Subparacompactness in Cartesian products of generalized ordered topological spaces (1)

by

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Abstract. By a GO_δ-space we understand a space with a G_δ-diagonal which can be embedded in a linearly ordered topological space. In this paper some results about subparacompactness in Cartesian products of GO_δ-spaces are proved. Among others it is shown that the product of two Lindelöf GO_δ-spaces is hereditarily subparacompact if and only if the Continuum Hypothesis holds. The Cartesian product of three GO_δ-spaces, whose closed subsets are G_δ-sets, is hereditarily subparacompact but this theorem is not true for products of four spaces. A necessary and sufficient condition for the product of four GO_δ-spaces to be subparacompact is that each pair of disjoint closed subsets of this product can be separated by G_δ-sets.

0. Introduction. The aim of this paper is to give some results about subparacompactness (2) in Cartesian products of generalized ordered topological spaces (abbreviated GO-spaces) (3).

Let us notice that many important examples connected with paracompactness in Cartesian products were obtained in the class of generalized topological spaces (see [12], [9], [10] and [11]). These examples have an additional property, namely they have G_δ-diagonals.

Our paper is organized in the following way: In section 1 we give basic definitions, section 2 contains technical lemmas and propositions which are used later, and the most important section 3 is devoted to a study of problem under what conditions the product $\boldsymbol{P}_{s\in S} X_s$ of GO-spaces with G_δ-diagonals (abbreviated GO_δ-spaces) is subparacompact or hereditarily subparacompact.

Let us recall that in [1] an example of a generalized ordered topological space X such that $X\times X$ is not subparacompact is constructed; if we assume the Continuum Hypothesis then the space X can be defined in such a way that the weight of X is $\aleph_1$. In both cases the space X has not

(1) This is a part of the authors doctoral dissertation written under the supervision of Professor R. Engelking.

(2) Subparacompactness has been extensively studied by D. Burke (in [3]).

(3) GO-spaces have been investigated by D. Lutzer (in [7]).

a G_δ-diagonal, so our restriction to generalized ordered topological spaces with G_δ-diagonals is natural in this context.

We shall often use the following theorem proved by D. Lutzer in [7] (Theorem 4.3):

(0.1) THEOREM. *Every GO_δ-space is hereditarily paracompact.*

We adopt the terminology and notation of [4].

1. Basic definitions.

(1.1) DEFINITION. A space X is *subparacompact* (*finitely subparacompact* or *countably subparacompact*) if every (finite or countable) open covering has a σ-discrete closed refinement (4).

(1.2) Remark. Let us notice that X is finitely (countably) subparacompact if and only if every finite (countable) open covering has a shrinking consisting of F_σ-sets.

(1.3) Remark. It is easy to see that finite subparacompactness is a generalization of normality, namely a space X is finitely subparacompact if and only if for each pair A, B of disjoint closed sets there exist disjoint G_δ-sets V and U such that $A \subset V$ and $B \subset U$.

(1.4) DEFINITION. By $\mathrm{GO}(K, \leqslant)$ we denote the class of GO-spaces of the form $(K, \tau, \leqslant)$, where $(K, \leqslant)$ is a fixed linearly ordered set (see [7], Definition 2.1).

(1.5) DEFINITION. We say that X is a GO_δ-*space* if X is a GO-space and has a G_δ-diagonal.

(1.6) DEFINITION. By $\mathrm{GO}_\delta(K, \leqslant)$ we mean the class of GO_δ-spaces of the form $(K, \tau, \leqslant)$, where $(K, \leqslant)$ is a fixed linearly ordered set.

(1.7) DEFINITION. By an *interval* in a generalized ordered space $(X, \tau, \leqslant)$ we mean any subset U of the form:

$$U = \{x \in X\colon a < x \leqslant b\}, \qquad U = \{x \in X\colon a \leqslant x < b\} \quad \text{or} \quad U = \{x \in X\colon a < x < b\}.$$

A one point subset of X which is not an interval will be called a *degenerate interval*.

(1.8) DEFINITION. We say that $\mathfrak{B}_a$ is a *canonical base* at a point $a \in X$, where X is a GO-space, if $\mathfrak{B}_a$ is a base at the point a consisting of intervals or degenerate intervals and satisfying the following conditions:

(1) if a is an isolated point then $\mathfrak{B}_a = \{\{a\}\}$,

(2) if a is an accumulation point and $\{x \in X\colon x \geqslant a\}$ is an open subset of X then $U \subset \{x \in X\colon x \geqslant a\}$ for every $U \in \mathfrak{B}_a$,

(4) We say that $\mathfrak{U} = \{U_s\}_{s \in S}$ is a *refinement* of $\mathfrak{V} = \{V_t\}_{t \in T}$ if for every $s \in S$ there exists a $t_s \in T$ such that $U_s \subset V_{t_s}$ and $\bigcup \mathfrak{U} = \bigcup \mathfrak{V}$.

(3) if a is an accumulation point and $\{x \in X\colon x \leqslant a\}$ is an open subset of X then $U \subset \{x \in X\colon x \leqslant a\}$ for every $U \in \mathfrak{B}_a$.

2. Technical lemmas. We start with a theorem which is an unpublished result of T. Przymusiński. The proof which we are going to present is simpler than the original one and is based on the Bing metrization theorem (see [2]): *a regular space is metrizable if and only if it is collectionwise normal and developable* (5).

(2.1) THEOREM. *If X is a GO_δ-space from the class $\mathrm{GO}(K, \leqslant)$, where K is a fixed linearly ordered set, then there exists a metrizable space M which belongs to $\mathrm{GO}(K, \leqslant)$ such that the topology of M is weaker than the topology of X.*

Proof. Let $\{G_n\}_{n=1}^\infty$ be a sequence of open subsets of $X \times X$ such that $\bigcap_{n=1}^\infty G_n = \{(x, x)\colon x \in X\}$. Let U_{nx} be, for $x \in X$ and $n \in N$ (N denotes the set of natural numbers), an arbitrary open subset of X which satisfies the following conditions:

(4) $\quad U_{nx} \in \mathfrak{B}_x$, where $\mathfrak{B}_x$ is a canonical base at the point x,

(5) $$U_{nx} \times U_{nx} \subset G_n,$$

(6) $$U_{n+1,x} \subset U_{nx}.$$

From (5) it follows that

(7) $$\bigcap_{n=1}^\infty \mathrm{St}(x, \mathfrak{U}_n) = \{x\}, \quad \text{where} \quad \mathfrak{U}_n = \{U_{nx}\}_{x \in X} \quad \text{for } n = 1, 2, \ldots$$

Let us put for $x \in X$ and $n = 1, 2 \ldots$

$$x_n^- = \begin{cases} x & \text{if } \inf \mathrm{St}(x, \mathfrak{U}_n) = x, \\ \text{an arbitrary point of } \mathrm{St}(x, \mathfrak{U}_n) \cap \{z \in X\colon z < x\} & \\ \text{if this set is not empty}. & \end{cases}$$

$$x_n^+ = \begin{cases} x & \text{if } \sup \mathrm{St}(x, \mathfrak{U}_n) = x, \\ \text{an arbitrary point of } \mathrm{St}(x, \mathfrak{U}_n) \cap \{z \in X\colon z > x\} & \\ \text{if this set is not empty}. & \end{cases}$$

($\inf A$ and $\sup A$ for $A \subset K$ are defined with respect to the linear order of K).

As a base at any point $x \in M = K$ we take the family

$$\{\{y \in M\colon x_n^- < y < x_n^+\} \cup \{x\}\}_{n=1}^\infty.$$

(5) A space X is *developable* if there exists a sequence $\{\mathfrak{U}_n\}_{n=1}^\infty$ of open coverings such that for every $x \in X$ and its open neighborhood U there is a natural number n satisfying $\mathrm{St}(x, \mathfrak{U}_n) \subset U$.

Let us put $\mathfrak{H}_n = \{\mathrm{Int}_M U\}_{U \in \mathfrak{U}_n}$ where $\mathrm{Int}_M U$ denotes the interior of U relatively to M.

From the Bing metrization theorem, Theorem 2.9 from [7] and the fact that a linearly topological space is hereditarily collectionwise normal (see [13]) it follows that the proof of Theorem (2.1) will be completed if we show that

(8) $$M \in \mathrm{GO}(K, \leqslant),$$

(9) the topology of M is weaker than the topology of X,

(10) $\{\mathfrak{H}_n\}$ is a development of M.

Ad (8). The base of M consists of intervals and degenerate intervals. Let $x \in U$, where U is an arbitrary set of the form $\{x \in K\colon a < x < b\}$, or $\{x \in K\colon a < x\}$, or else $\{x \in K\colon x < b\}$. Let $E = \{x \in K\colon x = \inf U$ if $\inf U$ exists or $x = \sup U$ if $\sup U$ exists$\}$. There exists $n(x) \in N$ such that $\mathrm{St}(x, U_{n(x)}) \subset K \setminus E$ (see (7) and (6)). It is easy to see that $\{x \in K\colon x^-_{n(x)} < x < x^+_{n(x)}\} \subset U$.

Ad (9). Let us notice that if the set

$$U = \{x \in K\colon x \leqslant a\} \quad (U = \{x \in K\colon x \geqslant a\})$$

is open in M, then it is open in X and because $M, X \in \mathrm{GO}(K, \leqslant)$, the topology of M is weaker than the topology of X.

