Expandability and its generalizations

by

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Abstract. A topological space (resp. a countably subparacompact space) X is said to be θ-expandable (resp. subexpandable), if for every locally finite (resp. discrete) collection $\{F_\lambda | \lambda \in \Lambda\}$ of subsets of X there exists a sequence

$$\{\mathfrak{G}_n = \{G_{\lambda,n} | \lambda \in \Lambda\} | n = 1, 2, \ldots\}$$

of collections of open subsets of X satisfying (1) $F_\lambda \subset G_{\lambda,n}$ for every λ and every n, (2) for every point x of X there exists some n for which x is in at most finitely many members (resp. at most one member) of $\mathfrak{G}_n$. The main results of this paper are as follows: (a) θ-refinable spaces, expandable spaces (in the sense of L. L. Krajewski) and subexpandable spaces are θ-expandable; (b) subparacompact spaces, expandable normal spaces (equivalently, countably paracompact collectionwise normal spaces) and perfectly normal spaces are subexpendable; (c) a θ-refinable space is subparacompact if and only if it is subexpandable; (d) various characterizations of expandability, θ-expandability and subexpandability in terms of coverings; (e) mapping, sum, product and subset theorems for θ-expandable spaces and subexpandable spaces.

In [7] Katětov proved the following useful theorem:

A normal space X is collectionwise normal and countably paracompact if and only if

(*) for every locally finite collection $\{F_\lambda | \lambda \in \Lambda\}$ of subsets of X there exists a locally finite collection $\{G_\lambda | \lambda \in \Lambda\}$ of open subsets of X such that $F_\lambda \subset G_\lambda$ for every $\lambda \in \Lambda$.

Recently, Krajewški [10] has called a topological space X expandable if X satisfies this condition (*). Smith and Krajewski [11] have introduced some generalizations of expandability, and they have obtained various results concerning these notions.

In this paper, we shall introduce new notions of θ-expandability, subexpandability, etc., and obtain analogous results. Furthermore, we shall study additional properties of expandable spaces, θ-expandable spaces, etc. The contents of this paper were announced in [9].

1. Definitions and elementary relations. First, let us recall the definitions of the terms of paracompact, metacompact, θ-refinable and subparacompact. A space X is *paracompact* (resp. *metacompact*), if every open covering of X has a locally finite (resp. point-finite) open refinement.

X is *θ-refinable* [21], if for every open covering $\mathfrak{U}$ of X there exists a sequence $\{\mathfrak{V}_n|\ n=1,2,\ldots\}$ of open refinements of $\mathfrak{U}$ such that for every point x of X there exists a positive integer n for which $\mathfrak{V}_n$ is point-finite at x (i.e., x is contained in at most finitely many members of $\mathfrak{V}_n$). X is *subparacompact* [3], if every open covering of X has a σ-discrete closed refinement.

Let $\mathfrak{m}$ be an infinite cardinal number. A space X is *$\mathfrak{m}$-paracompact* [16], if every open covering of X with power $\leqslant \mathfrak{m}$ has a locally finite open refinement. Similarly, *$\mathfrak{m}$-metacompact* spaces, *$\mathfrak{m}$-θ-refinable* spaces and *$\mathfrak{m}$-subparacompact* spaces are defined. In case $\mathfrak{m}=\aleph_0$ these are *countably paracompact* spaces, *countably metacompact* spaces, *countably θ-refinable* spaces and *countably subparacompact* spaces, respectively. A space X is *finitely subparacompact*, if every finite open covering of X has a σ-discrete closed refinement.

A space X is *expandable* [10] (resp. *almost expandable* [11]), if for every locally finite collection $\{F_\lambda|\ \lambda\in\Lambda\}$ of subsets of X there exists a locally finite (resp. point-finite) collection $\{G_\lambda|\ \lambda\in\Lambda\}$ of open subsets of X such that $F_\lambda\subset G_\lambda$ for every $\lambda\in\Lambda$.

A space X is *θ-expandable*, if for every locally finite collection $\{F_\lambda|\ \lambda\in\Lambda\}$ of subsets of X there exists a sequence $\big\{\mathfrak{G}_n=\{G_{\lambda,n}|\ \lambda\in\Lambda\}|\ n=1,2,\ldots\big\}$ of collections of open subsets of X satisfying the following:

(1) $F_\lambda\subset G_{\lambda,n}$ for each λ and each n.

(2) For each point x of X there is a positive integer n for which $\mathfrak{G}_n$ is point-finite at x.

A collection $\mathfrak{A}$ of subsets of a space X is *bounded locally finite*, if there exists a positive integer n such that every point of X has a neighborhood which intersects at most n members of $\mathfrak{A}$. Obviously, every discrete collection is bounded locally finite and every bounded locally finite collection is locally finite.

A space X is *boundedly* (resp. *discretely*) *expandable* [11], if for every bounded locally finite (resp. discrete) collection $\{F_\lambda|\ \lambda\in\Lambda\}$ of subsets of X there exists a locally finite collection $\{G_\lambda|\ \lambda\in\Lambda\}$ of open subsets of X such that $F_\lambda\subset G_\lambda$ for each $\lambda\in\Lambda$. *Almost boundedly* (*discretely*) *expandable* spaces [11] and *boundedly* (*discretely*) *θ-expandable* spaces are now easy to be understood.

A space X is *discretely subexpandable*, if for every discrete collection $\{F_\lambda|\ \lambda\in\Lambda\}$ of subsets of X there exists a sequence

$$\{\mathfrak{G}_n=\{G_{\lambda,n}|\ \lambda\in\Lambda\}|\ n=1,2,\ldots\}$$

of collections of open subsets of X satisfying the following:

(3) $F_\lambda\subset G_{\lambda,n}$ for each λ and each n.

(4) For every point x of X there is a positive integer n for which x is contained in at most one member of $\mathfrak{G}_n$.

A discretely subexpandable space is *subexpandable* or *boundedly subexpandable*, if it is countably subparacompact or finitely subparacompact respectively.

In the definitions of expandable spaces, θ-expandable spaces, subexpandable spaces, etc., if the power of the index set Λ is $\mathfrak{m}$, then we have the definitions of *$\mathfrak{m}$-expandable* spaces, *$\mathfrak{m}$-θ-expandable* spaces, *$\mathfrak{m}$-subexpandable* spaces, etc., respectively. In case $\mathfrak{m}=\aleph_0$ these are *countably expandable* spaces, *countably θ-expandable* spaces, *countably subexpandable* spaces, etc., respectively.

THEOREM 1.1 [10]. *A space is countably expandable if and only if it is countably paracompact.*

THEOREM 1.2. *The following are equivalent for a space X:*

(a) *X is almost countably expandable.*

(b) *X is countably metacompact.*

(c) *X is countably θ-expandable.*

(d) *X is countably θ-refinable.*

(e) *For every increasing countable open covering $\{U_n|\ n=1,2,\ldots\}$ there is a countable closed covering $\{V_n|\ n=1,2,\ldots\}$ such that $V_n\subset U_n$ for every n.*

Proof. The implications (a) → (c) and (b) → (d) are obvious. We shall show only the implications (a) → (b), (d) → (e) and (e) → (a); the proof of (c) → (d) is analogous to that of (a) → (b).

(a) → (b): Let $\mathfrak{U}=\{U_i|\ i=1,2,\ldots\}$ be a countable open covering of X. Let $F_1=U_1$ and $F_i=U_i-\bigcup_{k=1}^{i-1}U_k$ for each $i=2,3,\ldots$, then the collection $\{F_i|\ i=1,2,\ldots\}$ is a locally finite covering of X. Since X is almost countably expandable, there is a point-finite open collection $\{G_i|\ i=1,2,\ldots\}$ such that $F_i\subset G_i$ for each i. Let $V_i=U_i\cap G_i$ for each i, then the collection $\{V_i|\ i=1,2,\ldots\}$ is a point-finite open refinement of $\mathfrak{U}$. Hence X is countably metacompact.

(d) → (e): Let $\mathfrak{U}=\{U_i|\ i=1,2,\ldots\}$ be an increasing countable open covering of X. Since X is countably θ-refinable, we have a sequence $\{\mathfrak{V}_n|\ n=1,2,\ldots\}$ of open refinements of $\mathfrak{U}$ such that for every point x of X there is a positive integer n for which $\mathfrak{V}_n$ is point-finite at x. Let $W_{i,n}=\{x\in X|\ \mathrm{St}(x,\mathfrak{V}_n)\subset U_i\}$, where $\mathrm{St}(x,\mathfrak{V}_n)=\bigcup\{V\in\mathfrak{V}_n|\ V\ni x\}$, for each i and each n, then it is easily shown $\overline{W}_{i,n}\subset U_i$. Let $V_i=\bigcup_{n=1}^{i}\overline{W}_{i,n}$ for each i, then V_i is a closed subset and $V_i\subset U_i$ for each i. To show that the collection $\{V_i|\ i=1,2,\ldots\}$ is a covering of X, let x be a point of X. We have a positive integer $n(x)$ such that $\mathfrak{V}_{n(x)}$ is point-finite at x. Since

$\mathfrak{B}_{n(x)}$ refines $\mathfrak{U}$ and $\mathfrak{U}$ is an increasing covering, there is a positive integer $i(x)$ such that $\mathrm{St}(x, \mathfrak{B}_{n(x)}) \subset U_{i(x)}$. Hence $x \in W_{i(x),n(x)}$. Let $\max\{n(x), i(x)\} = m(x)$, then $W_{i(x),n(x)} \subset W_{m(x),n(x)} \subset V_{m(x)}$. Hence $\{V_i \mid i = 1, 2, \ldots\}$ is a covering of X.

