q_0$. Thus $f_\lambda$ defines a simplicial map $f_\lambda: (K_0, K_1) \to (Q, Q_0)$.
For any point \( x \) of \( X \) let \( \{a_1, \ldots, a_n\} \) be the totality of \( a \in \Omega_a \), such that \( x \in U_a \). Then \( f(x) \) and \( (f_\beta \circ \varphi_{\beta}) (x) \) lie in the simplex of \( Q \) which is spanned by \( f_\beta (g_{\alpha} (x)) \), \( i = 0, 1, \ldots, n \) and \( g_\beta \), \( j = 0, 1, \ldots, n \) where \( g_{\alpha} (x) \), \( \ldots, g_{\alpha} (x) \) is the carrier of \( f(x) \) in \( Q \). Hence we have \( f \equiv f_\beta \circ \varphi_{\beta} \colon (X, \Lambda) \to (Q, \Omega_a) \).

**Lemma 4.2.** Let \( \mathfrak{U}_\lambda \) be a locally finite normal open covering of \( X \) with \( \varphi \in \tilde{A} (\lambda) \), and let \( f_\beta \), \( g_\beta \colon (X, \Lambda) \to (Q, \Omega_a) \) be two continuous maps such that

\[ f_\beta \circ \varphi_\beta \equiv g_\beta \circ \varphi_\beta : (X, \Lambda) \to (Q, \Omega_a). \]

Then there exists a locally finite normal open covering \( \mathfrak{U}_\lambda \) of \( X \) with \( \lambda \in \Lambda (\lambda) \) such that

\[(a) \quad \mathfrak{U}_\lambda \subset \mathfrak{U}_\lambda,
(b) \quad f_\beta \circ \varphi_\beta \equiv g_\beta \circ \varphi_\beta: (K_\beta, L_\beta) \to (Q, \Omega_a).\]

**Proof.** Let \( (K_\beta, L_\beta) \) be a subdivision of \( (K_\beta, L_\beta) \) such that

\[ \{a_\beta\} \subset \partial \Omega_a, \quad \beta \in \Omega_a. \]

(1) \[ f_\beta^{-1} (U_\beta) \subset (S_{\pi_a} (K_\beta), \Omega_a) \subset \beta \in \Omega_a. \]

Here \( U_\beta \) is the same covering of \( X \) as in the proof of Lemma 4.1.

By assumption there is a continuous map \( \Phi: (X, \Lambda) \to (Q, \Omega_a) \) such that

\[(2) \quad \Phi (x, 0) = (f_\beta \circ \varphi_\beta) (x) \quad \text{for} \quad x \in X,
(3) \quad \Phi (x, 1) = (g_\beta \circ \varphi_\beta) (x) \quad \text{for} \quad x \in X.\]

Then by Theorem 2.5 there exists a locally finite normal open covering \( \mathfrak{U}_\lambda \) of \( X \) with \( \lambda \in \Lambda (\lambda) \) satisfying the following conditions:

\[(4) \quad \mathfrak{U}_\lambda \subset (U_\lambda, \alpha \in \Omega_a),
(5) \quad \mathfrak{U}_\lambda \supset \{g_\beta (S_{\pi_a} (U_\beta, \Omega_a)): \beta \in \Omega_a\},
(6) \quad \text{there exists a collection \( \{K_\beta, L_\beta\} \) of \( \Omega_a \) such that \( (U_\lambda, \alpha \in \Omega_a) \) is a covering of \( X \times I \) which refines \( \Phi^{-1} (U_\beta) \).}\]

For each covering \( \mathfrak{U}_\lambda \) there is a refinement of the form:

\[(t_{i-1}, t_{i+1}) \mid j = 0, 1, \ldots, n \}
\]

where \( 0 = t_0 < t_1 < \ldots < t_{n+1} = 1 \)

and \( t_{i-1}, t_{i+1} \) and \( t_{i-2}, t_{i+2} \) are respectively \( \{0, t_1, \ldots, t_n\} \) and \( \{t_1, \ldots, t_{n+1}\} \).

Let \( v_\beta \) be the vertex of \( K_\beta \) corresponding to \( U_\beta \). By (6), for each \( v_\beta \) and \( t_\beta \) \( \in I \) there is a vertex \( g_{\alpha} \) of \( Q \) such that

\[ (7) \quad (U_\beta \times (t_{i-1}, t_{i+1})) \subset (S_{\pi_a} (g_{\alpha}, \Omega_a)). \]

If \( U_\beta \times \Lambda \neq \emptyset \), then we have \( g_{\alpha} \neq Q \). Let us put

\[ \psi (v_\beta, t_\beta) = g_{\alpha} \]

Then \( g_{\alpha} \) and \( g_{\alpha} \) are vertices of a simplex of \( Q \).