Ad (10). From (4) it follows that if $x \notin \bigcup \{(\mathrm{Int}_K U)\}_{U \in \mathfrak{U}_n}$ then $U_{nx} = \mathrm{St}(x, \mathfrak{U}_n)$, therefore $x \in \mathrm{Int}_M U_{nx}$ and $\mathfrak{H}_n$ is the covering of M for $n = 1, 2, \ldots$

The proof of (10) will be finished if for

$$U = \{x' \in M\colon x^-_n < x' < x^+_n\} \cup \{x\}.$$

we find a natural number $m(x)$ such that $\mathrm{St}(x, \mathfrak{U}_{m(x)}) \subset U$. Let us put

$$m(x^-_n) = \begin{cases} n & \text{if} \quad x^-_n = x, \\ \text{an arbitrary natural number } m \text{ such that} & \\ \mathrm{St}(x, \mathfrak{U}_m) \subset K \setminus \{x^-_n\} & \text{if} \quad x^-_n \neq x. \end{cases}$$

$$m(x^+_n) = \begin{cases} n & \text{if} \quad x^+_n = x, \\ \text{an arbitrary natural number } m \text{ such that} & \\ \mathrm{St}(x, \mathfrak{U}_m) \subset K \setminus \{x^+_n\} & \text{if} \quad x^+_n \neq x. \end{cases}$$

The number $m(x) = \max\{m(x^+_n), m(x^-_n)\}$ has the required properties.

A metrizable GO_δ-space M which satisfies conditions of Theorem (2.1) will be called a *space associated* with the GO_δ-space X.

The following lemmas will play a fundamental role in the sequel.

(2.2) Lemma. *If $M = (K, \tau, \leqslant)$ is a metrizable* GO-*space and $\mathfrak{U}$ a family of intervals, then $\bigcup \mathfrak{U}$ is an F_σ-set in M.*

Proof. Let us put

$$V = \bigcup \{\mathrm{Int}_M U\colon U \in \mathfrak{U}\} \quad \text{and} \quad \mathfrak{U}_n = \{U \in \mathfrak{U}\colon \operatorname{diam} U > 1/n\},$$

where diam is taken with respect to the fixed metric ϱ of M, and $n = 1, 2, \ldots$ It is easy to see that $Z_n = (\bigcup \mathfrak{U}_n) \setminus V$ is closed in M, for $n = 1, 2, \ldots$, and that V is an F_σ-set in M. The proof of Lemma (2.2) follows from the equality $U = \bigcup_{n=1}^{\infty} Z_n \cup V$.

(2.3) Corollary. *If X is a GO_δ-space then X^d (the derived set) is perfect, i.e. every open subset of X^d is an F_σ-set in X^d.*

(2.4) Lemma. *If $\mathfrak{U}$ is an open family in $X_1 \times \ldots \times X_n$, where X_i is a GO_δ-space for $i = 1, 2, \ldots, n$, such that every $U \in \mathfrak{U}$ is of the form $U = U_1 \times \ldots \times U_n$, where one of the sets $U_1, \ldots, U_n$ is an interval in the corresponding X_i and others are open in M_j associated with X_j for $j \neq i$ then $\mathfrak{U}$ has a σ-discrete refinement.*

Proof. Let $\mathfrak{U}_i = \{U_1 \times \ldots \times U_n \in \mathfrak{U}\colon U_j$ is an open subset of M_j for $j \neq i\}$. The proof will be finished if we define a σ-discrete refinement $\mathfrak{E}_i$ of $\mathfrak{U}_i$ for $i = 1, 2, \ldots, n$. We can assume without loss of generality that $i = 1$. Let $\mathfrak{B}$ be a σ-discrete base of $M_2 \times M_3 \times \ldots \times M_n$ and $\mathfrak{V}$ an open refinement of $\mathfrak{U}_1$ such that if $V \in \mathfrak{V}$ then $V = V_1 \times V_2$, where V_1 is an interval in X and $V_2 \in \mathfrak{B}$. Let us put $\mathfrak{B}' = \{V_2\colon V_1 \times V_2 \in \mathfrak{V}\}$ and $\mathfrak{Z}(B) = \{V_1\colon V_1 \times B \in \mathfrak{V}\}$ for $B \in \mathfrak{B}'$. From Lemma (2.2) and Theorem (0.1) it follows that there exists a σ-discrete refinement $\mathfrak{F}(B)$ of $\mathfrak{Z}(B)$. The family $\mathfrak{E}_1 = \{F \times B\colon F \in \mathfrak{F}(B), B \in \mathfrak{B}'\}$ has the required properties.

3. Subparacompactness in the Cartesian product of GO_δ-spaces. In the present section we shall investigate the following problem: under what conditions the product $\boldsymbol{P}_{s \in S} X_s$ of GO_δ-spaces is subparacompact or hereditarily subparacompact?.

Let us recall that the Cartesian product of two GO_δ-spaces need not be paracompact (see [12], [9], [10] and [11]).

We shall consider three cases: $|S| > \aleph_0$, $|S| = \aleph_0$ and $|S| < \aleph_0$ ($|S|$ denotes cardinality of S).

Case 1. $|S| > \aleph_0$.

Theorem 3.1. *If the Cartesian product $\boldsymbol{P}_{s \in S} X_s$ is subparacompact, where X_s are GO_δ-spaces then the family $\{X_s\}_{s \in S}$ contains at most countably many non-compact spaces.*

Theorem 3.1 is equivalent to the result obtained by the author and R. Engelking (see [1], Theorem 2), which states that the Cartesian product of $\aleph_1$ copies of the discrete space of cardinality $\aleph_0$ is not subparacompact.

Case 2. $|S| = \aleph_0$.

The author does not know of any set of reasonably weak conditions which imply subparacompactness of $\overset{\infty}{\underset{n=1}{\boldsymbol{P}}} X_n$, where X_n is a GO_δ-space. For hereditary subparacompactness we have

(3.2) THEOREM. *If X_n is a GO_δ-space for $n = 1, 2, \ldots$, then the following conditions are equivalent:*

(a) *the Cartesian product $\overset{\infty}{\underset{n=1}{\boldsymbol{P}}} X_n$ is hereditarily countably subparacompact,*

(b) *the Cartesian product $\overset{\infty}{\underset{n=1}{\boldsymbol{P}}} X_n$ is hereditarily subparacompact.*

The proof of Theorem 3.2 will be preceded by a lemma.

(3.3) LEMMA. *If the Cartesian product $\overset{n}{\underset{i=1}{\boldsymbol{P}}} X_i$, where X_i is a GO_δ-space for $i = 1, 2, \ldots, n$, is hereditarily is countably subparacompact then it is hereditarily subparacompact.*

Proof. We shall prove Lemma (3.3) by induction on n. If $n = 1$ then Lemma (3.3) is obvious (see Theorem (0.1)). Let us assume that Lemma (3.3) holds for $n = k-1$. Let G be an arbitrary open subset of the hereditarily countably subparacompact Cartesian product $X_1 \times \ldots \times X_n$ and $\mathfrak{U}$ its open covering. We can assume without loss of generality that if $U \in \mathfrak{U}$ then $U = U_1 \times \ldots \times U_k$, where U_i is an interval or a degenerate interval in X_i.

Let us notice that

$$(11)\qquad |U \setminus \{[\mathrm{Int}_{M_1} U_1) \times U_2 \times \ldots \times U_k] \cup [U_1 \times (\mathrm{Int}_{M_2} U_2) \times U_3 \times \ldots \times U_k] \cup \cup \ldots \cup [U_1 \times \ldots \times U_{k-1} \times \mathrm{Int}_{M_k} U_k]\}| \leqslant 1,$$

where $U \in \mathfrak{U}$ and $U = U_1 \times \ldots \times U_k$, M_i is a space associated with X_i, for $i = 1, 2, \ldots, k$; $\mathfrak{U}$ is a covering of G hence the set

$$C = G \setminus \Big(\bigcup \{[(\mathrm{Int}_{M_1} U_1) \times U_2 \times \ldots \times U_k] \cup [U_1 \times (\mathrm{Int}_{M_2} U_2) \times U_3 \times \ldots \times U_k] \cup \cup \ldots \cup [U_1 \times \ldots \times U_{k-1} \times (\mathrm{Int}_{M_k} U_k)]\colon U_1 \times \ldots \times U_k \in \mathfrak{U}\}\Big)$$

is closed and discrete in G.

We shall show that C is a G_δ-set. To do this let us denote by $V^m(x)$, for $m = 1, 2, \ldots$ and $x \in C$, an arbitrary open neighborhood of x which satisfies the following conditions:

(12) $V^m(x) = V_1^m(x) \times \ldots \times V_k^m(x)$, where $V_i^m(x)$ is an interval or a degenerate interval in X_i for $i = 1, 2, \ldots, k$.

(13) $\operatorname{diam} V_i^m(x) < 1/m$ (diam is defined with respect to a metric in M_i) for $i = 1, 2, \ldots, k$.

(14) $\delta\{V^m(x)\colon m = 1, 2, \ldots, x \in C\} < \delta\mathfrak{U}$ (6).

