Let X be the space G of Michael [14, Example 2] which is a closed subset of the space F of Bing used in Example 5.3. This space is metacompact and perfectly normal but not collectionwise normal. Hence X is a space with the required properties.

References

[1] H. R. Bennett and D. J. Lutzer, *A note on weak θ-refinability*, Gen. Topology Appl. 2 (1972), pp. 49–54.
[2] R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. 3 (1951), pp. 175–186.
[3] D. K. Burke, *On subparacompact spaces*, Proc. Amer. Math. Soc. 23 (1969), pp. 655–663.
[4] — *On p-spaces and wΔ-spaces*, Pacific J. Math. 35 (1970), pp. 285–296.
[5] C. H. Dowker, *On countably paracompact spaces*, Canad. J. Math. 3 (1951), pp. 219–224.
[6] F. Ishikawa, *On countably paracompact spaces*, Proc. Japan Acad. 31 (1955), pp. 686–687.
[7] M. Katětov, *On extension of locally finite coverings*, Colloq. Math. 6 (1958), pp. 145–151.
[8] Y. Katuta, *On strongly normal spaces*, Proc. Japan Acad. 45 (1969), pp. 692–695.
[9] — *On expandability*, Proc. Japan Acad. 49 (1973), pp. 452–455.
[10] L. L. Krajewski, *On expanding locally finite collections*, Canad. J. Math. 23 (1971), pp. 58–68.
[11] J. C. Smith and L. L. Krajewski, *Expandability and collectionwise normality*, Trans. Amer. Math. Soc. 160 (1971), pp. 437–451.
[12] J. Mack, *Directed covers and paracompact spaces*, Canad. J. Math. 19 (1967), pp. 649–654.
[13] E. Michael, *A note on paracompact spaces*, Proc. Amer. Math. Soc. 4 (1953), pp. 831–838.
[14] — *Point-finite and locally finite coverings*, Canad. J. Math. 7 (1955), pp. 275–279.
[15] — *Yet another note on paracompact spaces*, Proc. Amer. Math. Soc. 10 (1959), pp. 309–314.
[16] K. Morita, *Paracompactness and product spaces*, Fund. Math. 50 (1961), pp. 223–236.
[17] K. Nagami, *Paracompactness and strong screenability*, Nagoya Math. J. 8 (1955), pp. 83–88.
[18] M. E. Rudin, *A normal space X for which $X\times I$ is not normal*, Fund. Math. 73 (1971), pp. 179–186.
[19] T. Shiraki, *M-spaces, their generalizations and metrization theorems*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A 11 (1971), pp. 57–67.
[20] J. W. Tukey, *Convergence and uniformity in topology*, Princeton 1940.
[21] J. M. Worrell, Jr. and H. H. Wicke, *Characterizations of developable topological spaces*, Canad. J. Math. 17 (1965), pp. 820–830.

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Some properties related to $[\mathfrak{a}, \mathfrak{b}]$-compactness

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Abstract. In this paper, three properties are studied which are closely related to $[\mathfrak{a}, \mathfrak{b}]$-compactness in the sense of complete accumulation points ($[\mathfrak{a}, \mathfrak{b}]$-compact^r) and $[\mathfrak{a}, \mathfrak{b}]$-compactness in the sense of open covers ($[\mathfrak{a}, \mathfrak{b}]$-compact).

§ 1. Introduction. The concept of $[\mathfrak{a}, \mathfrak{b}]$-compactness, which appears in many interesting results today, dates back to the work of P. Alexandroff and P. Urysohn in 1929. Since then many mathematicians have studied $[\mathfrak{a}, \mathfrak{b}]$-compactness, and several authors have introduced natural properties which they asserted were equivalent to $[\mathfrak{a}, \mathfrak{b}]$-compactness. Some of these properties, however, are not equivalent to $[\mathfrak{a}, \mathfrak{b}]$-compactness, although they are closely related to it. The purpose of this paper is to study the relations among several such properties, and to give some conditions under which they are equivalent. We believe that the consideration of these properties will aid in understanding $[\mathfrak{a}, \mathfrak{b}]$-compactness, in particular, $[\mathfrak{a}, \mathfrak{b}]$-compact product spaces. We will also point out some errors in the literature concerning three of these properties.

Let the letters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{m}$, and $\mathfrak{n}$ denote infinite cardinal numbers with $\mathfrak{a}\leqslant\mathfrak{b}$, and let $[\mathfrak{a}, \mathfrak{b}]$ stand for the set of all cardinals $\mathfrak{m}$ such that $\mathfrak{a}\leqslant\mathfrak{m}\leqslant\mathfrak{b}$. Let $|E|$ denote the cardinal number of a set E, and let $\mathfrak{m}^+$ denote the first cardinal strictly larger than $\mathfrak{m}$. The cofinality of $\mathfrak{m}$ is denoted by $\mathrm{cf}(\mathfrak{m})$. Use of the generalized continuum hypothesis will be denoted by [GCH].

Definitions. A space X is called $[\mathfrak{a}, \mathfrak{b}]$-*compact*^{r} if every open cover $\mathfrak{U}$ of X such that $|\mathfrak{U}|$ is a regular cardinal in $[\mathfrak{a}, \mathfrak{b}]$ has a subcover $\mathfrak{U}'\subset\mathfrak{U}$ with $|\mathfrak{U}'|<|\mathfrak{U}|$. This concept was introduced by Alexandroff and Urysohn [1]. The superscript r is a reminder of the "restriction of regularity" in the definition (see [5]). A space X is called $[\mathfrak{a}, \mathfrak{b}]$-*compact* if every open cover $\mathfrak{U}$ of X with $|\mathfrak{U}|\leqslant\mathfrak{b}$, has a subcover of cardinality strictly less than $\mathfrak{a}$. This idea was introduced in 1950 by Yu. Smirnov [13]. Essentially the same property was studied independently in 1957 by I. S. Gaal [3]. The work of Gaal mentioned in this paper [3, 4] has been reworded to coincide with the terminology of Alexandroff and Urysohn. A short survey of the theory of $[\mathfrak{a}, \mathfrak{b}]$-compactness is given in [15].