For \( t_{i+1} < t < t_{i+2} \) let \( \eta (U_\beta, t) \) be the point of \( Q \) which divides the segment from \( g_{\alpha} \) to \( g_{\alpha} \) in the ratio \( t \). (7)

Let \( (v_\beta, v_\beta, \ldots, v_\beta) \) be a simplex of \( K_{\beta} \) (resp. \( L_{\beta} \)). Then we have

\[ U_\beta \times (t \in \Phi^{-1} (S_{\pi_a} (v_\beta), \Omega_a) \cap \Phi^{-1} (S_{\pi_a} (v_{\beta+1}, \Omega_a))). \]

according as \( t = t_{i+1} \) or \( t = t_{i+2} \). Hence \( \psi (v_\beta, t) \) for \( i = 0, 1, \ldots, n \) are vertices or points on the 1-faces of a simplex of \( Q \) (resp. \( Q_\beta \)). Therefore the map \( \psi \) can be extended linearly over the simplex \( [v_\beta, \ldots, v_\beta] \) with \( t \) fixed, and we have a map

\[ \psi: (K, L_{\beta}) \to (Q, \Omega_a) \]

by using the same letter \( \psi \). The map \( \psi \) is clearly continuous over \( [v_\beta, v_\beta, \ldots, v_\beta] \times I \) and hence over \( K_{\beta} \times I \) since \( K_{\beta} \times I \) has the weak topology.

For each \( \alpha \in \Omega_a \) there is \( \beta \in \Omega_a \) such that

\[ U_\beta \subset \Phi^{-1} (S_{\pi_a} (v_\beta, \Omega_a)). \]

Let us define a simplicial map

\[ \Phi_\beta: (K_{\beta}, L_{\beta}) \to (K_{\beta}, L_{\beta}) \]

by \( \Phi (v_\beta, t) = \gamma_\beta (v_\beta, t) \). For each \( \beta \in \Omega_a \) there is a vertex \( f_\beta (v_\beta) \) of \( Q \) such that

\[ \Phi (v_\beta, t) \in \Phi^{-1} (S_{\pi_a} (v_\beta, \Omega_a)). \]

Hence we have a simplicial map \( f_\beta: (K_{\beta}, L_{\beta}) \to (Q, \Omega_a) \) which is a simplicial approximation of \( f_\beta \). Therefore we have by (7) and (9)

\[ U_\beta \subset \Phi^{-1} (S_{\pi_a} (v_\beta, \Omega_a)). \]

On the other hand, by (7) we have

\[ U_\beta \subset \Phi^{-1} (S_{\pi_a} (v_\beta, \Omega_a)). \]

and hence we get from (3)

\[ U_\beta \subset (f_\beta \circ \varphi_\beta)^{-1} (S_{\pi_a} (v_\beta, \Omega_a)). \]

Now let us put

\[ \psi_0 (y) = \psi (y, 0) \quad \text{for} \quad y \in K_{\beta}. \]

Then \( \psi_0: (K_{\beta}, L_{\beta}) \to (Q, \Omega_a) \) is a simplicial map and we have

\[ \psi_0 \circ f_\beta = \psi_0 \circ f_\beta. \]

Similarly, we have

\[ \psi_0 \circ g_\beta = \psi_0 \circ g_\beta. \]
where \( \varphi_1 : (K_1, L_1) \to (Q, Q_0) \) is defined by \( \varphi_1(y) = y(y, 1) \) for \( y \in K_1 \). Since \( \varphi_0 \simeq \varphi_1 \), we have

\[ f_* \circ \varphi_1^* \simeq g_* \circ \varphi_0^* : (K_1, L_1) \to (Q, Q_0). \]

Let \( \varphi_2 : (K_2, L_2) \to (K_1, L_1) \) be a simplicial map such that \( \text{St}(u^1_2; K_2) \subseteq \text{St}(u^1_0; K_1) \). Then we have

\[ U^1_0 \subseteq \varphi_2^{-1}(\text{St}(u^1_2; K_2); K_1) \supseteq \varphi_2^{-1}(\text{St}(u^1_0; K_1); K_1) \subseteq U^1_1, \]

where \( U^*_i \) is the set of \( U_i \) corresponding to \( \varphi_2^{-1}(u^*_i) \). Thus \( \varphi_2 \circ \varphi_1 = \varphi_2\circ \varphi_0 : K_1 \to K_2 \) is a canonical projection which will be denoted by \( \varphi_2 \). Since \( \varphi_0 \) is homotopic to the identity map, we have

\[ f_* \circ \varphi_2^* \simeq g_* \circ \varphi_0^* : (K_1, L_1) \to (Q, Q_0). \]

This completes the proof of Lemma 4.3.

As an immediate consequence of Lemmas 4.1 and 4.2 we obtain the following theorem.

**Theorem 4.3.** Let \( (X, A) \) be a pair of spaces and \( (Q, Q_0) \) a simplicial pair such that \( Q \) has at most \( m \) vertices, where \( m \) is an infinite cardinal number. Let \( \{U_i \mid \lambda \in \Lambda(m)\} \) be a cofinal subset of the directed set of all locally finite normal open covers of \( X \) of cardinality \( \leq m \). Let \( K_1 \) and \( L_1 \) be the nerves of \( U_1 \) and \( U_0 \), respectively, \( \varphi_0 : (K_1, A) \to (K_0, L_0) \) canonical maps and \( \varphi_0 : (K_1, L_1) \to (K_0, L_0) \) canonical projections, where \( \lambda, \nu \in \Lambda(m) \). Then \( \{ [K_1, L_1], U_i, Q_0, \varphi_0^* \mid \lambda \in \Lambda(m) \} \) is a direct system and \( \{ [\varphi_0^* \mid \lambda \in \Lambda(m) \} \) defines a bijective map:

\[ [X, A; Q, Q_0] \cong \lim \{ [K_1, L_1], U_i, Q_0, \lambda \in \Lambda(m) \}. \]