We shall show that $C = \bigcap_{m=1}^{\infty} \big(\bigcup \{V^m(x)\colon x \in C\}\big)$; suppose on the contrary that there exist a point y and a sequence of sets $\{V^m(x^m)\}_{m=1}^{\infty}$, where $x^m \in C$, such that

$$(15)\qquad y \in X_1 \times \ldots \times X_k \setminus C,$$

$$(16)\qquad y \in V^m(x^m) \quad \text{for} \quad m = 1, 2, \ldots$$

We can assume without loss of generality that if $m \neq m'$ then $x^m \neq x^{m'}$ (see (15) and (16)). Let p be the largest integer such that there are infinitely many points in the set $\{x^m\}_{m=1}^{\infty}$ which have p coordinates in common with the point $y = (y_1, \ldots, y_k)$. We can assume, passing if necessary to a subsequence of $(x^m)_{m=1}^{\infty}$, that

$$y_i = x_i^m \quad \text{for} \quad i = 1, 2, \ldots, p, \quad m = 1, 2, \ldots$$

and

$$y_i \neq x_i^m \quad \text{for} \quad i = p+1, \ldots, k, \quad m = 1, 2, \ldots,$$

where $x^m = (x_1^m, \ldots, x_k^m)$. Let us notice that if $x = (x_1, \ldots, x_k) \in C$ then $\{x_i\} = V_i^m(x) \setminus \mathrm{Int}_{M_i} V_i^m(x)$, hence the number $a = \inf_{i=p+1,\ldots,k} \big(\varrho_i(y_i, X_i \setminus V_i^1(x^1))\big)$ is positive. Let m' be a natural number such that $1/m' < a$. From the definition of a, (13) and (16) it follows that $V_i^{m'}(x^{m'}) \subset V_i^1(x^1)$ for $i = p+1, \ldots, k$, so we get $x^{m'} \in V^1(x^1)$; the contradiction with (11) proves that C is a G_δ-set. Let

$$\mathfrak{G}_i = \{U_1 \times \ldots \times U_{i-1} \times (\mathrm{Int}_{M_i} U_i) \times U_{i+1} \times \ldots \times U_k\colon U_1 \times \ldots \times U_k \in \mathfrak{U}\}$$

and let $\mathfrak{B}_i = \bigcup_{n=1}^{\infty} \mathfrak{B}_{in}$ be a base in M_i, for $i = 1, 2, \ldots, k$, where $\mathfrak{B}_{in}$ is discrete in M_i. Denote by $\mathfrak{V}_i$ an arbitrary open refinement of $\mathfrak{G}_i$ such that if $V \in \mathfrak{V}_i$ then $V = V_1 \times \ldots \times V_k$, where $V_i \in \mathfrak{B}_i$ and V_j is an interval or a degenerate interval in X_j for $j \neq i$. Let

$$\mathfrak{Z}_i(B) = \{V_1 \times \ldots \times V_{i-1} \times V_{i+1} \times \ldots \times V_k\colon V_1 \times \ldots \times V_{i-1} \times B \times V_{i+1} \times \ldots \times V_k \in \mathfrak{V}_i\}$$

and

$$\mathfrak{B}_i' = \{V_i\colon V_1 \times \ldots \times V_i \times \ldots \times V_k \in \mathfrak{V}_i\}$$

for $i = 1, 2, \ldots, k$. From the inductive assumption it follows that the set $\bigcup \mathfrak{Z}_i(B)$ is subparacompact hence there exists a refinement $\mathfrak{F}_i(B)$

(6) We shall write $\delta\mathfrak{U} < \delta\mathfrak{V}$, where $\mathfrak{U}$ and $\mathfrak{V}$ are families of subsets of X, if for every $U \in \mathfrak{U}$ there exists $V \in \mathfrak{V}$ such that $U \subset V$.

of $\mathfrak{Z}_i(B)$, which is σ-discrete in $\bigcup \mathfrak{Z}_i(B)$ for $B \in B'_i$ and $i = 1, 2, \ldots, k$. Let

$$Y_{in} = \bigcup \{V_1 \times \ldots \times V_k \in \mathfrak{B}_i\colon V_i \in \mathfrak{B}'_{in}\}, \quad \text{where} \quad \mathfrak{B}'_{in} = \mathfrak{B}'_i \cap \mathfrak{B}_{in}$$

for $i = 1, 2, \ldots, k$ and $n = 1, 2, \ldots$ The set G is hereditary countably subparacompact, $G \setminus C$ is an F_σ-set and $\{Y_{in}\}_{i=1}^{k}{}_{n=1}^{\infty}$ is open, hence there exists a shrinking $\{D_{in}\}_{i=1}^{k}{}_{n=1}^{\infty}$ of $\{Y_{in}\}_{i=1}^{k}{}_{n=1}^{\infty}$ (this means that $D_{in} \subset Y_{in}$ and $\bigcup_{i=1}^{k} \bigcup_{n=1}^{\infty} D_{in} = G \setminus C$) which consists of F_σ-sets in G. Thus $D_{in} = \bigcup_{m=1}^{\infty} D_{inm}$, where D_{inm} is closed in G. Let us put

$$\mathfrak{D} = \bigcup_{i=1}^{k} \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \mathfrak{D}_{inm} \cup \{\{x\}\}_{x \in C},$$

where

$$D_{inm} = \{D_m(F, B)\colon F \in \mathfrak{F}_i(B),\ B \in \mathfrak{B}'_{in}\}$$

and

$$D_m(F, B) = \{(x_1, \ldots, x_k) \in D_{inm}\colon (x, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) \in F,\ x_i \in B\}.$$

It is easy to verify that $\mathfrak{D}$ is a σ-discrete in G refinement of $\mathfrak{U}$, hence our lemma is proved.

Proof of Theorem (3.2). The implication (b) $\Rightarrow$ (a) is obvious. Let us show that (a) $\Rightarrow$ (b). Let $\mathfrak{U}$ be an arbitrary open covering of an open subset G of $\mathop{\boldsymbol{P}}_{n=1}^{\infty} X_n$. We can assume without loss of generality that $\mathfrak{U}$ consists of sets of the form $U = \mathop{\boldsymbol{P}}_{n=1}^{\infty} B_n$, where B_n is open in X_n and the set $\{n\colon B_n \neq X_n\}$ is finite. Let

$$\mathfrak{U}_i = \{U \in \mathfrak{U}\colon U = \mathop{\boldsymbol{P}}_{i=1}^{\infty} B_i, \text{ where } B_n = X_n \text{ for } n \geqslant i\}.$$

From Lemma (3.3) it follows that there exists a refinement $\mathfrak{F}_i$ of $\mathfrak{U}_i$ which is σ-discrete in $\bigcup \mathfrak{U}_i$. From the assumption that G is countably subparacompact it follows that $\{\bigcup \mathfrak{U}_i\}_{i=1}^{\infty}$ has a shrinking $\{D_i\}_{i=1}^{\infty}$ such that $D_i = \bigcup_{n=1}^{\infty} D_{in}$ and D_{in} is closed in G. The family $\mathfrak{K} = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} \mathfrak{K}_{in}$, where $\mathfrak{K}_{in} = \{K\colon K = D_{in} \cap F, F \in \mathfrak{F}_i\}$ is a refinement of $\mathfrak{U}$, which is σ-discrete in G, thus the proof is finished.

(3.4) Corollary (D. Lutzer [8]). *The Cartesian product of $\aleph_0$ copies of the Sorgenfrey line is hereditarily subparacompact.*

Proof. In [5] it is proved that the Cartesian product of $\aleph_0$ copies of the Sorgenfrey line is perfect. To prove our Corollary it is sufficient to notice that every perfect space is hereditarily countably subparacompact (see Remark (1.2)).

Case 3. $|S| < \aleph_0$.

(3.5) Theorem. *If X_i is a GO_δ-space such that $X_i = Y_i \cup Z_i$, where Y_i is perfect, $Z_i \subset X_i \setminus X_i^d$ and $|Z_i| \leqslant \aleph_1$, for $i = 1, 2$, then $X_1 \times X_2$ is hereditarily subparacompact.*

From Theorem (3.5) it follows that the Cartesian product $X_1 \times X_2$, where X_1 and X_2 are perfect GO_δ-spaces, is hereditarily subparacompact. As Theorem (3.6) shows it is sufficient that one of the factors is perfect.

(3.6) Theorem. *If X_i is a GO_δ-space for $i = 1, 2$ and X_1 is perfect then $X_1 \times X_2$ is hereditarily subparacompact.*

Theorems (3.5) and (3.6) will be proved simultaneously; the proof will be preceded by a lemma.

(3.7) Lemma. *If X_1 and X_2 satisfy assumptions of one of the above theorems, then the set $A = [(X_1 \setminus X_1^d) \times X_2] \cup [X_1 \times (X_2 \setminus X_2^d)]$ is hereditarily subparacompact.*

Proof. If X_1 is perfect then $A = A_1 \cup A_2$, where $A_1 = (X_1 \setminus X_1^d) \times X_2$ and $A_2 = A \setminus A_1$ are hereditarily subparacompact (Theorem 0.1) F_σ-sets, hence A is hereditarily subparacompact.

Suppose now that X_1 and X_2 satisfy assumptions of Theorem 3.5. We can assume without loss of generality that $|Z_1| = |Z_2| = \aleph_1$ and Z_i is disjoint to Y_i for $i = 1, 2$ (if the set Z_i is of cardinality $\aleph_0$ then X_i is perfect (see Corollary (2.3)). Let $Z_i = \{x_\alpha^i\}_{\alpha < \omega_1}$ for $i = 1, 2$.

