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INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

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An insertion theorem for real functions

by

J. M. Boyte and **E. P. Lane** (Boone, North Carolina)

Abstract. Characterizations of countably paracompact spaces and of normal countably paracompact spaces in terms of insertion of extended real-valued semi-continuous and continuous functions are given.

Dowker [1] and Katětov [2] proved that a topological space X is normal and countably paracompact if and only if for real-valued functions f and g defined on X such that f is lower semi-continuous and g is upper semi-continuous and $g(x) < f(x)$ for each x there exists a continuous real-valued function h on X such that $g(x) < h(x) < f(x)$ for each x. Mack [3] proved that a space is countably paracompact if and only if for each lower semi-continuous function g on X such that $g(x) > 0$ for each x there exist a lower semi-continuous function l and an upper semi-continuous function u such that $0 < l(x) \leqslant u(x) \leqslant g(x)$ for each x. This note generalizes these results by using extended real-valued functions.

The abbreviations lsc (resp. usc) for lower semi-continuous (resp. upper semi-continuous) are used, and we write $g \leqslant f$ (resp. $g < f$) in case $g(x) \leqslant f(x)$ (resp. $g(x) < f(x)$) for each x. Denote by L (resp. U) the set of extended real-valued lsc (resp. usc) functions defined on X. If f and g are extended real-valued functions defined on X, we write $g \ll f$ in case $g \leqslant f$ and if either $g(x)$ or $f(x)$ is a real number, then $g(x) < f(x)$.

THEOREM. *The following are equivalent:*

(α) *The space X is normal and countably paracompact.*

(β) *If $f \in L$, $g \in U$, and $g \ll f$, then there exist functions $f' \in L$ and $g' \in U$ such that $g \ll f' \leqslant g' \ll f$.*

(γ) *The space X is normal, and if $f \in L$, $g \in U$ and $g \ll f$, then there exists a function $h \in L$ such that $g \ll h \ll f$.*

(δ) *If $f \in L$, $g \in U$, and $g \ll f$, then there exists an extended real-valued continuous function b on X such that $g \ll b \ll f$.*

Proof. Observe that (δ) implies (β) trivially. The proof that (β) implies (α) is established as in the proof of Theorem 4 of [1]. In order to see that (γ) implies (δ), let $f \in L$, $g \in U$, and $g \ll f$. By (γ) there is a function $h \in L$ such that $g \ll h \ll f$. Then $-h \in U$, $-g \in L$ and $-h \ll -g$. Again by (γ), there exists a function $-k \in L$ such that $-h \ll -k \ll -g$. Then $g \ll k \ll h \ll f$. If $h^*(x) = h(x)/(1+|h(x)|)$ when $h(x)$ is finite, $h^*(x) = 1$ when $h(x) = +\infty$, and $h^*(x) = -1$ when $h(x) = -\infty$, then h^* is lsc. If k^* is defined similarly in terms of k, k^* is usc and $k^* \leqslant h^*$. By Theorem 2 of [4] there is a continuous function b^* on X such that $k^* \leqslant b^* \leqslant h^*$. If $b(x) = b^*(x)/(1-|b^*(x)|)$ when $|b^*(x)| < 1$, $b(x) = +\infty$ when $b^*(x) = 1$, and $b(x) = -\infty$ when $b^*(x) = -1$, then b is an extended real-valued continuous function on X. Since $k \leqslant b \leqslant h$, it follows that $g \ll b \ll f$. Thus (δ) holds. It therefore remains to prove that (α) implies (γ).

Let $f \in L$, let $g \in U$, and assume that $g \ll f$. Then $A = \{x: f(x) = g(x) = +\infty\}$ and $B = \{x: f(x) = g(x) = -\infty\}$ are closed sets, and the sets $U(n) = \{x: f(x) > n\}$ and $V(n) = \{x: g(x) < -n\}$ are open for each natural number n. Since X is normal it follows that there exist closed sets $A(n)$ and $B(n)$, $n = 1, 2, \ldots$, such that $A(1) \cap B(1) = \emptyset$ and

$$U(n) \supset A(n) \supset A(n)^0 \supset A(n+1) \supset A,$$
$$V(n) \supset B(n) \supset B(n)^0 \supset B(n+1) \supset B.$$

Take $A' = \bigcap \{A(n): n = 1, 2, \ldots\}$, $B' = \bigcap \{B(n): n = 1, 2, \ldots\}$ and $W = X - (A' \cup B')$. For $x \in W$, put $u(x) = -\infty$ if $x \notin A(1)$ and $u(x) = n$ if $x \in A(n) - A(n+1)$. Then u is usc and $f(x) > u(x)$ for each x in W. Hence also $u \vee g$ is usc and for $x \in W$, $(u \vee g)(x) < f(x)$. Since W is an F_σ-subspace of X, the space W is countably paracompact and normal; hence there exists a continuous function v on W such that $(u \vee g)(x) < v(x) < f(x)$ for x in W. If $h(x) = -\infty$ when $x \in B'$, $h(x) = v(x)$ when $x \in W$ and $h(x) = +\infty$ when $x \in A'$, then h is lsc and $g \ll h \ll f$.

The authors are indebted to the referee for the significantly shorter and more elegant proof of the implication (α) implies (γ) which is given above.

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APPALACHIAN STATE UNIVERSITY
Boone, North Carolina

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Čech cohomology and covering dimension for topological spaces

by

Kiiti Morita (Tokyo)

Abstract. For a topological space X let us define the covering dimension of X and the Čech cohomology groups of X by using only normal open coverings of X instead of arbitrary open coverings. Then it will be shown that some of the basic theorems concerning the Čech cohomology groups and covering dimension of CW complexes or paracompact spaces, such as the Hopf classification theorem and the product theorem on dimension for the case of one factor being σ-compact, can be generalized to the case of arbitrary topological spaces.

In discussing the topological invariants for topological spaces, such as the Čech cohomology groups and the covering dimension, which are defined by using open coverings, it seems natural to make a modification by restricting open coverings to normal ones.

For the covering dimension of Tychonoff spaces (= completely regular Hausdorff spaces) such a modification was made by M. Katětov [8] and Yu. Smirnov [20]; a nice exposition of their results is given in Engelking [4]. Applying their modification to a general case, we shall define the covering dimension of a topological space X, denoted by $\dim X$, to be the least integer n such that every finite normal open covering of X admits a finite normal open covering of order $\leqslant n+1$ as its refinement. In case X is a normal space, $\dim X$ defined here coincides with the covering dimension of X in the usual sense.

As for the nth Čech cohomology group $H^n(X; G)$ of a topological space X with coefficients in an abelian group G, we shall define it by using only normal open coverings of X. In case X is paracompact Hausdorff, $H^n(X; G)$ is the usual Čech cohomology group based on all open coverings.

The purpose of this paper is to show that with these definitions we can generalize some of the basic theorems concerning the Čech cohomology groups and dimension of paracompact Hausdorff spaces or CW complexes to the case of topological spaces.

Let X be a topological space, G an abelian group, and Z the additive group of all integers. Let $|K(G, n)|$ be the geometric realization of the