Let \( (Y, B) \) be a compact space. Then by a theorem of Minow the function space \( [Q, Q_0]^{(X, B)} \) has the homotopy type of a simplicial pair. Hence by Theorem 4.3 we have

**Theorem 4.4.** Let \( (X, A) \) be a pair of spaces, \( (Y, B) \) a pair of compact Hausdorff spaces, and \( (Q, Q_0) \) a pair of spaces dominated by a simplicial pair (1). Let \( \{U_i \mid \lambda \in \Lambda \} \) be a cofinal subset of the directed set of all locally finite normal open covers of \( X \) and let \( K_1, L_1, \varphi_1, and \varphi_2 \) be the same as in Theorem 4.3 except that \( \lambda, \nu \in \Lambda(m) \). Then \( \{ [K_1, L_1], B, Q_0, \varphi_1^* \mid \lambda \in \Lambda(m) \} \) is a direct system and \( \{ [\varphi_1^* \mid \lambda \in \Lambda(m) \} \) defines a bijective map:

\[ [(X, A) \times (Y, B); Q, Q_0] \cong \lim \{ [K_1, L_1] \times (Y, B); Q_0, \lambda \in \Lambda(m) \}. \]

**Remark.** It follows from Theorem 4.3 and [10] that \( [X, A; S^n, s_0] \) forms a group called the \( n \)-th cohomotopy group of \( (X, A) \), if \( \dim X/A \)

\[ < 2n-1 \text{ and } A \neq \emptyset \] (where \( S^n \) is an \( n \)-sphere). By virtue of Theorem 4.3, we can prove that the cohomotopy sequence of \( (X, A) \) is exact if \( X \) is a normal space of dimension \( < 2n-1 \) and \( A \) a closed subset of \( X \), since any countable normal open covering of \( A \) is refined by restriction of a countable normal open covering of \( X \). This settles a question stated in Morita [10, footnote 3] on p. 294.

The following theorem will be used in 

**Theorem 4.5.** Let \( S^n \) be an \( n \)-sphere \((n > 1)\) with base-point \( s_0 \). Let \( X \) be a space of dimension \( < n \) and \( s_0 \) its base-point. Then the suspension map \( S : [X, s_0; S^n, s_0] \to [SX, s_0; SS^n, s_0] \)

defined by \( S[f] = [Sf] \) is bijective.

Here \( S \) denotes the operation of taking the reduced suspension; that is, \( SX \) is the quotient space of \( X \times I \) by identifying all the points of \( X \times f \simeq X \times I \) to a single point which we consider as the base-point of \( SX \) and denote by the same letter \( s_0 \) as the base-point of \( X \).

**Proof.** With the notations of Theorem 4.3, the diagram

\[ [X, s_0; S^n, s_0] \xrightarrow{S} [SX, s_0; SS^n, s_0] \]

is commutative where without loss of cofinality we may restrict \( U_0 \) to those coverings which have exactly one member containing \( s_0 \) and \( h_1 \) is a vertex of \( K_1 \) corresponding to such a member. We can assume here that \( \dim K_1 \leq n \) for each \( \lambda \).

In case \( n > 2 \), it follows from the suspension theorem in Spanier [21, p. 408] that

\[ S : [K_1, h_1; S^n, s_0] \to [SK_1, h_1; SS^n, s_0] \]

is bijective. In case \( n = 1 \), this fact can be verified directly.

Therefore, by virtue of Theorem 4.4 we have Theorem 4.5.

**§ 5. The covering dimension.** Let \( G \) be the \( n \)-dimensional cube and \( I^n \) its boundary \((n > 1)\).

A continuous map \( f : X \to G \) is called essential if any continuous map \( g : X \to I^n \) such that \( g \circ f = f \) satisfies \( g(X) = I^n \); otherwise \( f \) is called inessential. Thus \( f : X \to I^n \) is essential if and only if there is no continuous map \( g : X \to I^n \) which is homotopic to \( f \) relative to \( f^{-1}(I^n) \) and \( g(X) \neq I^n \). Theorem 5.1 below is given by Smirnov [20] for the case of \( X \) being Tychonoff. In view of Lemmas 1.1 and 1.2 we have Theorem 5.1 for the general case.
Theorem 5.1. For a space $X$, $\dim X < n$ if and only if every continuous map $f: X \to \mathbb{I}^{n+1}$ is essential.

Hence, if $\dim X = n$, there is an essential map $f: X \to \mathbb{I}^n$.

Theorem 5.2. Let $X$ be a space such that $\dim X = n$. Let $f: X \to \mathbb{I}^n$ be an essential map. Then the continuous map $f \times 1: X \times I \to \mathbb{I} \times I$ is essential.