(e) → (a): Let $\{F_i \mid i = 1, 2, \ldots\}$ be a locally finite countable collection of subsets of X. We put $U_i = X - \bigcup_{k=i+1}^{\infty} \overline{F}_k$ for every positive integer i, then it is easily shown that $\{U_i \mid i = 1, 2, \ldots\}$ is an increasing open covering of X. By assumption, there exists a closed covering $\{V_i \mid i = 1, 2, \ldots\}$ of X such that $V_i \subset U_i$ for each i. Let $G_1 = X$ and $G_i = X - \bigcup_{k=1}^{i-1} V_k$ for $i = 2, 3, \ldots$, then $\{G_i \mid i = 1, 2, \ldots\}$ is a point-finite collection of open subsets of X. Since $F_i \cap V_k \subset F_i \cap U_k = \emptyset$ whenever $k < i$, we have $F_i \subset G_i$ for each i. Hence X is almost countably expandable.

THEOREM 1.3. *A space is countably subexpandable if and only if it is countably subparacompact.*

Proof. To complete the proof, it is sufficient to show that a countably subparacompact space is discretely countably subexpandable. Let X be a countably subparacompact space and let $\{F_i \mid i = 1, 2, \ldots\}$ be a discrete collection of subsets of X. Put $U_i = X - \bigcup_{k \neq i} \overline{F}_k$ for each positive integer i, then $\mathfrak{U} = \{U_i \mid i = 1, 2, \ldots\}$ is a countable covering of X. Hence $\mathfrak{U}$ has a σ-discrete closed refinement $\mathfrak{B}$. We may assume that $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$, $\mathfrak{B}_n = \{V_{i,n} \mid i = 1, 2, \ldots\}$, each $\mathfrak{B}_n$ is a discrete closed collection and $V_{i,n} \subset U_i$ for each i and each n. Let $G_{i,n} = X - \bigcup_{k \neq i} V_{k,n}$ for each i and each n and $\mathfrak{G}_n = \{G_{i,n} \mid i = 1, 2, \ldots\}$, then each $\mathfrak{G}_n$ is an open collection. It is easily proved that $F_i \subset G_{i,n}$ for each i and each n. Let x be a point of X. Pick up $i(x)$ and $n(x)$ such that $x \in V_{i(x),n(x)}$. Then, in the collection $\mathfrak{G}_{n(x)}$, only $G_{i(x),n(x)}$ contains x. Thus X is discretely countably subexpandable.

The following theorem is easily proved from Dowker [5, Theorem 2] and Theorems 1.1, 1.2 and 1.3.

THEOREM 1.4. *The following are equivalent for a normal space X.*

(a) *X is countably expandable.*
(b) *X is almost countably expandable.*
(c) *X is countably θ-expandable.*
(d) *X is countably subexpandable.*

The following theorem is a direct consequence of Theorems 2.1, 2.2, 2.3 and 2.4 below.

THEOREM 1.5. (a) *An $\mathfrak{m}$-paracompact space is $\mathfrak{m}$-expandable* [10].
(b) *An $\mathfrak{m}$-metacompact space is almost $\mathfrak{m}$-expandable* [11].

(c) *An $\mathfrak{m}$-θ-refinable space is $\mathfrak{m}$-θ-expandable.*
(d) *An $\mathfrak{m}$-subparacompact space is $\mathfrak{m}$-subexpandable.*

THEOREM 1.6. (a) *A space is boundedly $\mathfrak{m}$-expandable if and only if it is discretely $\mathfrak{m}$-expandable* [11].

(b) *A space is almost boundedly $\mathfrak{m}$-expandable if and only if it is almost discretely $\mathfrak{m}$-expandable* [11].

(c) *A space is boundedly $\mathfrak{m}$-θ-expandable if and only if it is discretely $\mathfrak{m}$-θ-expandable.*

Part (a) of the theorem was proved in [11], and (b) and (c) are proved by the same argument.

THEOREM 1.7. (a) *A space is $\mathfrak{m}$-expandable if and only if it is boundedly $\mathfrak{m}$-expandable and countably expandable* [11].

(b) *A space is almost $\mathfrak{m}$-expandable if and only if it is almost boundedly $\mathfrak{m}$-expandable and almost countably expandable* [11].

(c) *A space is $\mathfrak{m}$-θ-expandable if and only if it is boundedly $\mathfrak{m}$-θ-expandable and countably θ-expandable.*

This theorem will be proved in § 2.

THEOREM 1.8. *An $\mathfrak{m}$-subexpandable space is $\mathfrak{m}$-θ-expandable.*

Proof. A discretely $\mathfrak{m}$-subexpandable space is obviously discretely $\mathfrak{m}$-θ-expandable and hence boundedly $\mathfrak{m}$-θ-expandable by Theorem 1.6. It is easily shown that a countably subparacompact space X satisfies the condition (e) in Theorem 1.2. Therefore, by Theorem 1.7, an $\mathfrak{m}$-subexpandable space is $\mathfrak{m}$-θ-expandable.

A space X is $\mathfrak{m}$-*collectionwise normal* if for every discrete collection $\{F_\lambda \mid \lambda \in \Lambda\}$, with power $\leqslant \mathfrak{m}$, of subsets of X there exists a disjoint open collection $\{G_\lambda \mid \lambda \in \Lambda\}$ such that $F_\lambda \subset G_\lambda$ for every $\lambda \in \Lambda$. A space is *collectionwise normal* [2], if it is $\mathfrak{m}$-collectionwise normal for every cardinal number $\mathfrak{m}$. Evidently every $\mathfrak{m}$-collectionwise normal space is normal. It is easily seen that a normal space is $\aleph_0$-collectionwise normal.

THEOREM 1.9 ([7] or [11]). *A space is $\mathfrak{m}$-collectionwise normal if and only if it is boundedly $\mathfrak{m}$-expandable and normal.*

THEOREM 1.10. *An $\mathfrak{m}$-collectionwise normal space is boundedly $\mathfrak{m}$-subexpandable.*

Proof. Evidently every $\mathfrak{m}$-collectionwise normal space is discretely $\mathfrak{m}$-subexpandable and every normal space is finitely subparacompact.

THEOREM 1.11 [7]. *A normal space is $\mathfrak{m}$-expandable if and only if it is $\mathfrak{m}$-collectionwise normal and countably paracompact.*

COROLLARY 1.12. *An $\mathfrak{m}$-expandable normal space is $\mathfrak{m}$-subexpandable.*

THEOREM 1.13. *A space X in which every closed subset is a G_δ-subset is subexpandable.*

Proof. First we shall prove that X is discretely subexpandable. Let $\mathfrak{F} = \{F_\lambda \mid \lambda \in \Lambda\}$ be a discrete collection of subsets of X. By assumption, we have a collection $\{H_n \mid n = 1, 2, \ldots\}$ open subsets of X such that $\bigcap_{n=1}^{\infty} H_n = \overline{\bigcup_{\lambda \in \Lambda} F_\lambda}$. Let $G_{\lambda,n} = H_n - \overline{\bigcup_{\mu \neq \lambda} F_\mu}$ for each λ and each n, then $\mathfrak{G}_n = \{G_{\lambda,n} \mid \lambda \in \Lambda\}$ is an open collection for each n. Since $\mathfrak{F}$ is discrete, $\overline{F}_\lambda \cap \overline{\bigcup_{\mu \neq \lambda} F_\mu} = \emptyset$ for each λ. Hence $F_\lambda \subset G_{\lambda,n}$ for each λ and each n. Let x be a point of X. If $x \in \overline{\bigcup_{\lambda \in \Lambda} F_\lambda}$, there is an element $\lambda(x)$ of Λ such that $x \in \overline{F}_{\lambda(x)}$ and $x \notin \overline{\bigcup_{\mu \neq \lambda(x)} F_\mu}$. Hence, in any $\mathfrak{G}_n$, only $G_{\lambda(x),n}$ contains x. If $x \notin \overline{\bigcup_{\lambda \in \Lambda} F_\lambda}$, there is a positive integer $n(x)$ for which $x \notin H_{n(x)}$; consequently no member of $\mathfrak{G}_{n(x)}$ contains x. Therefore X is discretely subexpandable.

Next we shall prove that X is countably subparacompact. Let $\mathfrak{U} = \{U_i \mid i = 1, 2, \ldots\}$ be a countable open covering of X. By assumption, for each i, there is a collection $\{V_{i,n} \mid n = 1, 2, \ldots\}$ of closed subsets of X such that $U_i = \bigcup_{n=1}^{\infty} V_{i,n}$. Obviously the collection

$$\{V_{i,n} \mid i = 1, 2, \ldots;\ n = 1, 2, \ldots\}$$

is a σ-discrete closed refinement of $\mathfrak{U}$. Therefore X is countably subparacompact.