We will use the standard terminology concerning nets (Moore-Smith sequences) as found in Kelley's book [8]. In addition, a partially ordered set D is said to be $<\mathfrak{a}$-*directed* (cf. [4, Def. 1.2]) if each of its subsets, of cardinality strictly less than $\mathfrak{a}$, has an upper bound in D. The term $\leqslant \mathfrak{a}$-directed is defined similarly. Thus, every $<\aleph_0$-directed set is a directed set in the usual sense. For a set T, let

$$S_{\mathfrak{a}}(T) = \{H \subset T\colon\ |H| < \mathfrak{a}\}\,.$$

If $S_{\mathfrak{a}}(T)$ is used as the domain of a net, it is to be understood that the partial order on it is set inclusion. Recall that $\mathfrak{b}^{\underline{\mathfrak{a}}} = \sum\{\mathfrak{b}^{\mathfrak{m}}\colon \mathfrak{m} < \mathfrak{a}\}$.

We now define three properties which are related to $[\mathfrak{a}, \mathfrak{b}]$-compactness.

A space X is said to have *property* $\mathrm{S}[\mathfrak{a}, \mathfrak{b}]$ if every open cover of X of regular cardinality $\leqslant \mathfrak{b}$, has a subcover of cardinality $< \mathfrak{a}$. This property was introduced by Yu. Smirnov in [13] where it is incorrectly stated that $\mathrm{S}[\mathfrak{a}, \mathfrak{b}]$ is equivalent to $[\mathfrak{a}, \mathfrak{b}]$-compactness^r (see Example 1). (Professor Smirnov has informed me that a number of people have noticed this error.)

A space X is said to have *property* $\mathrm{G}[\mathfrak{a}, \mathfrak{b}]$ if every net in X whose domain D is $<\mathfrak{a}$-directed and $|D| \leqslant \mathfrak{b}$, has a cluster point. Essentially this condition was introduced by Gaal in [3]. It is incorrectly stated in [4, Theorem 1.2] that $\mathrm{G}[\mathfrak{a}, \mathfrak{b}]$ is equivalent to $[\mathfrak{a}, \mathfrak{b}]$-compactness (see Example 2).

A space is said to have *property* $\mathrm{N}[\mathfrak{a}, \mathfrak{b}]$ if every net in X whose domain is $S_{\mathfrak{a}}(\mathfrak{m})$ for some $\mathfrak{a} \leqslant \mathfrak{m} \leqslant \mathfrak{b}$ has a cluster point. Such nets were called $\mathfrak{b}$-$\mathfrak{a}$-sequences by N. Noble in [11] where it is erroneously stated [11, Prop. 1.3] that $\mathrm{N}[\mathfrak{a}, \mathfrak{b}]$ is equivalent to $[\mathfrak{a}, \mathfrak{b}]$-compactness (see Example 4). When $\mathfrak{b} = \infty$ is used in any of the five properties, it means that the property holds for all $\mathfrak{b} \geqslant \mathfrak{a}$.

The basic relations among these five properties are given in the following two results.

THEOREM 1. *Let X be a topological space. Each of the statements* (a)–(d) *below implies the one below it.*

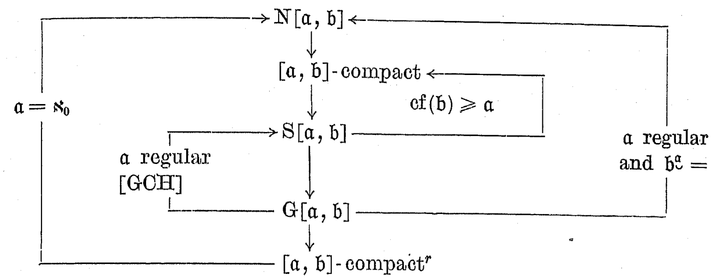
(a) *X has property* $\mathrm{N}[\mathfrak{a}, \mathfrak{b}]$.
(b) *X is* $[\mathfrak{a}, \mathfrak{b}]$*-compact.*
(c) *X has property* $\mathrm{S}[\mathfrak{a}, \mathfrak{b}]$.
(d) *X has property* $\mathrm{G}[\mathfrak{a}, \mathfrak{b}]$.
(e) *X is* $[\mathfrak{a}, \mathfrak{b}]$*-compact*^{r}.

As will be shown in § 4, none of these implications can be reversed for every value of $\mathfrak{a}$ and $\mathfrak{b}$. We are especially interested in conditions under which these properties are equivalent to $[\mathfrak{a}, \mathfrak{b}]$-compactness. In [5] some topological conditions were given under which the last four properties (b)–(e) would be equivalent. In the next result we give conditions on the cardinals $\mathfrak{a}$ and $\mathfrak{b}$ under which certain of the conditions of Theorem 1 are equivalent.

THEOREM 2.

A. *If $\mathfrak{a} = \aleph_0$, then the five conditions of Theorem* 1 *are equivalent.*
B. *If $\mathfrak{a}$ is regular and $\mathfrak{b}^{\underline{\mathfrak{a}}} = \mathfrak{b}$, then* (a)–(d) *of Theorem* 1 *are equivalent. In particular, they are equivalent when $\mathfrak{a}$ is regular and $\mathfrak{b} = \infty$.*
C. *If* $\mathrm{cf}(\mathfrak{b}) \geqslant \mathfrak{a}$ *then* $[\mathfrak{a}, \mathfrak{b}]$*-compactness is equivalent to* $\mathrm{S}[\mathfrak{a}, \mathfrak{b}]$.
D. [GCH]. *If $\mathfrak{a}$ is regular, then* $\mathrm{S}[\mathfrak{a}, \mathfrak{b}]$ *is equivalent to* $\mathrm{G}[\mathfrak{a}, \mathfrak{b}]$.

The following diagram summarizes the results in Theorems 1 and 2.