Proof. Let us put

$$A = f^{-1}(\mathbb{I}^n), \quad Y = X \times I, \quad B = (f \times 1)^{-1}(\mathbb{I}^{n+1}),$$

where $\mathbb{I}^{n+1} = (\mathbb{I} \times I) \cup (\mathbb{I} \times I)$. Then $B = (A \times I) \cup (X \times I)$. Let us construct quotient spaces and quotient maps as follows:

$$\varphi: X \to X/A = X_0, \quad \psi: Y \to Y/B = Y_0,$$

$$\alpha: \mathbb{I} \to \mathbb{I}/\mathbb{I}^n = S^n, \quad \beta: \mathbb{I} \times I \to (\mathbb{I} \times I)/(\mathbb{I} \times I) = \mathbb{I}^{n+1},$$

and put

$$a_0 = \varphi(A), \quad g_0 = \psi(B), \quad a_1 = \alpha(I^n), \quad b_1 = \beta(I^{n+1}).$$

Then we have two continuous maps

$$f_0: (X_0, a_0) \to (S^n, a_1), \quad g_0: (Y_0, g_0) \to (S^{n+1}, b_1)$$

so that the diagrams

$$\begin{array}{ccc}
(X, A) & \xrightarrow{f} & (X_0, a_0) \\
\downarrow & & \downarrow \\
(Y, B) & \xrightarrow{g} & (Y_0, g_0)
\end{array}$$

$$\begin{array}{ccc}
(\mathbb{I}, I) & \xrightarrow{\alpha} & (S^n, a_1) \\
\downarrow & & \downarrow \\
(\mathbb{I} \times I, I^{n+1}) & \xrightarrow{\beta} & (S^{n+1}, b_1)
\end{array}$$

are commutative.

As usual, let $S$ be the operation of taking the reduced suspension.

Then we have

$$X_0 = SX_0, \quad S^{n+1} = SS^n, \quad g_0 = Sg_0.$$

Lemma 5.3. $f_0: H^n(\mathbb{I}, I^n; \mathbb{Z}) \to H^n(X, A; \mathbb{Z})$ is not zero.

If we have proved Lemma 5.3, then $f_0$ is not null homotopic and hence by Theorem 4.5 $g_0$ is not null homotopic, and consequently

$$(f \times 1)_*: H^{n+1}(\mathbb{I} \times I, I^{n+1}) \to H^{n+1}(X \times I, Y \times I)$$

is not zero. Therefore $f \times 1$ is essential. This proves Theorem 5.2.

Proof of Lemma 5.3. If $\psi: X \to Y$ is a continuous map from $X$ onto a metric space $Y$ such that $f = g \circ \psi$ with some continuous map $g: Y \to \mathbb{I}^n$, then $g$ is essential. Hence by Theorems 3.2 and 3.3, it is sufficient to prove Lemma 5.3 for the case of $X$ being a metric space. In this case, in the commutative diagram

$$\begin{array}{ccc}
H^{n+1}(A, Z) & \xrightarrow{s} & H^n(X, A; Z) \\
\downarrow & & \downarrow \\
H^n(I^n; \mathbb{Z}) & \xrightarrow{\iota_{n_1}} & H^n(I^n, I^{n+1}; \mathbb{Z}) \xrightarrow{0}
\end{array}$$

the lower sequence is exact and $s(f|A)^n(i) \neq 0$ for a generator $i$ of $H^n(I^n, I^{n+1}; \mathbb{Z})$; this is seen by Hopf's extension theorem (cf. Dowker [2, Theorem 5.2]) since $f|A$ cannot be extended to a continuous map from $X$ to $I^n$. Thus, $f \times 1$ is essential. This proves Lemma 5.3.

Corollary 5.4. $\dim (X \times I) \geq \dim X + 1$.

If we use the special case of the Künneth formula:

$$H^{n+1}(\mathbb{I}, \mathbb{I} \times I) \times (I, \mathbb{I}) \times (I, \mathbb{I}) \cong H^{n+1}(\mathbb{I} \times I, \mathbb{I} \times I),$$

we have another proof of Corollary 5.4 by Lemma 5.3.

In a previous paper [15] we have proved the following theorem for the case of $X$ being a Tychonoff space, from which the theorem for the general case follows immediately in view of Lemmas 1.2 and 1.4.

Theorem 5.5. Let $X$ be a space and $Y$ a locally compact, paracompact Hausdorff space, then $\dim (X \times Y) \leq \dim X + \dim Y$.

Lemma 5.6. Let $X$ be a space and $Y$ a normal, Hausdorff $P$-space in the sense of Morita [13]. If a subset $B$ of $Y$ is locally compact, $\sigma$-compact and closed, then $X \times B$ is $\sigma$-embedded in $X \times Y$.

Proof. Let $f$ be any real-valued continuous function defined over $X \times B$. Then by Theorem 2.3 there is a continuous map $g: X \to T$ for some metric space $T$ such that $f = g \circ (x \times 1_B)$ for some real-valued continuous function $g$ defined over $T \times B$. Since $T \times Y$ is normal by Morita [13] and $T \times B$ is closed in $T \times Y$, there is a real-valued continuous function $h$ defined over $T \times Y$ such that $h(T \times Y) = g$. Then $h \circ (x \times 1_Y)$ is an extension of $f$.

Theorem 5.7. Let $X$ be a space and $Y$ a $\sigma$-compact regular Hausdorff space. Then $\dim (X \times Y) \leq \dim X + \dim Y$.

Before proceeding to the proof of Theorem 5.7 we state Lemmas 5.8 and 5.9 below, which were proved for the case of Tychonoff spaces by Katětov and Smirnov (cf. [4, p. 263]) and by Katětov (cf. [4, p. 285]) respectively.

Lemma 5.8. Let $A$ be a subspace of a space $X$ which is $\sigma$-embedded in $X$. If $\dim X < n$, then $\dim A < n$. 