To finish the proof it is enough to show that the closed subset $F = A \setminus (U_1 \cup U_2)$ of A, where $U_1 = (Y_1 \setminus X_1^d) \times X_2$ and $U_2 = X_1 \times (Y_2 \setminus X_2^d)$, is hereditarily subparacompact. To do this we shall define disjoint G_δ-sets G_1 and G_2 such that $G_i \supset B_i$ for $i = 1, 2$, where $B_1 = X_1^d \times Z_2$ and $B_2 = Z_1 \times X_2^d$. Let us put

$$K_\alpha = (\{x_\alpha^1\}) \times X_2 \setminus \bigcup_{\beta < \alpha} X_1 \times \{x_\beta^2\} \quad \text{and} \quad L_\alpha = (X_1 \times \{x_\alpha^2\}) \setminus \bigcup_{\beta \leqslant \alpha} (\{x_\beta^1\} \times X_2)$$

for $\alpha < \omega_1$. It is easy to verify that the sets

$$G_1 = (\bigcup_{\alpha < \omega_1} L_\alpha) \cap F \quad \text{and} \quad G_2 = (\bigcup_{\alpha < \omega_1} K_\alpha) \cap F$$

have the required properties.

The sets $F_i = F \setminus G_i$ are hereditarily subparacompact F_σ-sets for $i = 1, 2$, therefore F is hereditarily subparacompact and this completes the proof.

Proof of Theorems (3.5) and (3.6). Let $\mathfrak{U}$ be an arbitrary open covering of an open subset G of $X_1 \times X_2$. Let $\mathfrak{U}_1$ be an open family consisting of sets of the form $U = U_1 \times U_2$, where U_i is an interval in X_i

such that

(17) $$\bigcup \mathfrak{U}_1 \supset G \cap (X_1^d \times X_2^d) \quad \text{and} \quad \delta\mathfrak{U}_1 < \delta\mathfrak{U},$$

(18) $$\text{if} \quad U_1 \times U_2 \in U \quad \text{then} \quad U_i \setminus \mathrm{Int}_{M_i} U_i \subseteq \{x_i\} \subset X_i^d \quad \text{for} \quad i = 1, 2.$$

From Lemma (2.4) it follows that there exists a σ-discrete covering $\mathfrak{E}$ of $H_1 \cup H_2$, where

$$H_1 = \bigcup \{(\mathrm{Int}_{M_1} U_1) \times U_2 \colon U_1 \times U_2 \in \mathfrak{U}_1\}$$

and

$$H_2 = \bigcup \{U_1 \times (\mathrm{Int}_M U_2) \colon U_1 \times U_2 \in \mathfrak{U}_1\},$$

such that $\delta\mathfrak{E} < \delta\mathfrak{U}_1$.

The argument used in the proof of Lemma (3.3) shows that the set $C = \bigcup \mathfrak{U}_1 \setminus (H_1 \cup H_2)$ is discrete in $\bigcup \mathfrak{U}_1$. From the properties of $\mathfrak{U}_1$ it follows that $C \subset G \cap (X_1^d \times X_2^d) \subset \bigcup \mathfrak{U}_1$, therefore C is discrete and closed in G. The family $\mathfrak{E}_1 = \mathfrak{E} \cup \{\{x\}\}_{x \in C}$ is a σ-discrete in G refinement of $\mathfrak{U}_1$ (to obtain this we used only the fact that X_i are GO_δ-spaces). The set $F = G \setminus \bigcup \mathfrak{E}_1$ is closed in G and $F \subset A$, hence from Lemma (3.7) it follows that there exists a σ-discrete closed refinement $\mathfrak{F}$ of $\mathfrak{U}|F$; therefore $\mathfrak{E}_1 \cup \mathfrak{F}$ is a σ-discrete refinement of $\mathfrak{U}$ in G.

(3.8) Corollary. *If X_i is a GO_δ-space such that $|X_i \setminus X_i^d| \leqslant \aleph_1$ for $i = 1, 2$, then $X_1 \times X_2$ is hereditarily subparacompact.*

Corollary 3.8 follows from Corollary (2.3).

(3.9) Corollary. *The space $X \times Y$ defined by E. Michael in* [9] *is hereditarily subparacompact.*

(3.10) Corollary. *The Continuum Hypothesis is equivalent to the following statement: if X_1 and X_2 are GO_δ-spaces and $X_i = Y_i \cup Z_i$, where Y_i is perfect, $Z_i \subset X_i \setminus X_i^d$ and $|Z_i| \leqslant 2^{\aleph_0}$ for $i = 1, 2$, then the space $X_1 \times X_2$ is hereditarily subparacompact.*

Corollary (3.10) follows from Theorem (3.5) and the following

(3.11) Lemma. *Under the assumption that $\aleph_1 < 2^{\aleph_0}$ there exist Lindelöf GO_δ-spaces X_1, X_2 such that the space $X_1 \times X_2$ is not hereditarily subparacompact.*

Proof. Let X be a subset of the interval $I = [0, 1]$ such that $|X| = |I \setminus X| = 2^{\aleph_0}$ and both X and $I \setminus X$ considered as subspaces of I do not contain uncountable compact subsets (see [6], Theorem 1, p. 514). Let L be an arbitrary subset of I such that $|I \setminus L| = \aleph_1$.

Let us put $X = I_X$ and $X_2 = I_L$ (see [4], Example 5.1.2). It is easy to verify that X_1 and X_2 are Lindelöf GO_δ-spaces.

We shall show that $A = (I_X \setminus X) \times I_L \cup I_X \times (I_L \setminus L)$ is not subparacompact. To do this, it is enough to prove that $B_1 = X \times (I_L \setminus L)$ and $B_2 = (I_X \setminus X) \times L$ cannot be separated by G_δ-sets (see Remark (1.3)).

Suppose that $B_i \subset U_i$, for $i = 1, 2$, and U_i are disjoint G_δ-sets. From the properties of X and the inequality $2^{\aleph_0} > \aleph_1$ it follows that

(19) $$|[(I_X \times \{x\}) \setminus U_1]| \leqslant \aleph_0 \quad \text{for} \quad x \in I_L \setminus L$$

and

(20) $$|[(\{x\} \times I_L) \setminus U_2]| \leqslant \aleph_0 \quad \text{for} \quad x \in I_X \setminus X.$$

Let us put

$$B = \bigcap_{x \in (I \setminus L)} p[(I_X \times \{x\}) \cap U_1],$$

where p is the projection from $X_1 \times X_2$ onto the first factor. From (19) and the inequality $2^{\aleph_0} > \aleph_1$ it follows that $B \setminus X \neq \emptyset$. Let x be an arbitrary point of $B \setminus X$. It is easy to verify that $\{x\} \times (I_L \setminus L) \subset U_1$ but this contradicts (20), because $U_1 \cap U_2 = \emptyset$ (7).

(3.12) Corollary. *The Continuum Hypothesis is equivalent to the following statement: if X_1 and X_2 are Lindelöf GO_δ-spaces then the space $X_1 \times X_2$ is hereditarily subparacompact.*

To prove this fact it is enough to notice that $|M_i| \leqslant 2^{\aleph_0}$, where M_i is associated with X_i for $i = 1, 2$.

In Example (3.3) below we show that Theorem (3.5) does not hold for the Cartesian product of three spaces.

(3.3) Example. We shall define perfect GO_δ-spaces $X_1, X_2 \in \mathrm{GO}_\delta(I)$ such that the subspace $\Delta = \{(x, x) \colon x \in I$ of $X_1 \times X_2$ will be homeomorphic to I_X (see Lemma (3.11)).

Let x be an arbitrary point of I. If $x \in X$ then the base at x in X_1 and in X_2 is the same as in I, if $x \in X$ then the base at x in X_1 consists of sets of the form $U = \{y \in I \colon x \leqslant y < a\}$ and the base at x in X_2 of sets of the form $U = \{y \in I \colon a < y \leqslant x\}$. The space $X_1 \times X_2$ has the required properties (see Lemma (2.3)).

If we assume that $2^{\aleph_0} > \aleph_1$ then $X_1 \times X_2 \times I_L$ is not hereditarily subparacompact because $X_1 \times X_2 \times I_L \supset I_X \times I_L$.

In the case of three spaces we have the following

(3.4) Theorem. *If X_i is a perfect GO_δ-space for $i = 1, 2, 3$, then the space $X_1 \times X_2 \times X_3$ is hereditarily subparacompact.*

Proof. Let G be an arbitrary open subset of the Cartesian product $X_1 \times X_2 \times X_3$ and $\mathfrak{U}$ its open covering. Let $\mathfrak{U}_1$ be an open family consisting of sets of the form $U = U_1 \times U_2 \times U_3$, where U_i is an interval in X_i, for $i = 1, 2, 3$, such that

(21) $$\bigcup \mathfrak{U}_1 \supset G \cap (X_1^d \times X_2^d \times X_3^d) \quad \text{and} \quad \delta\mathfrak{U}_1 < \delta\mathfrak{U},$$

(22) $$\text{if} \quad U_1 \times U_2 \times U_3 \in \mathfrak{U}_1, \quad \text{then} \quad U_i \setminus (\mathrm{Int}_{M_i} U_i) \subseteq \{x_i\} \subset X_i^d,$$

where M_i is associated with X_i for $i = 1, 2, 3$.

(7) R. Pol proved that the space $X_1 \times X_2$ is subparacompact.