Corollary 1.14. *A perfectly normal space is subexpandable.*

From the results in this section together with well-known results, we have the following diagram:

paracompact →	metacompact →	θ-refinable	← subparacompact ←	paracompact and normal	⇄ fully normal
↓	↓	↓	↓	↓	↓
expandable	→ almost expandable	→ θ-expandable ←	subexpandable	← expandable and normal	⇄ strongly normal
↓	↓	↓	↓	↓	↓
boundedly expandable	→ almost boundedly expandable	→ boundedly θ-expandable	← boundedly subexpandable	← boundedly expandable and normal	⇄ collectionwise normal
↓↑	↓	↓↑	↓		
discretely expandable	→ almost discretely expandable	→ discretely θ-expandable	← discretely subexpandable		

2. Characterizations in terms of coverings. An open covering of a space is an *A-covering* (resp. *B-covering*) [8], if it has a locally finite (resp. bounded locally finite) refinement. Notice that the refinement is not necessarily open. Every B-covering is an A-covering and every countable open covering is an A-covering [8].

An open covering $\{U_\lambda \mid \lambda \in \Lambda\}$ is a *C-covering*, if it satisfies that $U_\lambda = \bigcap_{\mu \neq \lambda} \bigcup_{\nu \neq \mu} U_\nu$ for every $\lambda \in \Lambda$. Every C-covering is a B-covering. Indeed, for a C-covering $\mathfrak{U} = \{U_\lambda \mid \lambda \in \Lambda\}$ the collection $\mathfrak{B} = \{V_\lambda \mid \lambda \in \Lambda\} \cup \{X - \bigcup_{\lambda \in \Lambda} V_\lambda\}$ is a bounded locally finite refinement of $\mathfrak{U}$, where $V_\lambda = X - \bigcup_{\mu \neq \lambda} U_\mu$ for every $\lambda \in \Lambda$. Obviously every binary open covering is a C-covering.

A-coverings and B-coverings were used in [8] to characterize strongly normal spaces (i.e. countably paracompact, collectionwise normal spaces) and collectionwise normal spaces, respectively.

In this section, by using A-coverings, B-coverings and C-coverings respectively, we shall give various characterizations of

(A) expandability, almost expandability, θ-expandability,

(B) bounded expandability, almost bounded expandability, bounded θ-expandability, and

(C) bounded subexpandability.

For $\mathfrak{m}$-expandability, etc., too, quite analogous characterizations will be obtained under the condition the powers of coverings $\leqslant \mathfrak{m}$; these will be omitted, however.

A covering $\mathfrak{U}$ is *directed* [12], if it is directed by set inclusion.

Theorem 2.1. *The following are equivalent for a space X:*

(a) *X is expandable (boundedly expandable).*

(b) *Every A-covering (B-covering) of X has a locally finite open refinement.*

(c) *Every directed A-covering (B-covering) of X has a locally finite open refinement.*

(d) *Every directed A-covering (B-covering) of X has an open, locally star-refinement* (1).

(e) *Every directed A-covering (B-covering) of X has an open, cushioned refinement* [15].

Theorem 2.2. *The following are equivalent for a space X:*

(a) *X is almost expandable (almost boundedly expandable).*

(b) *Every A-covering (B-covering) of X has a point-finite open refinement.*

(c) *Every directed A-covering (B-covering) of X has a point-finite open refinement.*

(d) *Every directed A-covering (B-covering) of X has an open Δ-refinement.*

(1) Let $\mathfrak{U}$ and $\mathfrak{B}$ be two coverings of a space X. If every point x of X has a neighborhood $W(x)$ such that $\mathrm{St}(W(x), \mathfrak{B}) \subset U$ for some $U \in \mathfrak{U}$, then we say that the covering $\mathfrak{B}$ is a *locally star-refinement* of the covering $\mathfrak{U}$. Obviously, every open star-refinement [20] is a locally star-refinement and every locally star-refinement is a Δ-refinement [20].

(e) *Every directed A-covering (B-covering) of X has a cushioned refinement.*

THEOREM 2.3. *The following are equivalent for a space X:*

(a) *X is θ-expandable (boundedly θ-expandable).*

(b) *For every A-covering (B-covering)* $\mathfrak{U}$ *of X there exists a sequence* $\{\mathfrak{V}_n \mid n = 1, 2, \ldots\}$ *of open refinements of* $\mathfrak{U}$ *such that for every point x of X there exists a positive integer n for which* $\mathfrak{V}_n$ *is point-finite at x.*

(c) *For every directed A-covering (B-covering)* $\mathfrak{U}$ *of X there exists a sequence* $\{\mathfrak{V}_n \mid n = 1, 2, \ldots\}$ *of open refinements of* $\mathfrak{U}$ *such that for every point x of X there exists a positive integer n for which* $\mathfrak{V}_n$ *is point-finite at x.*

(d) *For every directed A-covering (B-covering)* $\mathfrak{U}$ *of X there exists a sequence* $\{\mathfrak{V}_n \mid n = 1, 2, \ldots\}$ *of open refinements of* $\mathfrak{U}$ *such that for every point x of X there exists a positive integer n and some* $U \in \mathfrak{U}$ *with* $\mathrm{St}(x, \mathfrak{V}_n) \subset U$.

(e) *Every directed A-covering (B-covering) of X has a σ-cushioned refinement.*

We prove only Theorem 2.3; the proofs of Theorems 2.1 and 2.2 are quite parallel to that of Theorem 2.3.

Proof of Theorem 2.3. (a) → (b): Let $\mathfrak{U}$ be an A-covering of X. Then $\mathfrak{U}$ has a locally finite refinement $\mathfrak{F} = \{F_\lambda \mid \lambda \in \Lambda\}$. Since X is θ-expandable, there exists a sequence $\{\mathfrak{G}_n = \{G_{\lambda,n} \mid \lambda \in \Lambda\} \mid n = 1, 2, \ldots\}$ of collections of open subsets of X which satisfies conditions (1) and (2) in the definition of θ-expandability. Since $\mathfrak{F}$ is a refinement of the open covering $\mathfrak{U}$, we may assume that every $\mathfrak{G}_n$ is also a refinement of $\mathfrak{U}$. Thus $\{\mathfrak{G}_n \mid n = 1, 2, \ldots\}$ is a required sequence of refinements of $\mathfrak{U}$.

(b) → (c): This is evident.

(c) → (d): Let $\mathfrak{U}$ be a directed covering of X and $\mathfrak{V}$ a refinement of $\mathfrak{U}$. If $\mathfrak{V}$ is point-finite at a point x of X, then there is some $U \in \mathfrak{U}$ such that $\mathrm{St}(x, \mathfrak{V}) \subset U$.

(d) → (e): If $\mathfrak{U}$ and $\mathfrak{V}$ are two open coverings of X, then the collection $\{W(U) \mid U \in \mathfrak{U}\}$ is cushioned in $\mathfrak{U}$, where

$$W(U) = \{x \in X \mid \mathrm{St}(x, \mathfrak{V}) \subset U\}.$$

(e) → (a): Let $\mathfrak{F} = \{F_\lambda \mid \lambda \in \Lambda\}$ be a locally finite collection of subsets of X, and let Γ be the set of all finite subsets of Λ. Put $U_\gamma = X - \bigcup_{\lambda \notin \gamma} \overline{F}_\lambda$ for each $\gamma \in \Gamma$, then $\mathfrak{U} = \{U_\gamma \mid \gamma \in \Gamma\}$ is evidently a directed open covering of X. If let $H_\gamma = \bigcap_{\lambda \in \gamma} \overline{F}_\lambda$ for each $\gamma \in \Gamma$, then it is easily shown that $\{U_\gamma \cap H_\gamma \mid \gamma \in \Gamma\}$ is a locally finite refinement of $\mathfrak{U}$. (If $\mathfrak{F}$ is bounded locally finite, then so also is it.) Hence $\mathfrak{U}$ is a directed A-covering of X. By assumption, $\mathfrak{U}$ has a σ-cushioned refinement $\mathfrak{V} = \bigcup_{n=1}^{\infty} \mathfrak{V}_n$ such that every $\mathfrak{V}_n$ is cushioned in $\mathfrak{U}$. Without loss of generality we may assume that $\mathfrak{V}_n = \{V_{\gamma,n} \mid \gamma \in \Gamma\}$ for every n, and $\overline{\bigcup_{\gamma \in \Gamma'} V_{\gamma,n}} \subset \bigcup_{\gamma \in \Gamma'} U_\gamma$ for every $\Gamma' \subset \Gamma$. Let $G_{\lambda,n} = X - \overline{\bigcup_{\gamma \not\ni \lambda} V_{\gamma,n}}$ for each $\lambda \in \Lambda$, then $\mathfrak{G}_n = \{G_{\lambda,n} \mid \lambda \in \Lambda\}$ is an open collection for each n. Since $F_\lambda \cap U_\gamma = \emptyset$ whenever $\lambda \notin \gamma$, we have $F_\lambda \subset X - \bigcup_{\gamma \not\ni \lambda} U_\gamma$ for each $\lambda \in \Lambda$. Hence, for each $\lambda \in \Lambda$ and each n,

$$G_{\lambda,n} = X - \overline{\bigcup_{\gamma \not\ni \lambda} V_{\gamma,n}} \supset X - \bigcup_{\gamma \not\ni \lambda} U_\gamma \supset F_\gamma .$$

Now, let x be a point of X. Since $\mathfrak{V}$ is a covering of X, we have a positive integer $n(x)$ and an element $\gamma(x)$ of Γ such that $x \in V_{\gamma(x),n(x)}$. Hence, if $x \in G_{\lambda,n(x)}$ then $\lambda \in \gamma(x)$; consequently $\mathfrak{G}_{n(x)}$ is point-finite at x, because $\gamma(x)$ is a finite subset of Λ. (If $\mathfrak{V}$ is an open covering, $\mathfrak{G}_{n(x)}$ is locally finite at x.) Thus X is θ-expandable.