LEMMA 5.9. Let \( \{A_i \mid i \in N\} \) be a countable covering of a space \( X \) such that each \( A_i \) is \( C^* \)-embedded in \( X \). If \( \dim A_i \leq n \) for each \( i \), then \( \dim X \leq n \).

These lemmas can be reduced to the case of Tychonoff spaces by Lemma 1.2 and the following lemma.

LEMMA 5.10. Let \( A \) be a subspace of a space \( X \) which is \( C^* \)-embedded in \( X \). Then \( \Phi_X(A) \) is \( C^* \)-embedded in \( \tau(X) \), and \( \dim \Phi_X(A) = \dim A \), where \( \Phi_X : X \rightarrow \tau(X) \) is the natural map defined in \( \S \).

Proof. The first assertion is obvious. Let \( f : X \rightarrow I \) be a continuous map. Then by Lemma 1.1 \( f \) is factored through \( \tau(X) \) such that \( f = g \circ \Phi_X \) for a continuous map \( g : \tau(X) \rightarrow I \), and, for \( a \in A \), \( \Phi_X(a) \in g^{-1}(0, 1]) \) if and only if \( a \in f^{-1}(0, 1]) \). It follows immediately from this fact that \( \dim \Phi_X(A) = \dim A \).

Now we shall return to the proof of Theorem 5.7.

Let \( \{B_i \mid i \in N\} \) be a countable closed covering of \( Y \) by compact subsets. Since \( Y \) is a normal \( F \)-space, by Lemma 5.6 each subspace \( X \times B_i \) is \( C^* \)-embedded in \( X \times Y \). Hence by Lemma 5.9 and Theorem 5.5 we have Theorem 5.7.

THEOREM 5.11. If \( X \) is a space and \( Y \) a countable CW complex, then \( \dim (X \times Y) = \dim X + \dim Y \).

Proof. Suppose that \( \dim Y = n \). Then there is a compact subset \( E \) of \( Y \) which is homeomorphic to the \( n \)-cube \( I^n \). By Lemma 5.6, \( X \times B \) is \( C^* \)-embedded in \( X \times Y \). Hence by Lemma 5.8 we have \( \dim (X \times Y) \geq \dim (X \times B) \).

On the other hand, by repeated application of Corollary 5.4 we have \( \dim (X \times B) \geq \dim X + n \).

These two inequalities, together with Theorem 5.7, imply the desired equality of the theorem.

THEOREM 5.12. If \( X \) is a \( \sigma \)-compact regular Hausdorff space and \( Y \) a space with \( \dim Y = 1 \), then \( \dim (X \times Y) = \dim X + \dim Y \).

Proof. In view of Lemmas 5.6, 5.8 and Theorem 5.7, it suffices to prove that \( \dim (X \times Y) \geq \dim X + 1 \) in case \( X \) is itself compact. Since \( \dim Y = 1 \) there exists a continuous map \( \lambda : Y \rightarrow I \) such that \( C \mathcal{V} - \mathcal{V} \neq \emptyset \) for any open set \( \mathcal{V} \) satisfying \( \mathcal{V} \subseteq Y \) and \( \lambda^{-1}(1) \).

Suppose that \( \dim X = n \). Then by Theorem 5.1 there is an essential map \( f : X \rightarrow I^n \). Let \( X_n \) be the space which is constructed in the proof of Theorem 5.2. Then we have an essential map \( f_n : (X_n, x_0) \rightarrow (I^n, 0) \). Let us define a continuous map \( \phi : X_n \times Y \rightarrow I^{n+1} \) by

\[ \phi(s, y) = (1 - h(y))f_n(s) + h(y)s_0 \quad \text{for} \quad s \in X_n, \; y \in Y, \]

where we regard \( S^n \) as the boundary \( I^{n+1} \) of \( I^{n+1} \).

Since the set \( A = f^{-1}(I^n) \) is a \( \sigma \)-closed set, there exist a countable number of compact sets \( A_i \) with \( i \in N \) such that

\[ X - A = \bigcup \{ A_i \mid i \in N \}. \]

Hence we have

\[ X_n - x_0 = \bigcup \{ \phi(A_i) \mid i \in N \}. \]

Here \( \phi : X \rightarrow X_n \) is the map defined in the proof of Theorem 5.2; we note that \( \phi(A_i) \) is homeomorphic to \( A_i \) for \( i \in N \).

Suppose that \( \dim (X \times Y) < n \) and \( n \geq 1 \); the theorem is obvious in case \( n = 0 \). Since

\[ X_n \times Y = x_0 \times Y \cup \left( \bigcup \{ \phi(A_i) \times Y \mid i \in N \} \right), \]

it follows from Lemmas 5.6, 5.8 and Theorem 5.7 that \( \dim (X \times Y) < n \).