From Lemma (2.4) it follows that there exists a σ-discrete covering $\mathfrak{E}$ of H, where

$$H = \bigcup \{[U_1 \times (\mathrm{Int}_{M_2} U_2) \times (\mathrm{Int}_{M_3} U_3)] \cup [(\mathrm{Int}_{M_1} U_1) \times U_2 \times (\mathrm{Int}_{M_3} U_3)] \cup \\ \cup [(\mathrm{Int}_{M_1} U_1) \times (\mathrm{Int}_{M_2} U_2) \times U_3]\colon\ U_1 \times U_2 \times U_3 \in \mathfrak{U}_1\}$$

such that $\delta\mathfrak{E} < \delta\mathfrak{U}_1$.

Let us notice that the set

$$C = \bigcup \mathfrak{U}_1 \setminus (\bigcup \{[(\mathrm{Int}_{M_1} U_1) \times U_2 \times U_3] \cup [U_1 \times (\mathrm{Int}_{M_2} U_2) \times U_3] \cup \\ \cup [U_1 \times U_2 \times (\mathrm{Int}_{M_3} U_3)]\colon\ U_1 \times U_2 \times U_3 \in \mathfrak{U}_1\})$$

is discrete and closed in $\bigcup \mathfrak{U}_1$, similarily as the set C from the proof of Lemma (3.3). From the properties of $\mathfrak{U}_1$ it follows that $C \subset G \cap \cap (X_1^d \times X_2^d \times X_3^d) \subset \bigcup \mathfrak{U}_1$, therefore C is discrete and closed in G.

Let us put

$$\mathfrak{D}_1 = \{V_1(U) \times D_2(U) \times D_3(U) \setminus H\colon\ U \in \mathfrak{U}_1\},$$

$$\mathfrak{D}_2 = \{D_1(U) \times V_2(U) \times D_3(U) \setminus H\colon\ U \in \mathfrak{U}_1\},$$

and

$$\mathfrak{D}_3 = \{D_1(U) \times D_2(U) \times V_3(U) \setminus H\colon\ U \in \mathfrak{U}_1\},$$

where $V_i(U) = \mathrm{Int}_{M_i} U_i$ and $D_i(U) = U_i \setminus \mathrm{Int}_{M_i}(U_i)$ for $U_1 \times U_2 \times U_3 = U$. Notice that

$$|D_i(U)| \leqslant 1 \quad \text{and} \quad U_1 \times U_2 \times U_3 \setminus (C \cup H) \subset \bigcup_{i=1}^{3} \bigcup \mathfrak{D}_i,$$

therefore

$$\bigcup \mathfrak{U}_1 \setminus (C \cup H) = \bigcup_{i=1}^{3} \bigcup \mathfrak{D}_i.$$

Let R_1 be the equivalence relation in $\mathfrak{D}_1$ defined by the conditions

$$(D_1 \times D_2 \times D_3) R_1 (D_1' \times D_2' \times D_3') \quad \text{if} \quad D_2 = D_2' \text{ and } D_3 = D_3'.$$

Relations R_2 in $\mathfrak{D}_2$ and R_3 in $\mathfrak{D}_3$ are defined in analogous way.

Let us put

$$\mathfrak{F}_i = \{F_i(D)\colon\ D \in \mathfrak{D}_i\}, \quad \text{where} \quad F_i(D) = \bigcup \{D'\colon\ D' R_i D\} \quad \text{for} \quad i = 1, 2, 3.$$

Let

$$\mathfrak{F}_i^n = \{F_i^n(D)\}_{F_i(D) \in \mathfrak{F}_i} \quad \text{for} \quad i = 1, 2, 3 \text{ and } n = 1, 2, \ldots,$$

where $F_i^n(D) = \{(x_1, x_2, x_3) \in F_i(D)$ there exists $U_1 \times U_2 \times U_3 \in \mathfrak{U}_1$ such that $\operatorname{diam} U_j > 1/n$, $\{x_j\} = U_j \setminus \mathrm{Int}_{M_j} U_j$ for $j \neq i$, and $x_i \in \mathrm{Int}_{M_i} U_i\}$.

We shall show that the family $\mathfrak{F}_i^n$, for $i = 1, 2, 3$ and $n = 1, 2, \ldots$ has the following properties:

(23) $F_i^n(D)$ is an F_σ-set in G.

(24) $\mathfrak{F}_i^n$ is discrete in $\bigcup \mathfrak{F}_i$.

(25) $\mathfrak{F}_i^n$ is discrete in $\bigcup_{i=1}^{3} \bigcup \mathfrak{F}_i$.

(26) $\bigcup \mathfrak{F}_i^n$ is an F_σ-set in $\bigcup \mathfrak{U}_1$.

(27) If $F_i^n(D) \in \mathfrak{F}_i^n$ then there exists a σ-discrete covering $\mathfrak{U}_i^n(D)$ of $\mathfrak{F}_i^n(D)$ such that $\delta\mathfrak{U}_i^n(D) < \delta\mathfrak{U}$.

Ad (23). Any set $F_i^n(D) \in \mathfrak{F}_i^n$ is an F_σ-set in G because it is an intersection of the F_σ-set $P = \bigcup \{P_1 \times P_2 \times P_3$ there exists $U_1 \times U_2 \times U_3 \in \mathfrak{U}_1$ such that $\operatorname{diam} U_j > 1/n$, $P_j = D_j = U_j \setminus \mathrm{Int}_{M_j}(U_j) = \{x_j\}$ for $j \neq i$ and $P_i = \mathrm{Int}_{M_i} U_i$, where $D = D_1 \times D_2 \times D_3\}$, and the set $G \setminus H$, which is closed in G.

Ad (24). It is easy to verify that we do not change a set $D \in \mathfrak{D}_i$ if we put in its definition $D_j(U) = U_j$ instead of $U_j \setminus \mathrm{Int}_{M_j}(U_j)$ for $j \neq i$, therefore $F_i(D) \in \mathfrak{F}_i$ is open in $\bigcup_{i=1}^{3} \bigcup \mathfrak{D}_i$. The family $\mathfrak{F}_i$ consists of pairwise disjoint sets and $\delta\mathfrak{F}_i^n < \delta\mathfrak{F}_i$, hence $\mathfrak{F}_i^n$ is discrete in $\bigcup \mathfrak{F}_i$.

Ad (25). From (24) it follows that it is sufficient to show that for an arbitrary point $x \in \bigcup \mathfrak{F}_j$ and $j \neq i$ there exists an open neighbourhood of x which intersects at most one element of $\mathfrak{F}_i^n$. We can assume without loss of generality that $i = 1$ and $j = 2$. Let us notice that the family $\mathfrak{F}_2$ is open in $\bigcup_{n=1}^{3} \bigcup \mathfrak{F}_n$ and if $F(D) \in \mathfrak{F}_2$ then $|p_k(F(D))| = 1$, for $k = 1, 3$, where p_k is the projection of $X_1 \times X_2 \times X_3$ onto X_k; therefore it is sufficient to show that the set $\{x_1\} \times X_2 \times \{x_3\} \cap G \cap (\bigcup \mathfrak{F}_1^n)$, where $x = (x_1, x_2, x_3)$, is closed in G, and this can easily be verified.

Ad (26). From (23) and (25) it follows, that $\bigcup \mathfrak{F}_i^n$ is an F_σ-set in $\bigcup_{i=1}^{3} \bigcup \mathfrak{F}_i$. It is easy to see that

$$\bigcup_{i=1}^{3} \bigcup \mathfrak{F}_i = \bigcup_{i=1}^{3} \bigcup \mathfrak{D}_i = \bigcup \mathfrak{U}_1 \setminus (C \cup H) \tag{28}$$

and $C \cup H$ is a G_δ-set, hence $\bigcup \mathfrak{F}_i^n$ is an F_σ-set in $\bigcup \mathfrak{U}_1$.

Ad (27). This is an obvious consequence of the definition of $\mathfrak{F}_i^n$ and metrizability of M_i.

From (25), (26), (27), the equality

$$\bigcup_{i=1}^{3} \bigcup \mathfrak{F}_i = \bigcup_{i=1}^{3} \bigcup_{n=1}^{\infty} \bigcup \mathfrak{F}_i^n, \tag{29}$$

(28), and the properties of $\mathfrak{E}$ and C it follows that there exists a σ-discrete in $\bigcup \mathfrak{U}_1$ refinement $\mathfrak{E}_1$ of $\mathfrak{U}_1$. Notice that to obtain this refinement we used only the assumption that X_i is a GO_δ-space for $i = 1, 2, 3$.

Let us put $\mathfrak{Z} = \mathfrak{E}_1|(X_1^d \times X_2^d \times X_3^d) \cap G$. From (21) it follows that $\mathfrak{Z}$ is a σ-discrete, in G, covering of $G \cap (X_1^d \times X_2^d \times X_3^d)$ and $\delta\mathfrak{Z} < \delta\mathfrak{U}$.

We shall show that $T = G \setminus (X_1^d \times X_2^d \times X_3^d)$ is a subparacompact F_σ-set in G. To do this let us put

$$A_1 = [(X_1 \setminus X_1^d) \times X_2 \times X_3] \cap G\,,$$

$$A_2 = [X_1 \times (X_2 \setminus X_2^d) \times X_3] \cap G$$

and

$$A_3 = [X_1 \times X_2 \times (X_3 \setminus X_3^d)] \cap G\,.$$

The space X_i is perfect, hence A_i is an F_σ-set in G and from Theorem (3.5) and the equality $T = A_1 \cup A_2 \cup A_3$ it follows that T is a subparacompact F_σ-set in G.