THEOREM 2.4. *The following are equivalent for a space X:*

(a) *X is boundedly subexpandable.*

(b) *For every C-covering* $\mathfrak{U}$ *of X there exists a sequence* $\{\mathfrak{V}_n \mid n = 1, 2, \ldots\}$ *of open refinements of* $\mathfrak{U}$ *such that for every point x of X there exists a positive integer n for which only one member of* $\mathfrak{V}_n$ *contains x.*

(c) *For every C-covering* $\mathfrak{U}$ *of X there exists a sequence* $\{\mathfrak{V}_n \mid n = 1, 2, \ldots\}$ *of open refinements of* $\mathfrak{U}$ *such that for every point x of X there exists a positive integer n and some* $U \in \mathfrak{U}$ *with* $\mathrm{St}(x, \mathfrak{V}_n) \subset U$.

(d) *Every C-covering of X has a σ-discrete closed refinement.*

(e) *Every C-covering of X has a σ-locally finite closed refinement.*

(f) *Every C-covering of X has a σ-closure-preserving closed refinement.*

(g) *Every C-covering of X has a σ-cushioned refinement.*

Proof. The implications (b) → (c) and (d) → (e) → (f) → (g) are obvious, and the proof of (c) → (g) is identical with that of (d) → (e) in Theorem 2.3.

(a) → (b): By assumption, X is discretely subexpandable and finitely subparacompact. Let $\mathfrak{U} = \{U_\lambda \mid \lambda \in \Lambda\}$ be a C-covering of X, and let $F_\lambda = X - \bigcup_{\mu \neq \lambda} U_\mu$ for each $\lambda \in \Lambda$. Then $\{F_\lambda \mid \lambda \in \Lambda\}$ is a discrete closed collection. By the discrete subexpandability of X, there exists a sequence $\{\mathfrak{G}_n = \{G_{\lambda,n} \mid \lambda \in \Lambda\} \mid n = 1, 2, \ldots\}$ of open collections satisfying (3) and (4) in the definition of a discretely subexpandable space. Since $F_\lambda \subset U_\lambda$ for each λ, we may assume $G_{\lambda,n} \subset U_\lambda$ for each λ and each n. Let $G_n = \bigcup_{\lambda \in \Lambda} G_{\lambda,n}$ for each n and let $H = X - \bigcup_{\lambda \in \Lambda} F_\lambda$, then $\{G_n, H\}$ is a binary open covering of X for each n. Since X is finitely subparacompact, we have a countable closed covering $\mathfrak{K}_n = \{K_{i,n} \mid i = 1, 2, \ldots\} \cup \{L_{i,n} \mid i = 1, 2, \ldots\}$ for each n such that $K_{i,n} \subset G_n$, $L_{i,n} \subset H$ and $K_{i,n} \cap L_{i,n} = \emptyset$ for each i. Let us put $V_{\lambda,i,n} = G_{\lambda,n} - L_{i,n}$ and $\mathfrak{V}_{i,n} = \{V_{\lambda,i,n} \mid \lambda \in \Lambda\} \cup \{H - K_{i,n}\}$ for each i and each n. It is easily shown that every $\mathfrak{V}_{i,n}$ is an open covering of X. Next, let x be a point of X, and let $n(x)$ be a positive integer such that x is contained in at most one element of $\mathfrak{G}_{n(x)}$. Since $\mathfrak{K}_{n(x)}$ is a covering of X, there is an $i(x)$ such that $x \in K_{i(x),n(x)}$ or $x \in L_{i(x),n(x)}$. Then x is in exactly one element of the covering $\mathfrak{B}_{i(x),n(x)}$. Finally, let us show that every $\mathfrak{B}_{i,n}$ refines $\mathfrak{U}$. To do this it is enough to show that some member of $\mathfrak{U}$ contains H. Really, every member U_λ of $\mathfrak{U}$ contains H; for

$$H = \bigcap_{\mu \in \Lambda} (X - F_\mu) \subset \bigcap_{\mu \neq \lambda} (X - F_\mu) = \bigcap_{\mu \neq \lambda} \bigcup_{\nu \neq \mu} U_\nu = U_\lambda .$$

Thus the sequence $\{\mathfrak{B}_{i,n} |\ i, n = 1, 2, \ldots\}$ is what we require.

(b) → (d): Let $\mathfrak{U}$ be a C-covering of X, and let $\{\mathfrak{B}_n |\ n = 1, 2, \ldots\}$ be a sequence of open refinements of $\mathfrak{U}$ with the property that, if $x \in X$, there is an n such that x is contained in exactly one member of $\mathfrak{B}_n$. We define, for every $V \in \mathfrak{B}_n$, $W(V) = X - \bigcup \{V' \in \mathfrak{B}_n |\ V' \neq V\}$, and let $\mathfrak{W}_n = \{W(V) |\ V \in \mathfrak{B}_n\}$ for every n. Then $\bigcup_{n=1}^{\infty} \mathfrak{W}_n$ is a σ-discrete closed refinement of $\mathfrak{U}$.

(g) → (a): First let us shown that X is discretely subexpandable. Let $\mathfrak{F} = \{F_\lambda |\ \lambda \in \Lambda\}$ be a discrete collection of X. If we put $U_\lambda = X - \bigcup_{\mu \neq \lambda} F_\mu$ for each $\lambda \in \Lambda$, then $\mathfrak{U} = \{U_\lambda |\ \lambda \in \Lambda\}$ is obviously an open covering of X. Since $\mathfrak{F}$ is discrete, $\bigcap_{\mu \neq \mu} \bigcup_{\nu \neq \mu} \overline{F}_\nu = \overline{F}_\lambda$ for each λ, and hence, $\bigcap_{\mu \neq \lambda} \bigcup_{\nu \neq \mu} U_\nu = U_\lambda$ for each λ. Therefore $\mathfrak{U}$ is a C-covering, so that, by assumption, it has a σ-cushioned refinement $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$. We may index $\mathfrak{B}_n = \{V_{\lambda,n} |\ \lambda \in \Lambda\}$ for each n such that, for each $\Lambda' \subset \Lambda$, $\overline{\bigcup_{\lambda \in \Lambda'} V_{\lambda,n}} \subset \bigcup_{\lambda \in \Lambda'} U_\lambda$. Let $G_{\lambda,n} = X - \overline{\bigcup_{\mu \neq \lambda} V_{\mu,n}}$ and $\mathfrak{G}_n = \{G_{\lambda,n} |\ \lambda \in \Lambda\}$ for each n, then each $\mathfrak{G}_n$ is an open collection. For each $x \in X$, if we choose $\lambda(x)$ and $n(x)$ such that $x \in V_{\lambda(x),n(x)}$, then, in $\mathfrak{G}_{n(x)}$, at most $G_{\lambda(x),n(x)}$ contains x. Furthermore, for each λ and each n, we have

$$G_{\lambda,n} = X - \overline{\bigcup_{\mu \neq \lambda} V_{\mu,n}} \supset X - \bigcup_{\mu \neq \lambda} U_\mu = \bigcap_{\mu \neq \lambda} \bigcup_{\nu \neq \mu} \overline{F}_\nu \supset F_\lambda .$$

Hence X is discretely subexpandable.

Next let us show that X is finitely subparacompact. Since a binary open covering is a C-covering, every binary open covering of X has a σ-cushioned refinement by assumption. Hence every finite open covering of X has also a σ-cushioned refinement. As is easily shown, a finite (more generally, countable) open covering with a σ-cushioned refinement has a σ-discrete closed refinement. Hence X is finitely subparacompact. Thus X is boundedly subexpandable, and this completes the proof.