Hence \( \Phi \) is inessential by Theorem 5.1. Thus, there exists a continuous map \( \Psi : (X_n, x_0) \times Y \rightarrow (S^n, s_0) \) such that

\[ \Psi((x, y), s) = f_n(x) \quad \text{for} \quad y \in h^{-1}(0), \]

\[ \Psi((x, y), s) = s_0 \quad \text{for} \quad y \in h^{-1}(1). \]

Let us put \( \Psi_0(y) = \Psi((x, y), s_0) \) and denote by \( V \) the set of points \( y \) of \( Y \) such that \( \Psi_0 : (X_n, x_0) \rightarrow (S^n, s_0) \) is not null homotopic. Then we have \( h^{-1}(0) \subseteq V \subseteq h^{-1}(1) \). For each point \( y \) of \( Y \) there is an open neighborhood \( U(y) \) such that \( \Psi_0 \) and \( \Psi \) are homotopic for any point \( y' \) of \( U(y) \). Since \( X_n \) is compact. Therefore \( V \) and \( Y - V \) are open. This contradicts the assumption that \( C \mathcal{V} - \mathcal{V} \neq \emptyset \). Thus, the proof of the theorem is completed.

The following lemma is sometimes useful.

LEMMA 5.13. Let \( (X, x_0) \) be a pointed space of dimension \( n \). Then \( \dim S^nX = n + 1 \).

Proof. For any point \( y_0 \) of \( I^n \), there is a homotopy \( F : I^n \times I \rightarrow I^n \) such that

\[ F(q, 0) = q \quad \text{for} \quad q \in I^n, \]

\[ F(q, 1) = y_0 \quad \text{for} \quad q \in I^n, \]

\[ F(0, 1) = y_0. \]

Hence there exists an essential map \( f : X \rightarrow I^n \) such that \( f(x_0) = I^n \). Let us apply the arguments in the proof of Theorem 5.2 to this map \( f \). Then we have the commutative diagram

\[ (X, B) \xrightarrow{f} (I^n, y_0) \]

\[ \xrightarrow{y_0} \]

\[ (S^nX, SA) \]
where \( g_{0} \) and \( g_{1} \) are quotient maps. Since \( \psi: H^{*+1}(Y, y_{0}; Z) \to H^{*+1}(Y, B; Z) \) is an isomorphism by Corollary 6.3 below,
\[
\psi: H^{*+1}(Y, y_{0}; Z) \to H^{*+1}(\delta X, S\delta; Z)
\]
is one-to-one. Since \( g_{0} \) is not null homotopic, \( H^{*+1}(Y, y_{0}; Z) \neq 0 \) by Lemma 5.14 and Theorem 6.7 below. Hence \( H^{*+1}(\delta X, S\delta; Z) \neq 0 \). This shows that \( \dim \delta X > n+1 \).

On the other hand, since \( \dim (X \times I) < n+1 \), we have \( \dim \delta X < n+1 \) by Lemma 5.14 below. Therefore, \( \dim \delta X = n+1 \).

**Lemma 5.14.** If \( A \) is a non-empty subset of a space \( X \) of finite dimension, then \( \dim X/A < \dim X \).

**Proof.** Let \( f: X \to X/A \) be the quotient map and let us put \( Y = X/A \) and \( y_{i} = f_{*}(A) \). Let \( X = \bigcup_{i} H_{i} \) be a finite covering of \( Y \) by cozero-sets. Let \( I \) be a subset of \( \{1, 2, ..., m\} \) such that \( f^{-1}(I) = \{y \in H_{i} \} \). Then \( \bigcap_{i} H_{i} \) is also a cozero-set and hence there is a continuous map \( \psi: X \to I \) such that \( \bigcap_{i} H_{i} = \{y \in X \mid \psi(y) > 0\} \). Let us put
\[
C = \{y \in X \mid \psi(y) \leq a\},
\]
\[
V_{i} = \{y \in X \mid \psi(y) > \frac{1}{a}\},
\]
\[
V_{i} = H_{i} - C \quad \text{for} \quad i = 1, 2, ..., m,
\]
where \( a = \psi(y_{0}) > 0 \). Then \( \{V_{i} \mid i = 0, 1, ..., m\} \) is a covering of \( X \) by cozero-sets and a refinement of \( X_{\varepsilon} \) and \( y_{i} \mid V_{i} \) for \( i > 0 \). Suppose that \( \dim X = n \). Since \( f^{-1}(V_{i}) \) is a finite covering of \( X \) by cozero-sets, there exists a finite covering \( G = \{G_{0}, G_{1}, ..., G_{n}\} \) of \( X \) by cozero-sets such that
\[
G_{i} \subset f^{-1}(V_{i}) \quad \text{for} \quad i = 0, 1, ..., m,
\]
and the order of \( G \) does not exceed \( n+1 \). Let \( \mathcal{L} \) be a normal open covering of \( X \) which is a star-refinement of \( \mathcal{G} \) and let us put
\[
M_{i} = \bigcup \{L \in \mathcal{L} \mid \text{St}(L, \mathcal{L}) \subset G_{i}\} \quad \text{for} \quad i = 0, 1, ..., m.
\]
Then we have
\[
\text{St}(M_{i}, \mathcal{L}) \subset G_{i} \quad \text{for} \quad i = 0, 1, ..., m,
\]
\[
X = \bigcup \{M_{i} \mid i = 0, 1, ..., m\}.
\]
Since \( \mathcal{L} \) is normal, for each \( i \) there is a continuous map \( \varphi_{i}: X \to I \) such that
\[
\varphi_{i}(x) = \begin{cases} 
1 & \text{for} \quad x \in M_{i}, \\
0 & \text{for} \quad x \in X - \text{St}(M_{i}, \mathcal{L}).
\end{cases}
\]
Since \( G_{i} \cap \mathcal{A} = \emptyset \) for \( i > 0 \), we have \( \text{St}(M_{i}, \mathcal{L}) \cap \mathcal{A} = \emptyset \) for \( i > 0 \) and \( \mathcal{A} \subset C_{\varepsilon} \). Let us put
\[
U_{i} = \{x \in X \mid \varphi_{i}(x) > 0\} \quad \text{for} \quad i = 0, 1, ..., m,
\]
\[
\varphi(x) = \varphi(f^{-1}(y)) \quad \text{for} \quad y \in Y.
\]
Then \( \psi: X \to I \) is single-valued for each \( i \geq 0 \) and \( \varphi_{i} = \varphi_{f} \). Hence \( \varphi_{i} \) is continuous. Since
\[
U_{i} = f^{-1}(U_{i}) \quad \text{and} \quad f(U_{i}) = \{y \in Y \mid \psi(y) > 0\},
\]
\( f^{-1}(U_{i}) \) is a finite covering of \( Y \) by cozero-sets which is a refinement of \( X_{\varepsilon} \) and has order \( < n+1 \). This proves Lemma 5.14. Thus we have a covering.