Theorem 2D points out the following interesting situation. Property $\mathrm{S}[\mathfrak{a}, \mathfrak{b}]$ was introduced in 1950 to characterize $[\mathfrak{a}, \mathfrak{b}]$-compactness^r, and $\mathrm{G}[\mathfrak{a}, \mathfrak{b}]$ was introduced in 1958 to characterize $[\mathfrak{a}, \mathfrak{b}]$-compactness. While neither achieves such characterizations, it is consistent to assume that $\mathrm{S}[\mathfrak{a}, \mathfrak{b}]$ and $\mathrm{G}[\mathfrak{a}, \mathfrak{b}]$ are equivalent when $\mathfrak{a}$ is regular. Further, from Theorem 2B and Lemma 4 in § 2, if both $\mathfrak{a}$ and $\mathfrak{b}$ are regular — regardless of the number of singular cardinals between them — then $\mathrm{G}[\mathfrak{a}, \mathfrak{b}]$, $\mathrm{S}[\mathfrak{a}, \mathfrak{b}]$, $[\mathfrak{a}, \mathfrak{b}]$-compactness and $\mathrm{N}[\mathfrak{a}, \mathfrak{b}]$ are all equivalent under the assumption of [GCH].

In § 2 we give some preliminary results which are used in § 3 to prove Theorems 1 and 2. The examples are given in § 4.

§ 2. Preliminary results. First we prove an equivalent form of the axiom of choice which has a number of uses. The first part of it is due to N. Howes [6, 7]. His proof [6] can be adapted to yield both parts of the result below, but we give our own proof which is slightly shorter.

LEMMA 1. *Let $(P, <^*)$ be a partially ordered set. There exists a cofinal subset $E \subset P$ and a well-ordering $<$ on E such that*

(1) *if* $d, d' \in E$, *and* $d <^* d'$, *then* $d < d'$,

(2) $(E, <)$ *is order isomorphic to the cardinal number which is the smallest cardinality of all cofinal subsets of* $(P, <^*)$.

Proof. Let D be a cofinal subset of $(P, <^*)$ having smallest possible cardinality $\mathfrak{m}$. Well-order D so that

$$D = \{d_0, d_1, \ldots, d_\alpha, \ldots:\ \alpha < \mathfrak{m}\}.$$

Define a map $f\colon \mathfrak{m} \to \mathfrak{m}$ by transfinite induction as follows. Let $f(0) = 0$, and define for each $\alpha < \mathfrak{m}$,

$$f(\alpha) = \min\{t:\ d_t \not\leqslant^* d_{f(\beta)} \text{ for all } \beta < \alpha\}.$$

The map f is well-defined since for each $\alpha < \mathfrak{m}$, clearly $|\{d_{f(\beta)}:\ \beta < \alpha\}| < \mathfrak{m}$, hence $H = \{d_{f(\beta)}:\ \beta < \alpha\}$ is not cofinal in P. Thus, there exists at least one $t < \mathfrak{m}$ such that $d_t \notin H$ and d_t is not exceeded in $(P, <^*)$ by any member of H. First we note that f is strictly increasing. If not, there exists $\beta < \alpha < \mathfrak{m}$ such that $f(\beta) \geqslant f(\alpha)$. Clearly, $f(\beta) \neq f(\alpha)$ so we have $f(\beta) > f(\alpha)$. By the definition of $f(\beta)$ as a minimum and the fact that $f(\alpha)$ is a smaller index that $f(\beta)$, we know there exists some $\tau < \beta < \alpha$ such that $d_{f(\alpha)} \leqslant^* d_{f(\tau)}$. But this contradicts the definition of $f(\alpha)$, and thus f is strictly increasing. Set $E = \{d_{f(\alpha)}:\ \alpha < m\}$, and well-order it by the well-order induced from its index set. Thus E is order isomorphic to $\mathfrak{m}$.

We now show that E is the desired cofinal subset of $(P, <^*)$. First of all, E is cofinal in D: Given any d in D, say $d = d_\beta$, since the image of f has cardinality $\mathfrak{m}$, there exists $\alpha < \mathfrak{m}$ such that $\beta < f(\alpha)$. By definition of $f(\alpha)$ there exists $\tau < \alpha$ such that $d_\beta \leqslant^* d_{f(\tau)}$. Thus E is cofinal in D and therefore cofinal in P. Finally, we show that property (1) holds. Let $d_{f(\alpha)}$ and $d_{f(\beta)}$ be members of E such that $d_{f(\beta)} <^* d_{f(\alpha)}$. We must show that $f(\beta) < f(\alpha)$. By definition of $f(\beta)$ we know that for all $\tau < \beta$ we have $d_{f(\beta)} \not\leqslant^* d_{f(\tau)}$, hence $\alpha < \beta$ is impossible, so $\beta < \alpha$. Since f is strictly increasing, $f(\beta) < f(\alpha)$. This completes the proof.

Lemma 2. *Let* $(P, <^*)$ *be a partially ordered set of cardinality* $\mathfrak{m}$, *where* $\mathfrak{m}$ *is a singular cardinal. If* P *is* $\leqslant \mathrm{cf}(\mathfrak{m})$*-directed, then* P *has a cofinal subset of cardinality strictly less than* $\mathfrak{m}$.

Proof. Assume false. By the preceding lemma, there exists a cofinal subset $E \subset P$ and a well-order $<$ on E such that $(E, <)$ satisfies (1) and (2) of Lemma 1. By our assumption, (2) implies that $(E, <)$ is order isomorphic to $\mathfrak{m}$. Let H be a cofinal subset of $(E, <)$ with $|H| = \mathrm{cf}(\mathfrak{m}) < \mathfrak{m}$. Since $(P, <^*)$ is $\leqslant \mathrm{cf}(\mathfrak{m})$-directed, there exists $p_0 \in P$ such that $h \leqslant^* p_0$ for all $h \in H$. Since E is cofinal in $(P, <^*)$, there exists $e_0 \in E$ such that $p_0 \leqslant^* e_0$. But this says that H has an upper bound in $(E, <)$ since given any $h \in H \subset E$, $h \leqslant^* e_0$ implies (by condition (1) of Lemma 1) that $h \leqslant e_0$. This is a contradiction since H is a cofinal subset of $(E, <) \cong \mathfrak{m}$ and has no upper bound in $(E, <)$.