Let $\mathfrak{T}$ be a σ-discrete in G covering of T such that $\delta\mathfrak{T} < \delta\mathfrak{U}$. The family $\mathfrak{T} \cup \mathfrak{Z}$ is a σ-discrete refinement of $\mathfrak{U}$ and the proof is finished.

By using Corollary (2.3) we infer

(3.5) Corollary. *If X_i is a GO_δ-space without isolated points, for $i = 1, 2, 3$, then the space $X_1 \times X_2 \times X_3$ is hereditarily subparacompact.*

Applying the method from Example (3.3), we easily notice that Theorem 3.4 does not hold for four spaces.

(3.16) Theorem. *If X_i is a GO_δ-space, for $i = 1, 2, 3, 4$, then the following conditions are equivalent:*

(a) *the Cartesian product $X_1 \times X_2 \times X_3 \times X_4$ is finitely subparacompact (hereditarily finitely subparacompact),*

(b) *the Cartesian product $X_1 \times X_2 \times X_3 \times X_4$ is subparacompact (hereditarily subparacompact).*

The proof of Theorem (3.6) will be preceded by two lemmas.

(3.17) Lemma. *If X_i is a GO_δ-space, for $i = 1, 2$, and the Cartesian product $X_1 \times X_2$ is finitely subparacompact, then $X_1 \times X_2$ is subparacompact.*

Proof. Let $\mathfrak{U}$ be an arbitrary open covering of $X_1 \times X_2$. We obtain, in the same way as in the proof of Theorem (3.5) and Theorem (3.6), a σ-discrete family $\mathfrak{E}_1$ such that $\delta\mathfrak{E}_1 < \delta\mathfrak{U}$, $\bigcup \mathfrak{E}_1 \supset X_1^d \times X_2^d$ and the set $\bigcup \mathfrak{E}_1$ is open in $X_1 \times X_2$. From the assumption that $X_1 \times X_2$ is finitely subparacompact it follows that there exist F_σ-sets F_1, F_2 and F_3 satisfying the following conditions

$$F_1 \cup F_2 \cup F_3 = X_1 \times X_2\,, \tag{30}$$

$$F_1 \subset (X_1 \setminus X_1^d) \times X_2\,, \tag{31}$$

$$F_2 \subset X_1 \times (X_2 \setminus X_2^d)\,, \tag{32}$$

$$F_3 \subset \bigcup \mathfrak{E}_1\,. \tag{33}$$

The sets F_1 and F_2 are subparacompact and the family $\mathfrak{E}_1|F_3$ is σ-discrete, hence $X_1 \times X_2$ is subparacompact.

(3.18) Lemma. *If X_i is a GO_δ-space, for $i = 1, 2, 3$, and the Cartesian product $X_1 \times X_2 \times X_3$ is finitely subparacompact then $X_1 \times X_2 \times X_3$ is subparacompact.*

Proof. Let $\mathfrak{U}$ be an arbitrary open covering of $X_1 \times X_2 \times X_3$. In the same way as in the proof of Theorem (3.4) we obtain a family $\mathfrak{E}_1$ which is σ-discrete in $\bigcup \mathfrak{E}_1$ such that $\delta\mathfrak{E}_1 < \delta\mathfrak{U}$, $\bigcup \mathfrak{E}_1 \supset X_1^d \times X_2^d \times X_3^d$ and the set $\bigcup \mathfrak{E}_1$ is open in $X_1 \times X_2 \times X_3$. From the assumption that $X_1 \times X_2 \times X_3$ is finitely subparacompact it follows that there exists F_σ-sets F_1, F_2, F_3 and F_4 which satisfy the following conditions:

$$F_1 \cup F_2 \cup F_3 \cup F_4 = X_1 \times X_2 \times X_3\,, \tag{34}$$

$$F_1 \subset (X_1 \setminus X_1^d) \times X_2 \times X_3\,, \tag{35}$$

$$F_2 \subset X_1 \times (X_2 \setminus X_2^d) \times X_3\,, \tag{36}$$

$$F_3 \subset X_1 \times X_2 \times (X_3 \setminus X_3^d)\,. \tag{37}$$

$$F_4 \subset \bigcup \mathfrak{E}_1\,. \tag{38}$$

The sets F_1, F_2, F_3 are subparacompact (see Lemma (3.7)) and the family $\mathfrak{E}_1|F_{4n}$, where F_{4n} are closed subsets of $X_1 \times X_2 \times X_3$ such that $\bigcup_{n=1}^{\infty} F_{4n} = F_4$, is discrete, hence $X_1 \times X_2 \times X_3$ is subparacompact.

Proof of Theorem (3.6). The implication (b) $\Rightarrow$ (a) is obvious; to prove Theorem (3.6) it is sufficient to show (a) $\Rightarrow$ (b).

Let $\mathfrak{U}$ be an arbitrary open covering of $X = X_1 \times X_2 \times X_3 \times X_4$. Let $\mathfrak{U}_1$ be an open family consisting of sets of the form $U = U_1 \times U_2 \times U_3 \times U_4$, where U_i is an interval in X_i, for $i = 1, 2, 3, 4$, such that

$$\bigcup \mathfrak{U}_1 \supset X_1^d \times X_2^d \times X_3^d \times X_4^d \quad \text{and} \quad \delta\mathfrak{U}_1 < \delta\mathfrak{U}\,, \tag{39}$$

$$\text{if} \quad U_1 \times U_2 \times U_3 \times U_4 \in \mathfrak{U}_1\,, \quad \text{then} \quad U_i \setminus \mathrm{Int}_{M_i} U_i \subset \{x_i\} \subseteq X_i^d\,. \tag{40}$$

Let $\mathfrak{T}_i = \{T \subset T_4\colon |T| = i\}$, where $i = 1, 2, 3, 4$ and $T_4 = \{1, 2, 3, 4\}$. Let us put for $T \in \mathfrak{T}_1$:

$$\mathfrak{K}_1(T) = \{D_1 \times D_2 \times D_3 \times D_4\colon D_t = U_t \text{ for } t \in T \text{ and } D_m = \mathrm{Int}_{M_m} U_m \text{ for } m \in T_4 \setminus T, \text{ where } U_1 \times U_2 \times U_3 \times U_4 \in \mathfrak{U}_1\}\,.$$

Suppose we have already defined $\mathfrak{K}_j(T)$ for $j < i$ and $T \in \mathfrak{T}_j$; let us put

for $T \in \mathfrak{T}_i$

$$\mathfrak{K}_i(T) = \{D_1 \times D_2 \times D_3 \times D_4 \setminus \bigcup_{j<i} \bigcup_{T' \in \mathfrak{T}_j} \bigcup \mathfrak{K}_j(T'):\ D_t = U_t \setminus \mathrm{Int}_{M_t} U_t \text{ for } t \in T$$

$$\text{and } D_m = \mathrm{Int}_{M_m} U_m \text{ for } m \in T_4 \setminus T, \text{ where } U_1 \times U_2 \times U_3 \times U_4 \in \mathfrak{U}_1\}.$$

The argument as in the proof of Lemma (3.3) shows that $\mathfrak{K}_4(T_4)$ is discrete in $\bigcup \mathfrak{U}_1$. From (39) and (40) it follows that $\bigcup \mathfrak{K}_4(T_4) \subset X_1^d \times X_2^d \times X_3^d \times X_4^d \subset \bigcup \mathfrak{U}_1$, hence $\mathfrak{K}_4(T_4)$ is discrete in $X_1 \times X_2 \times X_3 \times X_4$.

Applying Lemma (2.4) we find a covering $\mathfrak{E}$ of $\bigcup_{T \in \mathfrak{T}_1} \bigcup \mathfrak{K}_1(T)$ which is σ-discrete in $X_1 \times X_2 \times X_3 \times X_4$ and such that $\delta\mathfrak{E} < \delta\mathfrak{U}$.

We define the equivalence relation $R(i, T)$ in $\mathfrak{K}_i(T)$ where $T \in \mathfrak{T}_i$ and $i = 2, 3$, assuming

$$K\,R(i, T)\,K' \quad \text{if} \quad p(K') = p_j(K) \quad \text{for} \quad j \in T,$$

where p_j is the projection of $X_1 \times X_2 \times X_3 \times X_4$ onto X_j. Let $\mathfrak{F}(i, T) = \{F(K):\ K \in \mathfrak{K}_i(T)\}$ for $T \in \mathfrak{T}_i$, where $F(K) = \bigcup \{K':\ K'\,R(i, T)\,K\}$.

We shall show that the family $\mathfrak{F}(i, T)$, for $T \in \mathfrak{T}_i$ and $i = 2, 3$, has the following properties:

(41) elements of $\mathfrak{F}(i, T)$ are open in $\bigcup_{T \in \mathfrak{T}_i} \bigcup \mathfrak{F}(i, T)$,

(42) $\mathfrak{F}(i, T)$ is discrete in the $\bigcup \mathfrak{F}(i, T)$,

(43) $\bigcup \mathfrak{F}(3, T)$ is a G_δ-set in $X_1 \times X_2 \times X_3 \times X_4$,

(44) $\bigcup_{T \in \mathfrak{T}_i} \bigcup \mathfrak{F}(i, T)$ is an F_σ-set in $\bigcup \mathfrak{U}_1$.

(45) if $F(K) \in \mathfrak{F}(i, T)$ then there exists a σ-discrete covering $\mathfrak{U}(F(K))$ of $F(K)$ such that $\delta\mathfrak{U}(F(K)) < \delta\mathfrak{U}_1$.