In connection with Theorems 2.1–2.4, we raise the following problems:

PROBLEM 2.5. *Are the following equivalent for a space X?*

(a) *X is paracompact.*

(b) *Every directed open covering of X has an open, locally star-refinement.*

(c) *Every directed open covering of X has an open cushioned refinement.*

PROBLEM 2.6. *Are the following equivalent for a space X?*

(a) *X is metacompact.*

(b) *Every directed open covering of X has an open Δ-refinement.*

(c) *Every directed open covering of X has a cushioned refinement.*

PROBLEM 2.7. *Are the following equivalent for a space X?*

(a) *X is θ-refinable.*

(b) *For every directed open covering $\mathfrak{U}$ of X there exists a sequence $\{\mathfrak{B}_n |\ n = 1, 2, \ldots\}$ of open refinements of $\mathfrak{U}$ such that for every point x of X there exist some n and some $U \in \mathfrak{U}$ with* $\mathrm{St}(x, \mathfrak{B}_n) \subset U$.

(c) *Every directed open covering of X has a σ-cushioned refinement.*

PROBLEM 2.8. *Are the following equivalent for a space X?*

(a) *X is subparacompact.*

(b) *Every open covering of X has a σ-cushioned refinement.*

The implications (a) → (b) → (c) in Problems 2.5–2.7 and the implication (a) → (b) in Problem 2.8 are easily proved. An affirmative answer to Problem 2.7 would imply affirmative answers to the others (cf. Theorems 3.1 and 3.2 below).

For expandable spaces, furthermore, we have the following characterizations:

THEOREM 2.9. *The following are equivalent for a space X:*

(a) *X is expandable.*

(b) *For every directed A-covering $\mathfrak{U}$ of X, there exists a locally finite open covering $\mathfrak{B}$ of X such that $\{\overline{V} |\ V \in \mathfrak{B}\}$ refines $\mathfrak{U}$.*

(c) *Every directed A-covering of X has a locally finite closed refinement.*

(d) *Every directed A-covering of X has an open σ-cushioned refinement.*

Proof. The implications (b) → (c) and (b) → (d) are obvious. The proofs of (a) → (b) and (c) → (a) essentially due to Mack [12].

(a) → (b): Since X is expandable, every directed A-covering $\mathfrak{U}$ of X has a locally finite open refinement $\{G_\lambda |\ \lambda \in \Lambda\}$ by Theorem 2.1. Let Γ be the set of all finite subsets of Λ. Put $H_\gamma = X - \bigcup_{\lambda \notin \gamma} \overline{G}_\lambda$ for each $\gamma \in \Gamma$, and let $\mathfrak{H} = \{H_\gamma |\ \gamma \in \Gamma\}$. Then $\mathfrak{H}$ is evidently a directed open covering of X. Since $\{\bigcap_{\lambda \in \gamma} \overline{G}_\lambda \cap H_\gamma |\ \gamma \in \Gamma\}$ is a locally finite refinement of $\mathfrak{H}$, $\mathfrak{H}$ is an A-covering. Again, by the expandability of X, $\mathfrak{H}$ has a locally finite open refinement $\mathfrak{B}$. Now, since $\mathfrak{U}$ is directed, there is some $U \in \mathfrak{U}$ such that $\overline{H}_\gamma \subset X - \bigcup_{\lambda\notin\gamma} G_\lambda \subset \bigcup_{\lambda\in\gamma} G_\lambda \subset U$. Therefore $\{\overline{H} \mid H \in \mathfrak{H}\}$, and hence $\{\overline{V} \mid V \in \mathfrak{V}\}$ refines $\mathfrak{U}$.

(c) $\to$ (a): Let $\mathfrak{U} = \{U_\lambda \mid \lambda \in \Lambda\}$ be a directed Λ-covering of X, then by assumption there is a locally finite closed refinement $\{F_\lambda \mid \lambda \in \Lambda\}$ such that $F_\lambda \subset U_\lambda$ for each λ. As above-mentioned, if we put $H_\gamma = X - \bigcup_{\lambda\notin\gamma} F_\lambda$ for each γ, then $\{H_\gamma \mid \gamma \in \Gamma\}$ is a directed Λ-covering of X. And it has also a locally finite closed refinement $\{K_\gamma \mid \gamma \in \Gamma\}$ such that $K_\gamma \subset H_\gamma$ for each γ. Put $V_\lambda = U_\lambda - \bigcup_{\gamma \not\ni \lambda} K_\gamma$ for each $\lambda \in \Lambda$, then $\mathfrak{V} = \{V_\lambda \mid \lambda \in \Lambda\}$ is an open collection which refines $\mathfrak{U}$. Since $F_\lambda \cap K_\gamma \subset F_\lambda \cap H_\gamma = \emptyset$ whenever $\lambda \notin \gamma$, $F_\lambda \subset V_\lambda$ for each λ. Hence $\mathfrak{V}$ is a covering of X. To establish the locally finiteness of $\mathfrak{V}$, let x be an arbitrary point of X. Pick up $\gamma(x)$ in Γ for which $x \in K_{\gamma(x)}$. If $x \in V_\lambda$, then λ is in $\gamma(x)$ which is a finite subset of Λ. Hence $\mathfrak{V}$ is locally finite.

(d) $\to$ (a): For the moment, assume that X is countably paracompact. By (d), every directed Λ-covering $\mathfrak{U}$ of X has an open σ-cushioned refinement. As is easily shown, an open covering of a countably paracompact space with an open σ-cushioned refinement has an open cushioned refinement. Therefore $\mathfrak{U}$ has an open cushioned refinement, and hence, by Theorem 2.1, X is expandable. Thus, to establish the implication, it is enough to show that X is countably paracompact. To do this, let $\mathfrak{U} = \{U_i \mid i = 1, 2, \ldots\}$ be an increasing countable open covering of X, then it is obviously a directed Λ-covering. By assumption, $\mathfrak{U}$ has an open σ-cushioned refinement $\bigcup_{n=1}^{\infty} \mathfrak{V}_n$; we may index $\mathfrak{V}_n = \{V_{i,n} \mid i = 1, 2, \ldots\}$ such that $\overline{V}_{i,n} \subset U_i$ for each i and each n. Define $W_i = \bigcup_{j=1}^{i} \bigcup_{n=1}^{i} V_{j,n}$, then $\{W_i \mid i = 1, 2, \ldots\}$ is an open covering of X satisfying $\overline{W}_i \subset U_i$ for each i. Hence, by Ishikawa [6], X is countably paracompact.

Here we establish Theorem 1.7 which is left unfinished.

Proof of Theorem 1.7. We shall prove only parts (b) and (c); part (a) will be proved in parallel to part (b).

(b) To prove the nontrivial half, let X be almost boundedly $\mathfrak{m}$-expandable and almost countably expandable. Let $\mathfrak{U}$ be an Λ-covering of X with power $\leqslant \mathfrak{m}$. Then $\mathfrak{U}$ has a locally finite refinement $\mathfrak{F} = \{F_\lambda \mid \lambda \in \Lambda\}$ with $|\Lambda| \leqslant \mathfrak{m}$. For each positive integer n, let G_n be the set of all points of X contained in at most n members of $\overline{\mathfrak{F}}$, where $\overline{\mathfrak{F}} = \{\overline{F}_\lambda \mid \lambda \in \Lambda\}$. It is easy to show that $\{G_n \mid n = 1, 2, \ldots\}$ is an increasing countable open covering of X. Since X is almost countably expandable, by Theorem 1.2 there exists a countable closed covering $\{H_n \mid n = 1, 2, \ldots\}$ of X such that $H_n \subset G_n$ for each n. Then, for each n, the collection $\{F_\lambda \cap H_n \mid \lambda \in \Lambda\}$ is bounded locally finite. Hence, by the almost bounded $\mathfrak{m}$-expandability of X, for each n there exists a point-finite open collection $\mathfrak{V}_n = \{V_{\lambda,n} \mid \lambda \in \Lambda\}$ such that $F_\lambda \cap H_n \subset V_{\lambda,n}$ for each λ. Since $\mathfrak{F}$ refines $\mathfrak{U}$, we may assume that each $\mathfrak{V}_n$ refines $\mathfrak{U}$. Then $\bigcup_{n=1}^{\infty} \mathfrak{V}_n$ is obviously a σ-point-finite open refinement of $\mathfrak{U}$. Hence $\mathfrak{U}$ has a point-finite open refinement, because X is countably metacompact by Theorem 1.2. Thus X is almost $\mathfrak{m}$-expandable by Theorem 2.2.