Corollary. *If* $\mathfrak{U}$ *is an open cover of a space* X, $|\mathfrak{U}| = \mathfrak{m}$, $\mathfrak{m}$ *is singular, and* $\mathfrak{U}$ *is stable under all unions of cardinality* $\leqslant \mathrm{cf}(\mathfrak{m})$ *(i.e., if* $\mathfrak{U}' \subset \mathfrak{U}$ *and* $|\mathfrak{U}'| \leqslant \mathrm{cf}(\mathfrak{m})$, *then* $\bigcup \mathfrak{U}' \in \mathfrak{U}$*), then* $\mathfrak{U}$ *has a subcover of cardinality strictly less than* $\mathfrak{m}$.

Proof. By Lemma 2, the set $\mathfrak{U}$, partially ordered by set inclusion, has a cofinal subset of cardinality smaller than $|\mathfrak{U}|$.

Lemma 3. *If* D *is* $< \mathfrak{m}$*-directed, where* $\mathfrak{m}$ *is a singular cardinal, then* D *is* $\leqslant \mathfrak{m}$*-directed.*

Proof. We show that if $D' \subset D$ and $|D'| = \mathfrak{m}$ then D' has an upper bound in D. Let $D' = \bigcup\{D_\lambda:\ \lambda < \mathrm{cf}(\mathfrak{m})\}$ where $|D_\lambda| < \mathfrak{m}$ for each λ. Let d_λ be an upper bound for D_λ in D. Since $\mathrm{cf}(\mathfrak{m}) < \mathfrak{m}$, $\{d_\lambda:\ \lambda < \mathrm{cf}(\mathfrak{m})\}$ has an upper bound in D which is an upper bound for D'.

We will also need the following easy result concerning cardinal arithmetic (for example, see [12, p. 329]).

Lemma 4. [GCH]. *If* $\mathfrak{n} \leqslant \mathfrak{m}$ *and* $\mathfrak{m}$ *is regular, then* $\mathfrak{m}^{\underline{\mathfrak{n}}} = \mathfrak{m}$.

§ 3. Proofs of Theorem 1 and Theorem 2. For Theorem 1, all implications, except $\mathrm{S}[\mathfrak{a}, \mathfrak{b}] \to \mathrm{G}[\mathfrak{a}, \mathfrak{b}]$, are easy to prove directly from the definitions involved. Instead of this, however, we give a different proof based on the lemma below, which is of interest itself. It shows that four of the five properties have a characterization in terms of certain filter bases.

Let $\mathcal{F}$ be a collection of subsets of a space X. We say that $\mathcal{F}$ has the $< \mathfrak{a}$*-intersection property* provided for each $\mathcal{F}' \subset \mathcal{F}$ with $|\mathcal{F}'| < \mathfrak{a}$, we have $\bigcap \mathcal{F}' \neq \emptyset$. We say $\mathcal{F}$ is $< \mathfrak{a}$*-stable* if $\mathcal{F}$ consists of non-empty sets and for each $\mathcal{F}' \subset \mathcal{F}$ with $|\mathcal{F}'| < \mathfrak{a}$, there is an $F \in \mathcal{F}$ such that $F \subset \bigcap \mathcal{F}'$. Let $\bigcap \overline{\mathcal{F}}$ denote $\bigcap \{\overline{F}:\ F \in \mathcal{F}\}$.

Lemma 5. *Let* X *be a topological space.*

(a) *If* X *has property* $\mathrm{N}[\mathfrak{a}, \mathfrak{b}]$ *then for every* $\mathcal{F}$ *with the* $< \mathfrak{a}$*-intersection property and* $|\mathcal{F}| \leqslant \mathfrak{b}$, *we have* $\bigcap \{\overline{\bigcap \mathcal{F}'}:\ \mathcal{F}' \in S_\mathfrak{a}(\mathcal{F})\} \neq \emptyset$.

(b) X *is* $[\mathfrak{a}, \mathfrak{b}]$*-compact if and only if for every* $\mathcal{F}$ *with the* $< \mathfrak{a}$*-intersection property and* $|\mathcal{F}| \leqslant \mathfrak{b}$, *we have* $\bigcap \overline{\mathcal{F}} \neq \emptyset$.

(c) X *has property* $\mathrm{S}[\mathfrak{a}, \mathfrak{b}]$ *if and only if for every* $\mathcal{F}$ *with the* $< \mathfrak{a}$*-intersection property, where* $|\mathcal{F}|$ *is regular, and* $|\mathcal{F}| \leqslant \mathfrak{b}$, *we have* $\bigcap \overline{\mathcal{F}} \neq \emptyset$.

(d) X *has property* $\mathrm{G}[\mathfrak{a}, \mathfrak{b}]$ *if and only if for each* $\mathcal{F}$ *which is* $< \mathfrak{a}$*-stable and* $|\mathcal{F}| \leqslant \mathfrak{b}$, *we have* $\bigcap \overline{\mathcal{F}} \neq \emptyset$.

(e) X *is* $[\mathfrak{a}, \mathfrak{b}]$*-compact*^{r} *if and only if for every decreasing collection* $\mathcal{F} = \{F_\alpha:\ \alpha < \mathfrak{m}\}$, *such that*

$$F_0 \supset F_1 \supset \ldots \supset F_\alpha \supset \ldots, \qquad \alpha < \mathfrak{m}$$

where $\mathfrak{m}$ *is a regular cardinal with* $\mathfrak{a} \leqslant \mathfrak{m} \leqslant \mathfrak{b}$, *we have* $\bigcap \overline{\mathcal{F}} \neq \emptyset$.