**§ 6. Homotopical cohomology groups and the K"{u}nneth formula.** Let \( G \) be an abelian group and \( \mathcal{K} = \{K(G, n)\} \) the geometric realization of the Eilenberg-MacLane complex \( K(G, n) \); let \( k_{i} \) be the identity element of the weak abelian group \( K \). For any pair \( (X, A) \) of spaces let us define a map
\[
\mathcal{P}_{X,A}: [X, A; K, k_{0}] \to H^{n}(X, A; G)
\]
by \( \mathcal{P}_{X,A}(f) = f_{*}(C) \) where \( C \) is an \( n \)-characteristic element of \( H^{n}(K, k_{0}; G) \).

It is known that \( (X, A; K, k_{0}) \) is an abelian group and that \( \mathcal{P}_{X,A} \) is homomorphic. As is known, \( \mathcal{P}_{X,A} \) is an isomorphism if \( (X, A) \) is a simplicial pair. Since \( \mathcal{P}_{X,A} \) is natural with respect to a pair \( (X, A) \), by applying Theorem 4.3 we have the following.

**Theorem 6.1.** \( \mathcal{P}_{X,A} \) is an isomorphism for any pair \( (X, A) \).

In case \( A = \emptyset \), we can define a natural homomorphism
\[
\mathcal{P}_{X}: [X; K] \to H^{n}(X; G)
\]
similarly and we have

**Theorem 6.2.** \( \mathcal{P}_{X} \) is an isomorphism for any space \( X \).

For a pair \( (X, A) \) with \( A \) non-empty, let us denote by \( X/A \) the space obtained from \( X \) by contracting \( A \) to a point \( q_{A} \), and \( \psi: (X, A) \to (X/A, q_{A}) \) the quotient map. Then we have a commutative diagram:
\[
\begin{array}{ccc}
[X, A; K, k_{0}] & \to & H^{n}(X, A; G) \\
\downarrow & & \downarrow \psi^{*} \\
[X/A, q_{A}; K, k_{0}] & \to & H^{n}(X/A, q_{A}; G).
\end{array}
\]
Since \( \psi^{*} \) is bijective, \( \mathcal{P}_{X} \) is an isomorphism. Thus, we have

**Corollary 6.3.** \( H^{n}(X, A; G) \cong H^{n}(X/A, q_{A}; G) \). In particular, \( H^{n}(X, A; G) \cong H^{n}(X/A; G) \) for \( n > 1 \).
Corollary 6.4. If $f \cong g : (X, A) \to (Y, B)$, then $f^* = g^* : H^n(Y, B; G) \to H^n(X, A; G)$.

Proof. It is obvious from the fact that in this case we have

$$f^* = g^* : [Y, B; K, k_0] \to [X, A; K, k_0].$$

Lemma 6.5. If $(Y, B)$ is a compact pair, then $(K, k_0)^{Y, B}$ has the same homotopy type as

$$[X, Y; K, k_0] \cong H^n(X, Y; G; k_0).$$

Proof. Let $Y/B$ be the quotient space of $Y$ obtained by identifying all the points of $B$ to a single point $q_B$. Then $(K, k_0)^{Y, B} \cong (K, k_0)^{Y, B}$. Hence we have only to deal with pointed spaces; we do so without mentioning base-points.

Let $L = |K(G, n+1)|$ be the geometric realization of the Eilenberg-MacLane complex $K(G, n+1)$. Then $L^n$ is a weak abelian group. Let $M$ be the path-component of the identity element of the total singular complex $S(L^n)$ of $L^n$. Then $M$ is a connected group complex. Since we have

$$n_{q*}(M) = n_{q+1}(L^n) \cong [S^{q+1}; L^n] \cong [Y, L^{q+1}]$$

$$\cong [Y, |K(G, n+1)|] \cong H^n(Y; G)$$

for $0 \leq q < n$ and $n_{q+1}(M) = 0$ for $q > n$, by Moore [16, Theorem 3.29] we have

$$M \cong \bigotimes_{q=0}^n H^n(Y; G, q+1).$$

Hence by taking the geometric realization we have

$$[M] = \bigotimes_{q=0}^n H^n(Y; G, q+1) \cong \bigotimes_{q=0}^n |K|^n(Y; G, q+1) \cong \bigotimes_{q=0}^n |K|^n(Y; G).$$

On the other hand, since $L^n$ has the homotopy type of a CW complex, by taking loop spaces we have

$$\Omega(M) = \Omega\bigotimes_{q=0}^n H^n(Y; G) \cong \bigotimes_{q=0}^n \Omega H^n(Y; G) \cong \bigotimes_{q=0}^n \Omega.L^n \cong \Omega L^n = \Omega^n.$$

Thus, Lemma 6.5 is proved.