(c) To prove the nontrivial half, let X be boundedly $\mathfrak{m}$-θ-expandable and countably θ-expandable. Let $\mathfrak{F} = \{F_\lambda \mid \lambda \in \Lambda\}$ be a locally finite collection of subsets of X with $|\Lambda| \leqslant \mathfrak{m}$. For each positive integer n, as stated in (b), let G_n be the set of all points of X contained in at most n members of $\overline{\mathfrak{F}}$, then $\{G_n \mid n = 1, 2, \ldots\}$ is an increasing countable open covering of X. Since X is countably θ-expandable, by Theorem 1.2 there is a countable closed covering $\{H_n \mid n = 1, 2, \ldots\}$ of X such that $H_n \subset G_n$ for each n. Then, for each n, $\{F_\lambda \cap H_n \mid \lambda \in \Lambda\}$ is a bounded locally finite collection. Hence, by the bounded $\mathfrak{m}$-θ-expandability of X, for each n there exists a sequence $\{\mathfrak{V}_{n,p} = \{V_{\lambda,n,p} \mid \lambda \in \Lambda\} \mid p = 1, 2, \ldots\}$ of collections of open subsets of X such that $F_\lambda \cap H_n \subset V_{\lambda,n,p}$ for each λ and each p, and such that for each point x of X there is some p for which $\mathfrak{V}_{n,p}$ is point-finite at x. Put $W_{\lambda,n,p} = V_{\lambda,n,p} \cup (X - H_n)$ for each triple (λ, n, p), and let $\mathfrak{W}_{n,p} = \{W_{\lambda,n,p} \mid \lambda \in \Lambda\}$. Obviously $W_{\lambda,n,p}$ is open and $F_\lambda \subset W_{\lambda,n,p}$. To show that the sequence $\{\mathfrak{W}_{n,p} \mid n = 1, 2, \ldots;\ p = 1, 2, \ldots\}$ satisfies condition (2) of the definition of θ-expandability, let x be a point of X. Take out positive integers $n(x)$ and $p(x)$ such that $x \in H_{n(x)}$ and $\mathfrak{V}_{n(x),p(x)}$ is point-finite at x. Then it is obvious that $\mathfrak{W}_{n(x),p(x)}$ is point-finite at x. Hence X is $\mathfrak{m}$-θ-expandable.

3. A characterization of subparacompact spaces. In collectionwise normal spaces, it is known that paracompactness is equivalent to the following properties:

(i) metacompactness (Michael [14] or Nagami [17]),

(ii) θ-refinability (Worrell and Wicke [21]),

(iii) subparacompactness (Burke [3]).

(Since metacompactness as well as subparacompactness implies θ-refinability [4], case (ii) covers the others.)

In [10] and [11] Krajewski and Smith proved the following:

THEOREM 3.1 ([10] and [11]). (a) *A space X is paracompact if and only if it is expandable and θ-refinable.*

(b) *A space X is metacompact if and only if it is almost expandable and θ-refinable.*

Similarly, using the techniques of Michael [14], we shall prove the following:

THEOREM 3.2. *A space X is subparacompact if and only if it is subexpandable and θ-refinable.*

First we prove the following:

LEMMA 3.3. *Let X be a discretely subexpandable, and let N be the set of all positive integers. Then for every open covering $\mathfrak{U} = \{U_\lambda|\ \lambda \in \Lambda\}$ of X, there exists a family $\{\mathfrak{B}(i_1, \dots, i_n)|\ i_1, \dots, i_n \in N;\ n \in N\}$ of collections of open subsets of X satisfying the following:*

(a) *$\mathfrak{B}(i_1, \dots, i_n) = \{V_\gamma(i_1, \dots, i_n)|\ \gamma \in \Gamma_n\}$, where Γ_n denotes the set of all sets consisting of exactly n elements of Λ (i.e. $\Gamma_n = \{\gamma \subset \Lambda|\ |\gamma| = n\}$).*

(b) *$V_\gamma(i_1, \dots, i_n) \subset \bigcap_{\lambda \in \gamma} U_\lambda$, for every $\gamma \in \Gamma_n$.*

(c) *If $x \in X$ belongs to at most n members of $\mathfrak{U}$, then, for every $(i_1, \dots, i_n)$, $x \in \bigcup_{k=1}^{n} V(i_1, \dots, i_k)$, where $V(i_1, \dots, i_k) = \bigcup_{\gamma \in \Gamma_k} V_\gamma(i_1, \dots, i_k)$.*

(d) *For every $x \in X$ and every $(i_1, \dots, i_{n-1})$ there exists some $i_n \in N$ such that x is in at most one member of $\mathfrak{B}(i_1, \dots, i_{n-1}, i_n)$.*

Proof. Suppose that $\{\mathfrak{B}(i_1, \dots, i_k)|\ i_1, \dots, i_k \in N;\ k = 1, \dots, n-1\}$ has been constructed, and let us construct $\{\mathfrak{B}(i_1, \dots, i_n)|\ i_1, \dots, i_n \in N\}$. For every $(i_1, \dots, i_{n-1})$ and for every $\gamma \in \Gamma_n$, define

$$F_\gamma(i_1, \dots, i_{n-1}) = \Big(X - \bigcup_{k=1}^{n-1} V(i_1, \dots, i_k)\Big) \cap \Big(X - \bigcup_{\lambda \notin \gamma} U_\lambda\Big).$$

(In case $n = 1$, $F_{\{\lambda\}} = X - \bigcup_{\mu \neq \lambda} U_\mu$ for every $\{\lambda\} \in \Gamma_1$.) Let us prove that $\mathfrak{F} = \{F_\gamma(i_1, \dots, i_{n-1})|\ \gamma \in \Gamma_n\}$ is discrete for every $(i_1, \dots, i_{n-1})$. Let x be a point of X. If x is in at most $n-1$ members of $\mathfrak{U}$, then $x \in \bigcup_{k=1}^{n-1} V(i_1, \dots, i_k)$ by (c). Hence $\bigcup_{k=1}^{n-1} V(i_1, \dots, i_k)$ is a neighborhood of x which intersects no member of $\mathfrak{F}$. If x is in at least n members of $\mathfrak{U}$, there exists some $\gamma(x) \in \Gamma_n$ such that $x \in \bigcap_{\lambda \in \gamma(x)} U_\lambda$. Then $\bigcap_{\lambda \in \gamma(x)} U_\lambda$ is a neighborhood of x, and it intersects at most $F_{\gamma(x)}(i_1, \dots, i_{n-1})$ in $\mathfrak{F}$. Thus $\mathfrak{F}$ is a discrete collection.

Since X is discretely subexpandable, for each $(i_1, \dots, i_{n-1})$ there exists a sequence $\{\mathfrak{B}(i_1, \dots, i_{n-1}, i_n)|\ i_n \in N\}$ of open collections satisfying conditions (a), (d) and the condition $F_\gamma(i_1, \dots, i_{n-1}) \subset V_\gamma(i_1, \dots, i_{n-1}, i_n)$ for each $\gamma \in \Gamma_n$ and each $i_n \in N$. From the definition of $F_\gamma(i_1, \dots, i_{n-1})$, it is obvious that $F_\gamma(i_1, \dots, i_{n-1}) \subset \bigcap_{\lambda \in \gamma} U_\lambda$ for each γ. Hence we may suppose that (b) is satisfied. To see (c), fix $(i_1, \dots, i_n)$ and let x be a point of X contained in at most n members of $\mathfrak{U}$. If $x \notin \bigcup_{k=1}^{n-1} V(i_1, \dots, i_k)$, then $x \in F_\gamma(i_1, \dots, i_{n-1})$ for some $\gamma \in \Gamma_n$ and hence $x \in V(i_1, \dots, i_n)$. This completes the proof.

Next, using Lemma 3.3, we prove the following:

LEMMA 3.4. *Every point-finite open covering of a subexpandable space X has a σ-discrete closed refinement.*

Proof. Let $\mathfrak{U} = \{U_\lambda|\ \lambda \in \Lambda\}$ be a point-finite open covering of X. To show that $\mathfrak{U}$ has a σ-discrete closed refinement, it is enough to construct a sequence $\{\mathfrak{H}_s|\ s = 1, 2, \dots\}$ of open refinements of $\mathfrak{U}$ such that for every $x \in X$ there is some s for which x is in at most one member of $\mathfrak{H}_s$ (see the proof of the implication (b) → (d) in Theorem 2.4). Since X is discretely subexpandable, there exists a family $\{\mathfrak{B}(i_1, \dots, i_n)|\ i_1, \dots, i_n \in N;\ n \in N\}$ satisfying (a)–(d) in Lemma 3.3. Set $\Omega = \{(i_1, \dots, i_n)|\ i_1, \dots, i_n \in N\colon n \in N\}$, then it is a countable set.

For a moment, fix $\omega = (i_1, \dots, i_n) \in \Omega$. For each $k \in N$ and $\gamma \in \Gamma_k$, put $W_{\gamma,k}(\omega) = V_\gamma(i_1, \dots, i_k)$ and $W_k(\omega) = V(i_1, \dots, i_k)$; where, if $k > n$, $i_n = i_{n+1} = \dots = i_k$. Of course $W_k(\omega) = \bigcup_{\gamma \in \Gamma_k} W_{\gamma,k}(\omega)$. Remembering that $\mathfrak{U}$ is point-finite, we see that $\mathfrak{W}(\omega) = \{W_k(\omega)|\ k = 1, 2, \dots\}$ is a countable open covering of X by (c) in Lemma 3.3. Since X is countably subparacompact, $\mathfrak{W}(\omega)$ has a σ-discrete closed refinement. Hence, as is easily shown, there exists a sequence $\{\mathfrak{G}_p(\omega)|\ p = 1, 2, \dots\}$ of open coverings of X such that $\mathfrak{G}_p(\omega) = \{G_{k,p}(\omega)|\ k = 1, 2, \dots\}$, $G_{k,p}(\omega) \subset W_k(\omega)$, and, moreover, for each $x \in X$ there is some p for which x is in exactly one member of $\mathfrak{G}_p(\omega)$. For each k, each p and each $\gamma \in \Gamma_k$, define $H_{\gamma,k,p}(\omega) = G_{k,p}(\omega) \cap W_{\gamma,k}(\omega)$ and $\mathfrak{H}_p(\omega) = \{H_{\gamma,k,p}(\omega)|\ \gamma \in \Gamma_k,\ k = 1, 2, \dots\}$. Then every $\mathfrak{H}_p(\omega)$ is obviously an open covering of X, and it is a refinement of $\mathfrak{U}$ by (b) in Lemma 3.3.