Proof. Gaal proved (b) in [4], and the proof of (c) is similar. Statement (e) is due to Alexandroff and Urysohn [1, p. 20]. Our proof of (d) uses the fact that if $\mathcal{F}$ is $<\mathfrak{a}$-stable, then $\mathcal{F}$, partially ordered by reverse inclusion is $<\mathfrak{a}$-directed. We sketch a proof of (a). Let X satisfy property $\mathrm{N}[\mathfrak{a}, \mathfrak{b}]$ and $\mathcal{F}$ have the $<\mathfrak{a}$-intersection property with $|\mathcal{F}| = \mathfrak{m} \leqslant \mathfrak{b}$. Well-order $\mathcal{F} = \{F_\alpha\colon \alpha < \mathfrak{m}\}$ and define a net $f\colon S_\mathfrak{a}(\mathfrak{m}) \to X$ by choosing a point $f(H)$ in $\bigcap \{F_\alpha\colon \alpha \in H\}$ for each $H \in S_\mathfrak{a}(\mathfrak{m})$. Any cluster point of f is in the closure of each set $(\bigcap \{F_\alpha\colon \alpha \in H\})$ for each $H \in S_\mathfrak{a}(\mathfrak{m})$. Conversely, if X satisfies the condition described in (a), then X need not satisfy property $\mathrm{N}[\mathfrak{a}, \mathfrak{b}]$ as can be seen from Example 4. This completes the proof of the lemma.

Now from this lemma, it is clear in Theorem 1 that (a) $\to$ (b), (b) $\to$ (c), and (d) $\to$ (e). To prove that (c) $\to$ (d), we first prove Theorem 2C.

Proof of Theorem 2C. Since it is obvious that every $[\mathfrak{a}, \mathfrak{b}]$-compact space has property $\mathrm{S}[\mathfrak{a}, \mathfrak{b}]$, we prove the converse with our additional hypothesis. First we assume that $\mathfrak{b}$ is regular. Suppose X has property $\mathrm{S}[\mathfrak{a}, \mathfrak{b}]$ and is not $[\mathfrak{a}, \mathfrak{b}]$-compact. Then there is an open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leqslant \mathfrak{b}$, such that $\mathcal{U}$ has no subcover of cardinality strictly less than $\mathfrak{a}$. Let $\mathfrak{m}$ be the smallest cardinal for which X has such an open cover $\mathcal{U}$ of cardinality $\mathfrak{m}$. Note that $\mathcal{U}$ does not have a subcover of cardinality strictly less than $\mathfrak{m}$, and further, since X has property $\mathrm{S}[\mathfrak{a}, \mathfrak{b}]$, $\mathfrak{m}$ is a singular cardinal. We consider another open cover of X. Set

$$\mathcal{W} = \{\bigcup \mathcal{U}'\colon \mathcal{U}' \subset \mathcal{U} \text{ and } |\mathcal{U}'| \leqslant \mathrm{cf}(\mathfrak{m})\}.$$

Clearly $\mathcal{W}$ is stable under unions of cardinality $\leqslant \mathrm{cf}(\mathfrak{m})$. If $|\mathcal{W}| = \mathfrak{m}$, then by the corollary to Lemma 2, $\mathcal{W}$ has a subcover $\mathcal{W}'$ with $|\mathcal{W}'| < |\mathcal{W}|$. Now each $W \in \mathcal{W}'$ is a union of no more than $\mathrm{cf}(\mathfrak{m})$ elements of $\mathcal{U}$. Thus, $\mathcal{U}$ has a subcover of cardinality no more than $\mathrm{cf}(\mathfrak{m}) \cdot |\mathcal{W}'| < \mathfrak{m}$. This is impossible, so $|\mathcal{W}| > \mathfrak{m}$. Let $\mathcal{W}'$ be a subcover of $\mathcal{W}$ such that $|\mathcal{W}'| = \mathfrak{m}^+$. Now X has property $\mathrm{S}[\mathfrak{a}, \mathfrak{b}]$ and $\mathfrak{b}$ is regular, thus $\mathfrak{m}^+ \leqslant \mathfrak{b}$. Hence $\mathcal{W}'$ has a subcover $\mathcal{W}''$ of cardinality strictly less than $\mathfrak{a}$. Thus $\mathcal{U}$ has a subcover of cardinality no more than $\mathrm{cf}(\mathfrak{m}) \cdot |\mathcal{W}''| < \mathfrak{m}$. This is a contradiction and completes the proof when $\mathfrak{b}$ is regular. We now assume that $\mathfrak{a} \leqslant \mathrm{cf}(\mathfrak{b})$ and $\mathfrak{b}$ is singular. From what was just proved, we see that X is $[\mathfrak{a}, \mathfrak{m}]$-compact for all $\mathfrak{a} \leqslant \mathfrak{m} < \mathfrak{b}$ because X satisfies property $\mathrm{S}[\mathfrak{a}, \mathfrak{m}^+]$ for $\mathfrak{a} \leqslant \mathfrak{m} < \mathfrak{b}$ and $\mathfrak{m}^+$ is regular. Thus, to show that X is $[\mathfrak{a}, \mathfrak{b}]$-compact, we need only show that every open cover of X of cardinality $\mathfrak{b}$ has a subcover of smaller cardinality. Let $\mathcal{U} = \{U_\alpha\colon \alpha < \mathfrak{b}\}$ be an open cover of X with $|\mathcal{U}| = \mathfrak{b}$. Set $W_\alpha = \bigcup \{U_\beta\colon \beta < \alpha\}$ for each $\alpha < \mathfrak{b}$. Then

$$W_0 \subset W_1 \subset \ldots \subset W_\alpha \subset \ldots, \qquad \alpha < \mathfrak{b}$$

is an increasing open cover of X. By picking a cofinal subset of it, we have a subcollection $\mathcal{W} \subset \{W_\alpha\colon \alpha < \mathfrak{b}\}$ such that $\mathcal{W}$ covers X and $|\mathcal{W}| = \mathfrak{n} \leqslant \mathrm{cf}(\mathfrak{b}) < \mathfrak{b}$. Since X is $[\mathfrak{a}, \mathfrak{m}]$-compact for $\mathfrak{a} \leqslant \mathfrak{m} < \mathfrak{b}$, $\mathcal{W}$ has a subcover $\mathcal{W}'$ with $|\mathcal{W}'| < \mathfrak{a}$. By hypothesis, $|\mathcal{W}'| < \mathrm{cf}(\mathfrak{b})$. Now the set of indices of the elements of $\mathcal{W}'$ has an upper bound $\alpha < \mathfrak{b}$. Thus $\{U_\beta\colon \beta < \alpha\}$ is a subcover of $\mathcal{U}$ having cardinality less than $\mathfrak{b}$. This completes the proof of Theorem 2C.