We are now in a position to prove the following Künneth formula.

Theorem 6.6. If $(Y, B)$ is a compact pair, then

$$H^n((X, A) \times (Y, B); G) \cong \bigotimes_{q=0}^n H^q(X, A; G) \times \bigotimes_{q=0}^n H^q(Y, B; G)$$

for any pair $(X, A)$.

Proof. By Lemma 6.5 we have

$$[(X, A) \times (Y, B); K, k_0] \cong [X, A; (K, k_0)^{Y, B}, 0, 0]$$

$$\cong \bigotimes_{q=0}^n [X, A; (K, k_0)^n(Y, B; G); 0],$$

and hence Theorem 6.6 follows from Theorem 6.1.

Next, let $S^n$ be an $n$-sphere and $p_n$ a point of $S^n$. For a pair $(X, A)$ of spaces with $A$ closed we define a map

$$\theta_{X,A}: [X, A; S^n, p_n] \to H^n(X, A; Z)$$

by $\theta_{X,A}(f) = f^*(i)$, where $i$ is a generator of $H^n(S^n, Z)$; $\theta_{X,A}$ is natural with respect to a pair $(X, A)$. Then we have the following generalization of the Hopf classification theorem.

Theorem 6.7. If $(X, A)$ is a pair of spaces with $A$ non-empty and $\dim X \leq n$, then $\theta_{X,A}$ is an isomorphism for $n \geq 1$.

Proof. Let $\varphi : (X, A) \to (X/A, \Delta_A)$ be the quotient map. Then we have a commutative diagram

$$\begin{array}{ccc}
[X, A; S^n, p_n] & \longrightarrow & H^n(X, A; Z) \\
\varphi & \downarrow & \varphi^* \\
[X/A, \Delta_A; S^n, p_n] & \longrightarrow & H^n(X/A, \Delta_A; Z)
\end{array}$$

and $\varphi^*$ is bijective. Hence it is sufficient to deal with the case of pointed spaces. It is known that if $X$ is a simplicial complex, $A \subseteq X$ and $\dim X \leq n$, then $\theta_{X,A}$ is an isomorphism for $n \geq 1$. Hence we have the theorem by Theorem 1.3 and 4.3.

Finally, we shall prove the following theorem.

Theorem 6.8. Let $(X, A)$ be a pair of spaces and $G$ an abelian group which is generated by at most $m$ elements $(m \geq n)$. Then we have

$$H^n(X, A; G) \cong \lim_{\longrightarrow m} \bigotimes_{q=0}^n H^q(K_{n+1}, K; G),$$

where $\{K_{n+1} \in A(m)\}$, $K_{n+1}$ and $K_{n+2}$ are the same as in Theorem 4.3.

Proof. The theorem follows from Theorems 4.3 and 6.1, since $|K(G, n)|$ is chosen to be a simplicial complex with at most $m$ vertices.

References

Measurable relations

by

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Abstract. The measurability properties of relations (= set valued functions) are developed. First the logical relations among the various definitions of measurability are worked out and used to determine sufficient conditions for the intersection of measurable relations to be measurable. These results are then used to generalize the selection theorems of Kuratowski and Ryll-Nardzewski, Czaja, and Aumann, to generalize Filippov’s implicit function theorem, and to prove the existence of a measurable selector extending a given measurable partial selector. The paper concludes with some applications to relations with values in a locally convex space.

1. Introduction. Measurable relations, i.e., set valued functions which assign to each element $t$ of a measurable space $T$ a subset of a topological space $X$ in a manner satisfying any one of several possible definitions of measurability, have been studied extensively in recent years by numerous authors (Aumann [A-1, 2], Czaja [C], Debreu [D], Jacob [J], Kuratowski and Ryll-Nardzewski [K], McShane and Warfield [M], Rockafellar [R], Van Vleck and the author [HV-1, 2, 3] and many others.) Much of this work either assumes that the measurable structure on $T$ is that of a Radon measure on a locally compact space or that $X$ is a very special kind of space, say compact metric or Euclidean. The purpose of this paper is to develop the properties of measurable relations in the general situation where $T$ is an abstract measurable space and $X$ is separable metric. It turns out that to work with $T$ this general we must usually (but not always) introduce compactness somewhere, either in $X$ or in the values of a multifunction with values in $X$. Alternatively, we obtain a similar body of results assuming that $X$ is a Souslin space and that a $\sigma$-finite measure is defined on the measurable subsets of $X$.

In the main, we will confine our attention to the general properties of measurable relations, and to selection, extension, and implicit function theorems.

In Section 2 we give most of the necessary definitions and terminology, and state without proof some trivial but often used properties of measurable relations. Section 3 is an account of the logical relationships among the various definitions of measurability. Section 4 is concerned with measurability of the intersection, complement, and boundary of