Ad (41). It is easy to see that we do not change a set $F(K) \in \mathfrak{F}(i, T)$ if we put in its definition $D_t = U_t$ instead of $U_t \setminus \mathrm{Int}_{M_t}(U_t)$ for $t \in T$, hence $F(K) \in \mathfrak{F}(i, T)$ is open in $\bigcup_{T \in \mathfrak{T}_i} \bigcup \mathfrak{F}(i, T)$ as the union of open subsets of $\bigcup_{T \in \mathfrak{T}_i} \bigcup \mathfrak{F}(i, T)$.

Ad (42). The elements of $\mathfrak{F}(i, T)$ are pairwise disjoint hence from (41) it follows that $\mathfrak{F}(i, T)$ is discrete in $\bigcup \mathfrak{F}(i, T)$.

Ad (43). From (39) and (40), the definition of $\mathfrak{F}(3, T)$, and Lemma (2.2) it follows that the elements of $\mathfrak{F}(3, T)$ are of the form

(46) $F(K) = P_1 \times P_2 \times P_3 \times P_4$, where $|P_t| = 1$ for $t \in T$ and P_m is a G_δ-set in M_m for $m \in T_4 \setminus T$.

We can assume without loss of generality that $T = \{1, 2, 3\}$ and $m = T_4 \setminus T = \{4\}$.

Let $V^n(x)$ be an arbitrary neighborhood of $x = (x_1, x_2, x_3, x_4) \in \bigcup \mathfrak{F}(3, T)$ satisfying the following conditions for $n = 1, 2, 3, \ldots$

(47) $V^n(x) = V_1^n(x) \times V_2^n(x) \times V_3^n(x) \times V_4^n(x)$, where $V_i^n(x)$ is an interval in X_i for $i = 1, 2, 3$ and $V_4^n(x)$ is an open subset of M_4,

(48) $$\operatorname{diam} V_i^n(x) < 1/n,$$

(49) $$\delta\{V^n(x):\ x \in \bigcup \mathfrak{F}(3, T),\ n = 1, 2, \ldots\} < \delta\mathfrak{U}_1,$$

(50) if $x \in F(K) = P_1 \times P_2 \times P_3 \times P_4 \in \mathfrak{F}(3, T)$, then $V_4^n(x) \subset G_n$, where G_n is open subset of M_4 such that $\bigcap_{n=1}^{\infty} G_n = P_4$.

Let us assume that the equality

$$\mathfrak{F}(3, T) = \bigcap_{n=1}^{\infty} \Big(\bigcup \{V^n(x):\ x \in \bigcup \mathfrak{F}(3, T)\}\Big)$$

does not hold. Then there exist a point y and a sequence of sets $(V^n(x^n))_{n=1}^{\infty}$, where $x^n \in \bigcup \mathfrak{F}(3, T)$, such that

(51) $$y \in X_1 \times X_2 \times X_3 \times X_4 \setminus \Big(\bigcup \mathfrak{F}(3, T)\Big),$$

(52) $$y \in V^n(x^n) \quad \text{for} \quad n = 1, 2, \ldots$$

From (51), (52), (46) and from the properties of the sets $V^n(x)$ it follows that there exists an infinite subset of $\{x^n\}_{n=1}^{\infty}$, such that its intersection with each element of $\mathfrak{F}(3, T)$ is of cardinality $\leqslant 1$. We can assume without loss of generality that

(53) $$|F(K) \cap \{x^1, x^2, \ldots\}| \leqslant 1 \quad \text{for} \quad F(K) \in \mathfrak{F}(3, T) \text{ and } x^n \neq x^m \text{ if } n \neq m.$$

Let k be the maximal integer such that there exist infinitely many points in $\{x^n\}_{n=1}^{\infty}$ which have k coordinates among the first three coordinates in common with the point $y = (y_1, y_2, y_3, y_4)$. We can assume, passing if necessary to a subsequence of $(x^n)_{n=1}^{\infty}$, that for $n = 1, 2, \ldots$ $y_i = x_i^n$ if $i \leqslant k$ and $y_i \neq x_i^n$ if $k < i \leqslant 4$.

Let us notice that if $x = (x_1, x_2, x_3, x_4) \in \bigcup \mathfrak{F}(3, T)$, where $T = \{1, 2, 3\}$, then $\{x_i\} = V_i^n(x) \setminus \mathrm{Int}_{M_i} V_i^n(x)$ for $i = 1, 2, 3$, so $a = \inf_{k<i} \varrho_i\big(y_i, X_i \setminus V_i^1(x^1)\big) > 0$. Let n be a natural number such that $1/n < a$. From the definition of a, (48) and (52) it follows that $V_i^n(x^n) \subset V_i^1(x^1)$ for $k < i \leqslant 4$; as $x_i^n = x_i^1$ for $i \leqslant k$, hence $x^n \in V^1(x^1)$, but it is impossible because

$$V^1(x^1) \cap \{x^n:\ n = 2, 3, \ldots\} = \emptyset,$$

therefore $\bigcup \mathfrak{F}(3, T)$ is a G_δ-set.

Ad (44). It is easy to verify that

$$\bigcup_{T \in \mathfrak{T}_2} \bigcup \mathfrak{F}(2, T) = \Big(\bigcup \mathfrak{U}_1\Big) \setminus \Big(\bigcup_{T \in \mathfrak{T}_1} \bigcup \mathfrak{K}(1, T) \cup \bigcup_{T \in \mathfrak{T}_3} \bigcup \mathfrak{F}(3, T) \cup \bigcup \mathfrak{K}(4, T_4)\Big),$$

and

$$\bigcup_{T\in\mathfrak{T}_3}\bigcup\mathfrak{F}(3,T)=\bigcup\mathfrak{U}_1\setminus\Big(\bigcup_{i=1}^{2}\bigcup_{T\in\mathfrak{T}_i}\bigcup\mathfrak{K}(i,T)\cup\bigcup\mathfrak{K}(4,T_4)\Big).$$

Let us notice that the sets $\bigcup_{T\in\mathfrak{T}_1}\bigcup\mathfrak{K}(1,T)$ and $\bigcup_{i=1}^{2}\bigcup_{T\in\mathfrak{T}_i}\bigcup\mathfrak{K}(i,T)$ are open, and $\bigcup\mathfrak{F}(3,T)$ and $\mathfrak{K}(4,T_4)$ are G_δ-sets; therefore (44) holds.

Ad (45). This is an obvious consequence of metrizability of M_i and the definition of $F(K)$.

The Cartesian product $X_1\times X_2\times X_3\times X_4$ is finitely subparacompact, therefore there exist F_σ-sets F_1, F_2, F_3, F_4, and F_5 such that

(54) $$\bigcup_{i=1}^{5}F_i=X_1\times X_2\times X_3\times X_4\,,$$

(55) $$F_1\subset(X_1\setminus X_1^d)\times X_2\times X_3\times X_4\,,$$

(56) $$F_2\subset X_1\times(X_2\setminus X_2^d)\times X_3\times X_4\,,$$

(57) $$F_3\subset X_1\times X_2\times(X_3\setminus X_3^d)\times X_4\,,$$

(58) $$F_4\subset X_1\times X_2\times X_3\times(X_4\setminus X_4^d)\,,$$

(59) $$F_5\subset\bigcup\mathfrak{U}_1\,.$$

From the fact that finite subparacompactness is hereditary with respect to F_σ-sets and from (44), (41), (42) and (45) it follows that there exists a σ-discrete family $\mathfrak{E}(i,T)$ for $T\in\mathfrak{T}_i$ such that

$$\bigcup_{T\in\mathfrak{T}_i}\bigcup\mathfrak{E}(i,T)=F_5\cap\Big(\bigcup_{T\in\mathfrak{T}_i}\bigcup\mathfrak{F}(i,T)\Big)\quad\text{and}\quad\delta\mathfrak{E}(i,T)<\delta\mathfrak{U}_1\quad\text{for}\quad i=2,3\,.$$

Let

$$\mathfrak{E}_1=\bigcup_{T\in\mathfrak{T}_2}\mathfrak{E}(2,T)\cup\bigcup_{T\in\mathfrak{T}_3}\mathfrak{E}(3,T)\cup\mathfrak{E}\cup\{x\}_{x\in K_4(4,T_4)}\,.$$

It is easy to see that E_1 is a σ-discrete refinement of $\mathfrak{U}_1|F_5$ and this completes the proof because F_i for $i\leqslant 4$ is subparacompact and an F_σ-set (see Lemma 3.18).

The proof of the part in brackets is similar; it is sufficient to replace the product $X_1\times X_2\times X_3\times X_4$ by its arbitrary open subset.

The author does not know whether Theorem (3.16) holds for the product of five spaces.

(3.19) THEOREM. *If the Cartesian product $X_1\times\ldots\times X_n$ of GO_δ-spaces is collectionwise normal then it is subparacompact, and a fortiori paracompact.*

Proof. Let $\mathfrak{U}$ be an arbitrary open covering of $X=X_1\times\ldots\times X_n$. We can assume without loss of generality that $\mathfrak{U}$ consists of sets of the form $U=U_1\times\ldots\times U_n$, where U_i is an interval or a degenerate interval in X_i, for $i=1,2,\ldots,n$.

Let $\mathfrak{T}_i=\{T\subset T_n\colon |T|=i\}$, where $i=1,2,\ldots,n$, $T_n=\{1,2,\ldots,n\}$ and $T_0=\{\emptyset\}$.