Now we complete the proof of Theorem 1. We need only show that $\mathrm{S}[\mathfrak{a}, \mathfrak{b}] \to \mathrm{G}[\mathfrak{a}, \mathfrak{b}]$. If $\mathfrak{b}$ is regular, then $\mathrm{S}[\mathfrak{a}, \mathfrak{b}] \to [\mathfrak{a}, \mathfrak{b}]$-compact by Theorem 2C, and clearly $[\mathfrak{a}, \mathfrak{b}]$-compactness implies $\mathrm{G}[\mathfrak{a}, \mathfrak{b}]$. Thus we assume that $\mathfrak{b}$ is singular. Now if $\mathfrak{b}$ is singular and X satisfies $\mathrm{S}[\mathfrak{a}, \mathfrak{b}]$, then X is $[\mathfrak{a}, \mathfrak{m}]$-compact (hence satisfies $\mathrm{G}[\mathfrak{a}, \mathfrak{m}]$) for all $\mathfrak{m} < \mathfrak{b}$. To show that X has property $\mathrm{G}[\mathfrak{a}, \mathfrak{b}]$ we need only show that for every $<\mathfrak{a}$-stable collection $\mathcal{F}$ with $|\mathcal{F}| = \mathfrak{b}$, we have $\bigcap \overline{\mathcal{F}} \neq \emptyset$. If $\mathfrak{a} \leqslant \mathrm{cf}(\mathfrak{b})$, then by Theorem 2C, X is $[\mathfrak{a}, \mathfrak{b}]$-compact (hence satisfies $\mathrm{G}[\mathfrak{a}, \mathfrak{b}]$). Thus, we assume that $\mathrm{cf}(\mathfrak{b}) < \mathfrak{a}$. From this we see that $\mathcal{F}$, partially ordered with reverse inclusion, is $\leqslant \mathrm{cf}(\mathfrak{b})$-directed. By Lemma 2, $\mathcal{F}$ has a cofinal subset $\mathcal{F}' \subset \mathcal{F}$ with $|\mathcal{F}'| = \mathfrak{m} < \mathfrak{b}$. Since $\mathcal{F}'$ is cofinal in $\mathcal{F}$, it is $<\mathfrak{a}$-stable and $\bigcap \overline{\mathcal{F}'} = \bigcap \overline{\mathcal{F}}$. Since X is $[\mathfrak{a}, \mathfrak{m}]$-compact, we have that $\bigcap \overline{\mathcal{F}'} \neq \emptyset$, and this completes the proof of Theorem 1.

Proof of Theorem 2A. Alexandroff and Urysohn [1] proved that $[\aleph_0, \mathfrak{b}]$-compactness^r and $[\aleph_0, \mathfrak{b}]$-compactness are equivalent, and S. Mrówka [10, Lemma 2] proved that $N[\aleph_0, \mathfrak{b}]$ is equivalent to $[\aleph_0, \mathfrak{b}]$-compactness. The important property of $\aleph_0$ here is that $|S_{\aleph_0}(\mathcal{F})| = |\mathcal{F}|$.

Proof of Theorem 2B. Let $\mathcal{F}$ have the $<\mathfrak{a}$-intersection property and $|\mathcal{F}| \leqslant \mathfrak{b}$. We must show that $\bigcap \{\overline{\bigcap \mathcal{F}'}\colon \mathcal{F}' \in S_\mathfrak{a}(\mathcal{F})\} \neq \emptyset$. Now $\mathcal{G} = \{\bigcap \mathcal{F}'\colon \mathcal{F}' \in S_\mathfrak{a}(\mathcal{F})\}$ has cardinality $\leqslant \mathfrak{b}^\mathfrak{a} = \mathfrak{b}$ and, since $\mathfrak{a}$ is regular, $\mathcal{G}$ is $<\mathfrak{a}$-stable. Since we are assuming that X has property $G[\mathfrak{a}, \mathfrak{b}]$ we have that $\bigcap \overline{\mathcal{G}} \neq \emptyset$.

Proof of Theorem 2D. We assume [GCH], that $\mathfrak{a}$ is regular and that $\mathrm{G}[\mathfrak{a}, \mathfrak{b}]$ holds. We show that $\mathrm{S}[\mathfrak{a}, \mathfrak{b}]$ holds. Let $\mathcal{F}$ have the $<\mathfrak{a}$-intersection property, and suppose that $\mathfrak{a} \leqslant |\mathcal{F}| = \mathfrak{m}$ is regular, and $|\mathcal{F}| \leqslant \mathfrak{b}$. We must show that $\bigcap \overline{\mathcal{F}} \neq \emptyset$. By Lemma 4, we have that $\mathfrak{m}^\mathfrak{a} = \mathfrak{m}$. Since $\mathfrak{m}$ is regular and X satisfies $\mathrm{G}[\mathfrak{a}, \mathfrak{m}]$, we know from Theorem 2B that X satisfies $\mathrm{S}[\mathfrak{a}, \mathfrak{m}]$ (in fact, $\mathrm{N}[\mathfrak{a}, \mathfrak{m}]$), thus $\bigcap \overline{\mathcal{F}} \neq \emptyset$. This completes the proof of Theorem 2.

§ 4. Examples. We now give the examples which show that none of the implications in Theorem 1 may be reversed for every value of $\mathfrak{a}$ and $\mathfrak{b}$. First we describe an example due to A. Miščenko [9], which we will use below. Let ω_1 denote the first uncountable cardinal, ω_2 the second uncountable cardinal, and so on. For each natural number $i \geqslant 1$, let ω_1+1

denote the set of ordinals less than or equal to ω_i. Miščenko's space is

$$R^* = \bigcup_{k=1}^{\infty} \Big(\prod_{i=1}^{k} (\omega_i+1) \times \prod_{i=k+1}^{\infty} \omega_i\Big)$$

considered as a subspace of the compact product space $\prod_{i=1}^{\infty} (\omega_i+1)$. Miščenko proved that R^* is $[\aleph_1, \infty]$-compact^r, and if $\mathfrak{U}$ is an open cover of R^* with $|\mathfrak{U}| < \aleph_\omega$, then $\mathfrak{U}$ has a countable subcover (for an easy proof of this see [5]). Further, R^* has an open cover of cardinality $\aleph_\omega$ which does not have a subcover of smaller cardinality.