Now, for every $\omega \in \Omega$ we construct $\{\mathfrak{H}_p(\omega)|\ p \in N\}$ as described above, and let us prove that for every $x \in X$ there exist some $\omega \in \Omega$ and some $p \in N$ such that x is contained in exactly one member of $\mathfrak{H}_p(\omega)$. By the point-finiteness of $\mathfrak{U}$, x is contained in at most finitely many member of $\mathfrak{U}$; assume that x is in exactly n members of $\mathfrak{U}$. By using (d) in Lemma 3.3 repeatedly, we can select positive integers $i_1, \dots, i_n$ such that x in at most one member of $\mathfrak{B}(i_1, \dots, i_k)$ for every k with $1 \leqslant k \leqslant n$. Let $\omega(x) = (i_1, \dots, i_n)$, and for this $\omega(x)$ pick up $k(x)$, $p(x) \in N$ such that $x \in G_{k(x),p(x)}(\omega(x))$ and $x \notin G_{j,p(x)}(\omega(x))$ for all $j \neq k(x)$. Then $x \in W_{\gamma,k(x)}(\omega(x))$ for some $\gamma \in \Gamma_{k(x)}$, and $x \in \bigcap_{\lambda \in \gamma} U_\lambda$ by (b) in Lemma 3.3. Hence, from the assumption for x, we have $k(x) \leqslant n$. Therefore, from the way of selecting $\omega(x)$ and $p(x)$, it follows that x belongs to exactly one member of the covering $\mathfrak{H}_{p(x)}(\omega(x))$. Thus $\{\mathfrak{H}_p(\omega)|\ \omega \in \Omega,\ p \in N\}$ is a required sequence of open refinements of $\mathfrak{U}$ (observe that $\Omega \times N$ is a countable set). This complets the proof.

Finally, using Lemma 3.4, we establish Theorem 3.2.

Proof of Theorem 3.2. If X is subparacompact, then it is subexpandable by Theorem 1.5 and it is θ-refinable by Burke [4, p. 288].

Conversely, suppose that X is subexpandable and θ-refinable. Let $\mathfrak{U}$ be an open covering of X. By Worrell and Wicke [21, p. 824], there exist a countable closed covering $\{F_n|\ n=1,2,\ldots\}$ of X and a sequence $\{\mathfrak{B}_n|\ n=1,2,\ldots\}$ of open refinements of $\mathfrak{U}$ such that for every n $\mathfrak{B}_n$ is point-finite at every point of F_n. Since every closed subset of a subexpandable space is also subexpandable as a subspace, F_n is subexpandable for every n. Hence, by Lemma 3.4, the point-finite open covering $\{V\cap F_n|\ V\in\mathfrak{B}_n\}$ of F_n has a σ-discrete closed refinement $\bigcup_{i=1}^{\infty}\mathfrak{W}_{i,n}$. Then $\bigcup_{n=1}^{\infty}\bigcup_{i=1}^{\infty}\mathfrak{W}_{i,n}$ is obviously a σ-discrete closed refinement of $\mathfrak{U}$. Therefore X is subparacompact, and this completes the proof of Theorem 3.2.

Theorem 3.2 as well as the result of Bennett and Lutzer [1, Theorem 5] covers the result of Shiraki [19, Theorem 5.9], because a space in which every closed subset is a G_δ-subset is subexpandable by Theorem 1.13. However, each of our Theorem 3.2 and the result of Bennett and Lutzer does not cover the other.

In Theorems 3.1 and 3.2, we may attach "$\mathfrak{m}$-" to paracompact, metacompact, subparacompact, θ-refinable, expandable, almost expandable and subexpandable.

4. Mapping, sum, product and subset theorems. All mapping in this section are continuous and onto.

THEOREM 4.1. *Let $f\colon X\to Y$ be a closed mapping.*

(a) *If X is almost (discretely) expandable, then so also is Y* ([11]).

(b) *If X is (discretely) θ-expandable, then so also is Y.*

(c) *If X is (discretely, boundedly) subexpandable, then so also is Y.*

Part (a) of the theorem was proved in [11], and parts (b) and (c) are proved by the same argument. (It is nearly obvious that the image of a countably (finitely) subparacompact space under a closed mapping is so also.)

THEOREM 4.2. ([10]). *Let $f\colon X\to Y$ be a quasi-perfect mapping (i.e. a closed mapping such that $f^{-1}(y)$ is countably compact for every $y\in Y$). If X is (discretely) expandable, then so also is Y.*

THEOREM 4.3. *Let $f\colon X\to Y$ be a quasi-perfect mapping.*

(a) *If Y is expandable, so also is X* ([10]).

(b) *If Y is almost expandable, so also is X* ([11]).

(c) *If Y is θ-expandable, so also is X.*

Parts (a) and (b) of the theorem were proved in [10] and [11], respectively, and (c) is proved by the same argument.

THEOREM 4.4. *Let $\{A_i|\ i=1,2,\ldots\}$ be a countable closed covering of a space X.*

(a) *If all A_i are (discretely) θ-expandable, then so also is X.*

(b) *If all A_i are (discretely, boundedly) subexpandable, then so also is X.*

Proof. (a) Let $\{F_\lambda|\ \lambda\in\Lambda\}$ is a locally finite (discrete) collection of subsets of X. For every i, by the (discrete) θ-expandability of A_i, there exists a sequence $\{\mathfrak{G}_{i,n}=\{G_{\lambda,i,n}|\ \lambda\in\Lambda\}|\ n=1,2,\ldots\}$ of collections of open subsets of A_i such that $F_\lambda\cap A_i\subset G_{\lambda,i,n}$ for every λ and every n, and such that for every $x\in X$ there is some n for which $\mathfrak{G}_{i,n}$ is point-finite at x. Define $H_{\lambda,i,n}=G_{\lambda,i,n}\cup(X-A_i)$ and $\mathfrak{H}_{i,n}=\{H_{\lambda,i,n}|\ \lambda\in\Lambda\}$, then $F_\lambda\subset H_{\lambda,i,n}$ for every triple (λ,i,n). And, since all A_i are closed in X, all $H_{\lambda,i,n}$ are open in X. Now, let $x\in X$, and choose $i(x)$ such that $x\in A_{i(x)}$. Then, for this $i(x)$, there is some $n(x)$ for which $\mathfrak{H}_{i(x),n(x)}$ is point-finite at x. Therefore X is (discretely) θ-expandable.

(b) We can prove the case of "discretely subexpandable" by the same argument as in (a). It is obvious that if all A_i are countably (finitely) subparacompact, then so also is X.

THEOREM 4.5. *Let $\mathfrak{A}$ be a locally finite closed covering of a space X.*

(a) *If all members of $\mathfrak{A}$ are (discretely) expandable, then so also is X* [10].

(b) *If all members of $\mathfrak{A}$ are almost (discretely) expandable, then so also is X* [11].

(c) *If all members of $\mathfrak{A}$ are (discretely) θ-expandable, then so also is X.*

(d) *If all members of $\mathfrak{A}$ are (discretely, boundedly) subexpandable, then so also is X.*

Parts (a) and (b) of the theorem were proved in [11], and (c) and (d) are proved by the same argument.

COROLLARY 4.6. *Let $\mathfrak{A}$ be a σ-locally finite closed covering of a space X.*

(a) *If all members of $\mathfrak{A}$ are (discretely) θ-expandable, then so also is X.*

(b) *If all members of $\mathfrak{A}$ are (discretely, boundedly) subexpandable, then so also is X.*

THEOREM 4.7. *Let X be a space and Y be a compact space.*

(a) *If X is expandable, then so also is $X\times Y$* [10].

(b) *If X is almost expandable, then so also is $X\times Y$* [11].

(c) *If X is θ-expandable, then so also is $X\times Y$.*

Proof. Since Y is compact, the projection $X\times Y\to X$ is a (quasi-) perfect mapping. Hence the theorem is an immediate consequence of Theorem 4.3.

COROLLARY 4.8. [11]. *Let X be a space and Y be a locally compact, paracompact Hausdorff space.*

(a) *If X is expandable, then so also is $X\times Y$.*

(b) *If X is almost expandable, then so also is $X\times Y$.*

COROLLARY 4.9. *Let X and Y be spaces and $\{A_i|\ i=1,2,\ldots\}$ be a countable closed covering of Y. If X is θ-expandable and all A_i are locally compact, paracompact Hausdorff spaces, then $X\times Y$ is θ-expandable.*

Proof. As is easily shown, a locally compact, paracompact Hausdorff space has a locally finite closed covering whose every member is compact. Therefore this corollary is an immediate consequence of Theorem 4.7 and Corollary 4.6.