Example 1. An $[\mathfrak{a}, \mathfrak{b}]$-compact^r space which does not satisfy $G[\mathfrak{a}, \mathfrak{b}]$. The space is R^*, $\mathfrak{a} = \aleph_1$, and $\mathfrak{b}$ may be taken to be any cardinal such that $\mathfrak{b} \geqslant |R^*|$ or $\mathfrak{b} = \infty$. In light of the above remarks concerning R^*, we need only show that R^* does not satisfy $G[\mathfrak{a}, \mathfrak{b}]$. Let $D = \prod_{i=1}^{\infty} \omega_i$, and give D the partial order defined by $d \leqslant d'$ if and only if $d(k) \leqslant d'(k)$ for all integers $k \geqslant 1$ (where $d(k)$ denotes the kth coordinate of d). It is easy to check that D is $< \aleph_1$-directed and $|D| = |R^*|$. We consider the inclusion map $f\colon D \to R^*$ as a net and show that it does not have a cluster point in R^*. For each x in R^* there exists an integer k such that $x(k) < \omega_k$. Now consider the open set $U = \{y \in R^*\colon y(k) \leqslant x(k)\}$ and the element d in D defined by

$$d(i) = \begin{cases} x(k)+1 & \text{if } i = k, \\ 0 & \text{otherwise}. \end{cases}$$

There is no member $d' \in D$ such that $d' \geqslant d$ and $f(d') \in U$. Thus x is not a cluster point of f. Since x was an arbitrary point in R^*, we have shown that R^* does not satisfy $G[\aleph_1, |R^*|]$. This also gives another proof of Miščenko's result that R^* is not $[\mathfrak{a}, \mathfrak{b}]$-compact.

Example 2. A space X which satisfies $G[\mathfrak{a}, \mathfrak{b}]$ but not $S[\mathfrak{a}, \mathfrak{b}]$. Let X be a discrete space of cardinality $\aleph_\omega$. Take $\mathfrak{a} = \aleph_\omega$ and let $\mathfrak{b}$ be any cardinal with $\mathfrak{b} \geqslant \mathfrak{a}^+$ or $\mathfrak{b} = \infty$. Clearly X does not satisfy $S[\mathfrak{a}, \mathfrak{b}]$. We show that X does have property $G[\mathfrak{a}, \mathfrak{b}]$. Let $f\colon D \to X$ be a net where D is $< \aleph_\omega$-directed. By Lemma 3, D is $\leqslant \aleph_\omega$-directed. If f has no cluster point in X then for each $x \in X$, there exists $d_x \in D$ such that $f(d) \neq x$ for all $d \geqslant d_x$. The set $\{d_x\colon x \in X\}$ has an upper bound, say d, in D but $f(d) \neq x$ for all $x \in X$. This contradiction shows that f has a cluster point in X. We do not know if Theorem 2D still holds without the assumption of [GCH].

Example 3. A space which satisfies $S[\mathfrak{a}, \mathfrak{b}]$ but is not $[\mathfrak{a}, \mathfrak{b}]$-compact. The space is R^*, $\mathfrak{a} = \aleph_1$ and $\mathfrak{b} = \aleph_\omega$. We note that in accordance with Theorem 2C we have that $\mathrm{cf}(\mathfrak{b}) < \mathfrak{a}$.

Example 4. An $[\mathfrak{a}, \mathfrak{b}]$-compact space which does not satisfy $N[\mathfrak{a}, \mathfrak{b}]$. The space is the set of natural numbers $N = \{0, 1, 2, \ldots\}$ with the discrete topology. We take $\mathfrak{a} = \aleph_\omega$ and $\mathfrak{b}$ any cardinal $\geqslant \mathfrak{a}$ or $\mathfrak{b} = \infty$. Since N is a Lindelöf space, N is clearly $[\aleph_\omega, \infty]$-compact. We show that N does not satisfy $N[\aleph_\omega, \aleph_\omega]$. Define a net $f\colon S_{\aleph_\omega}(\aleph_\omega) \to N$ by $f(H) = n$ where $\aleph_0 \cdot |H| = \aleph_n$. It is easy to see that f has no cluster point in N.

We conclude with some results concerning $[\mathfrak{a}, \mathfrak{b}]$-compact product spaces which involve properties $G[\mathfrak{a}, \mathfrak{b}]$ and $N[\mathfrak{a}, \mathfrak{b}]$. In [14] we introduced a property for topological spaces which we called (tentatively) property $(2)_{\mathfrak{a},\mathfrak{b}}$, and we proved [14, Thm. 3.4] that a countable product of spaces, each of which satisfies $(2)_{\mathfrak{a},\mathfrak{b}}$, where $\mathfrak{a}$ is regular and $\mathfrak{b}^{\underline{\mathfrak{a}}} = \mathfrak{b}$, is $[\mathfrak{a}, \mathfrak{b}]$-compact.

In light of the characterization of property $G[\mathfrak{a}, \mathfrak{b}]$ given in Lemma 5, it appears that property $G[\mathfrak{a}, \mathfrak{b}]$ plays the main role in the proof of [14, Thm. 3.4], rather than $[\mathfrak{a}, \mathfrak{b}]$-compactness, and we, therefore, can strengthen this theorem by using some results in this paper. First we need to change slightly the definition of $(2)_{\mathfrak{a},\mathfrak{b}}$. We say that a space has property $(3)_{\mathfrak{a},\mathfrak{b}}$ if for every $< \mathfrak{a}$-stable collection $\mathcal{F}$ on X with $|\mathcal{F}| \leqslant \mathfrak{b}$, there exists a compact set K and a collection $\mathcal{G}$ which is $< \mathfrak{a}$-stable and $|\mathcal{G}| \leqslant \mathfrak{b}$ such that $\mathcal{G}$ is finer than $\mathcal{F}$ and the filter base of all open sets containing K.

The proof of [14, Thm. 3.4] yields.