THEOREM 4.10. *Let X be an F_σ-subset of a space Y.*
(a) *If Y is almost expandable, then so also is X.*
(b) *If Y is (discretely) θ-expandable, then so also is X.*
(c) *If Y is (discretely, boundedly) subexpandable, then so also is X.*

Proof. (a) Since X is an F_σ-subset of Y, we have a collection $\{C_n \mid n = 1, 2, \ldots\}$ of closed subsets of Y such that $X = \bigcup_{n=1}^{\infty} C_n$. Let $\mathfrak{U} = \{U_\lambda \mid \lambda \in \Lambda\}$ be an Λ-covering of X. Choose open subsets V_λ of Y such that $V_\lambda \cap X = U_\lambda$ for all λ, and let $\mathfrak{B}_n = \{V_\lambda \mid \lambda \in \Lambda\} \cup \{Y - C_n\}$ for every n. Obviously every $\mathfrak{B}_n$ is an open covering of Y. Furthermore every $\mathfrak{B}_n$ is an Λ-covering of Y; for, let $\mathfrak{F}$ be a locally finite (in X) refinement of $\mathfrak{U}$, then the collection $\{F \cap C_n \mid F \in \mathfrak{F}\} \cup \{Y - C_n\}$ is a locally finite (in Y) refinement of $\mathfrak{B}_n$. Hence, by Theorem 2.2 and by the assumption Y is almost expandable, there exists a point-finite open covering $\{W_{\lambda,n} \mid \lambda \in \Lambda\} \cup \{D_n\}$ such that $W_{\lambda,n} \subset V_\lambda$ and $D_n \subset Y - C_n$ for each λ and each n. Then it is easily proved that the collection $\{W_{\lambda,n} \cap X \mid \lambda \in \Lambda,\ n = 1, 2, \ldots\}$ is a σ-point-finite open covering of X refining $\mathfrak{U}$. Therefore, if we assume that X is countably metacompact, $\mathfrak{U}$ has a point-finite open refinement, and hence X is almost expandable by Theorem 2.2.

It remains to show that X is countably metacompact. Let $\{G_i \mid i = 1, 2, \ldots\}$ be an increasing countable open covering of X. If we put $H_{i,n} = G_i \cup (Y - C_n)$, then $\{H_{i,n} \mid i = 1, 2, \ldots\}$ is an increasing open covering of Y for each n. Since Y is almost expandable, it is naturally almost countably expandable. Hence, by Theorem 2.1, for each n there exists a closed covering $\{K_{i,n} \mid i = 1, 2, \ldots\}$ of Y such that $K_{i,n} \subset H_{i,n}$ for each i. Define $L_i = \bigcup_{k=1}^{i} \bigcup_{n=1}^{i} (K_{k,n} \cap C_n)$ for each i, then $\{L_i \mid i = 1, 2, \ldots\}$ is a closed covering of X such that $L_i \subset G_i$ for each i. Hence X is countably metacompact by Theorem 1.2.

Parts (b) and (c) are immediately proved from Theorem 4.4. (It is evident that the properties of θ-expandability, subexpandability, etc. are hereditary with respect to closed subsets.)

COROLLARY 4.11. *Let X be a generalized F_σ-subset* [13] *of a space Y.*
(a) *If Y is almost expandable, then so also is X.*
(b) *If Y is (discretely) θ-expandable, then so also is X.*
(c) *If Y is (discretely, boundedly) subexpandable, then so also is X.*

The corollary follows immediately from Theorem 4.10 and the following easily proved lemma:

LEMMA 4.12. *Let X be a subset of a space Y. Let $\mathfrak{F}$ be a collection of subsets of X which is locally finite (discrete) in X. Then there exists an open subset of Y such that $X \subset G$ and $\mathfrak{F}$ is locally finite (discrete) in G.*

THEOREM 4.13. *Let X be an F_σ-subset of a space Y. If Y is expandable and X is countably paracompact, then X is expandable.*

The theorem is proved by the same argument as in the proof of part (a) of Theorem 4.10.

THEOREM 4.14. (a) *Every subset of X is (discretely) expandable if and only if every open subset of X is so* [10].

(b) *Every subset of X is almost (discretely) expandable if and only if every open subset of X is so* [11].

(c) *Every subset of X is (discretely) θ-expandable if and only if every open subset of X is so.*

(d) *Every subset of X is (discretely, boundedly) subexpandable if and only if every open subset of X is so.*

The theorem follows immediately from Lemma 4.12.

All results in this section are true, even if we attach "$\mathfrak{m}$-" to expandable, θ-expandable, etc.

5. Examples.

EXAMPLE 5.1. A boundedly expandable space which is not countably θ-expandable.

Let X be the space constructed by Rudin [18]. This space is collectionwise normal but not countably paracompact. Hence X is boundedly expandable by Theorem 1.9, and X is not countably θ-expandable by Theorems 1.1 and 1.4.

EXAMPLE 5.2. An almost expandable space which is not subexpandable.

Let X be the space described by Burke [3, Example 4.2]. This space is metacompact but not subparacompact. Hence X is almost expandable by Theorem 1.5, and X is not subexpandable by Theorem 3.2.

EXAMPLE 5.3. A subexpandable space which is not almost expandable.

Let X be the space F of Bing [2, Example H]. This space is perfectly normal but not collectionwise normal. Hence X is subexpandable by Corollary 1.14, and X is not expandable by Theorem 1.9. Michael [14, Example 1] pointed out that every point-finite open covering of X has a locally finite open refinement, so that X is not almost expandable by Theorems 2.1 and 2.2.

EXAMPLE 5.4. An almost expandable, subexpandable space which is not expandable.

Let X be the space G of Michael [14, Example 2] which is a closed subset of the space F of Bing used in Example 5.3. This space is metacompact and perfectly normal but not collectionwise normal. Hence X is a space with the required properties.

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Some properties related to $[\mathfrak{a},\mathfrak{b}]$-compactness

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Abstract. In this paper, three properties are studied which are closely related to $[\mathfrak{a},\mathfrak{b}]$-compactness in the sense of complete accumulation points ($[\mathfrak{a},\mathfrak{b}]$-compact^r) and $[\mathfrak{a},\mathfrak{b}]$-compactness in the sense of open covers ($[\mathfrak{a},\mathfrak{b}]$-compact).

§ 1. Introduction. The concept of $[\mathfrak{a},\mathfrak{b}]$-compactness, which appears in many interesting results today, dates back to the work of P. Alexandroff and P. Urysohn in 1929. Since then many mathematicians have studied $[\mathfrak{a},\mathfrak{b}]$-compactness, and several authors have introduced natural properties which they asserted were equivalent to $[\mathfrak{a},\mathfrak{b}]$-compactness. Some of these properties, however, are not equivalent to $[\mathfrak{a},\mathfrak{b}]$-compactness, although they are closely related to it. The purpose of this paper is to study the relations among several such properties, and to give some conditions under which they are equivalent. We believe that the consideration of these properties will aid in understanding $[\mathfrak{a},\mathfrak{b}]$-compactness, in particular, $[\mathfrak{a},\mathfrak{b}]$-compact product spaces. We will also point out some errors in the literature concerning three of these properties.

Let the letters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{m}$, and $\mathfrak{n}$ denote infinite cardinal numbers with $\mathfrak{a}\leqslant\mathfrak{b}$, and let $[\mathfrak{a},\mathfrak{b}]$ stand for the set of all cardinals $\mathfrak{m}$ such that $\mathfrak{a}\leqslant\mathfrak{m}\leqslant\mathfrak{b}$. Let $|E|$ denote the cardinal number of a set E, and let $\mathfrak{m}^+$ denote the first cardinal strictly larger than $\mathfrak{m}$. The cofinality of $\mathfrak{m}$ is denoted by $\mathrm{cf}(\mathfrak{m})$. Use of the generalized continuum hypothesis will be denoted by [GCH].

DEFINITIONS. A space X is called $[\mathfrak{a},\mathfrak{b}]$-*compact*^{r} if every open cover $\mathfrak{U}$ of X such that $|\mathfrak{U}|$ is a regular cardinal in $[\mathfrak{a},\mathfrak{b}]$ has a subcover $\mathfrak{U}'\subset\mathfrak{U}$ with $|\mathfrak{U}'|<|\mathfrak{U}|$. This concept was introduced by Alexandroff and Urysohn [1]. The superscript r is a reminder of the "restriction of regularity" in the definition (see [5]). A space X is called $[\mathfrak{a},\mathfrak{b}]$-*compact* if every open cover $\mathfrak{U}$ of X with $|\mathfrak{U}|\leqslant\mathfrak{b}$, has a subcover of cardinality strictly less than $\mathfrak{a}$. This idea was introduced in 1950 by Yu. Smirnov [13]. Essentially the same property was studied independently in 1957 by I. S. Gaal [3]. The work of Gaal mentioned in this paper [3, 4] has been