Theorem 3. *A countable product of spaces, each of which satisfies property* $(3)_{\mathfrak{a},\mathfrak{b}}$, *satisfies* $G[\mathfrak{a}, \mathfrak{b}]$.

In other words, the effect of the restrictions on the cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$ in [14, Thm. 3.4] is to have $G[\mathfrak{a}, \mathfrak{b}]$ imply $[\mathfrak{a}, \mathfrak{b}]$-compactness. Thus, by Theorem 2B we may improve [14, Thm. 3.4].

Corollary. *A countably product of spaces, each of which satisfies property* $(3)_{\mathfrak{a},\mathfrak{b}}$, *where* $\mathfrak{a}$ *is regular and* $\mathfrak{b}^{\underline{\mathfrak{a}}} = \mathfrak{b}$, *satisfies* $N[\mathfrak{a}, \mathfrak{b}]$.

Let $\mathfrak{m} = \sum_{i=1}^{\infty} \mathfrak{m}_i$ be a countable sum of smaller cardinals, and for each positive integer i, let X_i be a discrete space of cardinality $\mathfrak{m}_i$. In [14, Example 3.6] we showed that the product space $X = \prod \{X_i\colon i = 1, 2, \ldots\}$ is not $[\mathfrak{m}, \infty]$-compact. We point out that each X_i satisfies property $(3)_{\mathfrak{m},\infty}$, and thus by Theorem 3 the space X satisfies property $G[\mathfrak{m}, \infty]$.

Added in proof. We have recently developed a theory of "totally Ω-compact spaces," which includes property $(3)_{\mathfrak{a},\mathfrak{b}}$ as a special case, and allows product theorems like Theorem 3 above. The details will appear elsewhere.

References

[1] P. Alexandrov and P. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verh. Kon. Akad. Van Weten. Te Amsterdam 14 (1929), pp. 1–96.

[2] — — *On compact topological spaces*, Trudy Mat. Inst. Steklov 31 (95) (1950).

[3] I. S. Gaal, *On a generalized notion of compactness I-II*, Proc. Nederl. Akad. Wetensch 60 (1957), pp. 421–435.
[4] — *On the theory of* (m, n)*-compact spaces*, Pacific J. Math. 8 (1958), pp. 721–734.
[5] R. E. Hodel and J. E. Vaughan, *A note on* $[\mathfrak{a}, \mathfrak{b}]$*-compactness*, General Topology and Appl. 4 (1974), pp. 179–189.
[6] N. Howes, *Well-ordered sequences*, Dissertation, Texas Christian University, 1968.
[7] — *A theorem on partially ordered sets and a new theory of convergence*, Notices Amer. Math. Soc. 15 (1968), p. 346.
[8] J. L. Kelley, *General Topology*, Princeton 1955.
[9] A. Miščenko, *Finally compact spaces*, Soviet Math. 3 (4) (1962), pp. 1199–1202.
[10] S. Mrówka, *Compactness and product spaces*, Colloq. Math. 7 (1959), pp. 19–22.
[11] N. Noble, *Products with closed projections II*, Trans. Amer. Math. Soc. 160 (1971), pp. 169–183.
[12] J. E. Rubin, *Set Theory for the Mathematician*, San Francisco 1967.
[13] Yu. M. Smirnov, *On topological spaces, compact in a given interval of powers*, Akad. Nauk SSSR Izvest. Ser. Mat. 14 (1950), pp. 155–178.
[14] J. E. Vaughan, *Product spaces with compactness-like properties*, Duke Math. J. 39 (1972), pp. 611–617.
[15] — *Some recent results in the theory of* $[\mathfrak{a}, \mathfrak{b}]$*-compactness*, Preceedings of the Second Pittsburgh International Conference on General Topology and its Applications, Lecture Notes in Mathematics 378, Berlin 1974, pp. 534–550.

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On the descriptive set theory of the lexicographic square

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Abstract. Analytic and descriptive Borel subsets of the lexicographic square S are characterized. A sigma-compact subset is found not to be descriptive Borel. All analytic subsets are seen to be images under a three-valued semi-continuous mapping from the set I of irrationals and some are not two-valued such images. A first-countable separable compact subset is seen to be a two-valued such image of I but not single-valued. Two Borelian hierarchies in S (one derived from compact sets, the other from descriptive Borel sets) are studied. An absolutely closed space which is not sigma-descriptive Borel is constructed.

Introduction and definitions. Let S be the unit square $[0,1]^2$ ordered lexicographically (so that $\langle x_1, x_2\rangle < \langle x_1', x_2'\rangle$ if and only if either $x_1 < x_1'$ or both $x_1 = x_1'$ and $x_2 < x_2'$) and endowed with the topology generated by this ordering. S is compact and first-countable (compare [4, pp. 52–53]). Our investigations below of the analytic and descriptive Borel subsets of S (shortly to be defined) uncover an interesting (perhaps "exemplary") divergence of descriptive set theory in S from the classical situation in Polish spaces. For example, the compact subset $[0,1]\times\{0,1\}$, which is first-countable and separable (it contains $Q\times\{0,1\}$ as a dense subset, where Q denotes the rationals of $[0,1]$), is the image of the set I of irrationals under a two-valued semi-continuous mapping, as indeed is any compact, separable, ordered space, however it is not the image of I under a single-valued, semi-continuous mapping. The compact set $[0,1]\times\{0,\frac{1}{2},1\}$ is the image of I under a three-valued, semi-continuous mapping but not under a two-valued such mapping. The set $S\setminus[0,1]\times\{\frac{1}{2}\}$ is a "naturally occurring" example of a sigma-compact subset which is not descriptive Borel (compare the example given by Z. Frolík in [2, p. 166]).

Let $\mathcal{K}$ be a family of sets in a space X. We denote by Borelian-$\mathcal{K}$ the smallest family of sets of X to include $\mathcal{K}$ and closed under countable unions and countable intersections. We characterize two hierarchies of Borelian-$\mathcal{K}$ sets (see § 3 for definitions), one for $\mathcal{K}$ consisting of the compact sets $\mathcal{K}$ of S, the other for $\mathcal{K}$ consisting of the descriptive Borel sets, finding them cofinal in one another with respect to inclusion. These considerations