Let $\mathfrak{K}_0(\emptyset)=\{(\mathrm{Int}_{M_1}U_1)\times(\mathrm{Int}_{M_2}U_2)\times\ldots\times\mathrm{Int}_{M_n}(U_n)\colon U_1\times\ldots\times U_n\in\mathfrak{U}\}$. Suppose we have already defined $\mathfrak{K}_j(T)$ for $j<i$ and put for $T\in\mathfrak{T}_i$

$$\mathfrak{K}_i(T)=\{D_1\times\ldots\times D_n\setminus\Big(\bigcup_{j<i}\bigcup_{T'\in\mathfrak{T}_j}\bigcup\mathfrak{K}_j(T')\Big)\colon D_t=U_t\setminus\mathrm{Int}_{M_t}U_t\ \text{ for }\ t\in T,$$
$$D_j=\mathrm{Int}_{M_j}U_j\ \text{ for }\ j\in T_n\setminus T,\ \text{ where }\ U_1\times\ldots\times U_n\in\mathfrak{U}\}\,.$$

We define a relation $R(i,T)$ in $\mathfrak{K}_i(T)$, where $T\in\mathfrak{T}_i$ and $i=1,2,\ldots,n$, assuming $K\,R(i,T)\,K'$ if $p_j(K')=p_j(K)$ for $j\in T$, where p_j is the projection from $\mathop{\boldsymbol{P}}_{i=1}^{n}X_i$ onto X_j.

Let $\mathfrak{F}(i,T)=\{F(K)\colon K\in\mathfrak{K}_i(T)\}$, where $F(K)=\bigcup\{K'\colon K'R(i,T)K\}$.

It is easy to verify that $\mathfrak{F}(i,T)$ for $T\in\mathfrak{T}_i$ and $i=1,2,\ldots,n$ has the following properties:

(60) $\mathfrak{F}(n,T_n)$ is discrete,

(61) $\mathfrak{F}(i,T)$ is discrete in $\bigcup\mathfrak{F}(i,T)$ and consists of sets open in $\bigcup_{T\in\mathfrak{T}_i}\bigcup\mathfrak{F}(i,T)$,

(62) $\bigcup_{T\in\mathfrak{T}_i}\bigcup\mathfrak{F}(i,T)$ is a closed subset of $\bigcup_{j=1}^{i}\bigcup_{T\in\mathfrak{T}_j}\bigcup\mathfrak{F}(j,T)$,

(63) $\bigcup_{i=1}^{n}\bigcup_{T\in\mathfrak{T}_i}\bigcup\mathfrak{F}(i,T)\cup\bigcup\mathfrak{K}_0(\emptyset)=X$,

(64) if $F(K)\in\mathfrak{F}(i,T)$, then there exists a σ-discrete open family $\mathfrak{U}(F(K))$ such that $\bigcup\mathfrak{U}(F(K))\supset F(K)$ and $\delta\mathfrak{U}(F(K))<\delta\mathfrak{U}$.

The family $\mathfrak{K}_0(\emptyset)$ consists of open subsets of $M_1\times\ldots\times M_n$, hence the proof will be finished if we define for $i=1,2,\ldots,n$ a σ-discrete open family $\mathfrak{B}_i$ such that $\delta\mathfrak{B}_i<\delta\mathfrak{U}$ and $\bigcup\mathfrak{B}_i\supset\bigcup_{T\in\mathfrak{T}_i}\bigcup\mathfrak{F}(i,T)$.

From (60) and collectionwise normality of X it follows that there exists $\mathfrak{B}_n$ which satisfies both required properties.

Suppose we have already defined $\mathfrak{B}_j$ for $j<i$. Let

$$Z=\Big(\big(\bigcup_{T\in\mathfrak{T}_i}\bigcup\mathfrak{F}(i,T)\big)\setminus\big(\bigcup\mathfrak{K}_0(\emptyset)\cup\bigcup\mathfrak{B}_n\cup\ldots\cup\bigcup\mathfrak{B}_{i+1}\big)\,.$$

It is easy to verify that Z is closed (see (63) and (62)), therefore from the normality of X and (61) it follows that there exists a family $\mathfrak{Z}=\{Z(T)\colon T\in\mathfrak{T}_i\}$ of closed sets such that

(65) $$Z(T)\subset\bigcup\mathfrak{F}(i,T)\setminus\big(\bigcup\mathfrak{K}_0(\emptyset)\cup(\bigcup\mathfrak{B}_n\cup\ldots\cup\bigcup\mathfrak{B}_{i+1})$$

and

(66) $$\bigcup\mathfrak{Z}=Z\,.$$

From (61), (64), collectionwise normality of X, and (65) it follows that there exists an open σ-discrete family $\mathfrak{B}(T)$ for $T \in \mathfrak{T}_i$ such that $\delta\mathfrak{B}(T) < \delta\mathfrak{U}$ and $\bigcup \mathfrak{B}(T) \supset Z(T)$.

Applying (66) we see that $\mathfrak{B}_i = \mathfrak{B}_n \cup \ldots \cup \mathfrak{B}_{i+1} \cup \mathfrak{K}_0(\emptyset) \cup \bigcup_{T \in \mathfrak{T}_i} \mathfrak{B}(T)$ has the required properties, and the proof is finished.

Let us notice that using similar arguments we can prove that if X is hereditarily collectionwise normal, then X is hereditarily paracompact.

(3.20) Corollary. *Let us assume that $2^{\aleph_0} < 2^{\aleph_1}$. If X_i is a hereditarily Lindelöf GO_δ-space for $i = 1, 2, \ldots, n$ then the following conditions are equivalent:*

(a) *the Cartesian product $X_1 \times \ldots \times X_n$ is normal (hereditarily normal),*

(b) *the Cartesian product $X_1 \times \ldots \times X_n$ has the Lindelöf property (is hereditarily Lindelöf).*

Proof. The implication (b) $\Rightarrow$ (a) is obvious; to prove Corollary (3.20) it is sufficient to show that (a) $\Rightarrow$ (b).

Let us notice that if U is an open subset of X_i and $U \cap X_i^d \neq \emptyset$ then $\mathrm{Int}_{M_i} U_i \neq \emptyset$, hence from the inequalities $w(M_i) \leqslant \aleph_0$, where $w(M_i)$ denotes the weight of M_i, and $|M_i \setminus M_i^d| \leqslant \aleph_0$ it follows that $X_1 \times \ldots \times X_n$ is separable as a finite product of separable spaces. By the standard arguments (see [4], Example 1.52 or 2.1.2) we can prove that X is collectionwise normal.

(3.21) Corollary. *If X_i is a hereditarily Lindelöf GO_δ-space for $i = 1, 2, \ldots, n$ then the equivalence of the conditions* (a) *and* (b) *of Corollary* (3.20) *is independent of the usual axioms of set theory.*

The proof follows from Corollary (3.20) and from the fact that it is consistent with the usual axioms of set theory to assume that there exists a hereditarily Lindelöf GO_δ-space X such that the Cartesian product $X \times X$ is hereditarily normal but not paracompact (see [11]).

(3.22) Corollary. *Let us assume that $2^{\aleph_0} < 2^{\aleph_1}$. If X_i is a Lindelöf GO_δ-space for $i = 1, 2, \ldots, n$ then following conditions are equivalent:*

(a) *the Cartesian product $X_1 \times \ldots \times X_n$ is normal (hereditarily normal),*

(b) *the Cartesian product $X_1 \times \ldots \times X_n$ is paracompact (hereditarily paracompact).*

Proof. The implication (b) $\Rightarrow$ (a) is obvious; to prove Corollary (3.22) it is sufficient to show that (a) $\Rightarrow$ (b).

We shall prove the implication (a) $\Rightarrow$ (b) by induction on n. For $n = 1$ the implication is trivial. Let us assume that the implication (a) $\Rightarrow$ (b) holds if $n = k-1$. Let $\mathfrak{U}$ be an arbitrary open covering of $X = X_1 \times \ldots \times X_k$. From Corollaries (2.3) and (3.20) it follows that there exists a countable open family $\mathfrak{B}$ such that $\delta\mathfrak{B} < \delta\mathfrak{U}$ and $\bigcup \mathfrak{B} \supset X_1^d \times \ldots \times X_k^d$.

Let H be an open set such that $X^d \times \ldots \times X_k^d \subset H \subset \bar{H} \subset \bigcup \mathfrak{B}$ and let $Z = X \setminus H$. From normality of X it follows that there exist open subsets $P_1, \ldots, P_k$ of Z such that $\bar{P}_j \subset G_j$, where $G_j = \{x \in Z\colon x_j \in X_j \setminus X_j^d\}$, and $\bigcup_{j=1}^{k} P_j = Z$. From the inductive assumption it follows that there exists a σ-discrete open covering $\mathfrak{U}_j$ of $\bar{P}_j$ such that $\delta\mathfrak{U}_j < \delta\mathfrak{U}$ for $j = 1, 2, \ldots, k$. The family $\mathfrak{H} = V \cup \bigcup_{j=1}^{k} \{U \cap P_j \cap (X \setminus \bar{H})\colon U \in \mathfrak{U}_j\}$ is an open, σ-discrete refinement of $\mathfrak{U}$. The proof of the part in brackets is similar.

E. Michael showed that if $2^{\aleph_0} = \aleph_1$ then for $n = 1, 2, \ldots$, there exists a GO_δ-space $X(n)$ such that the Cartesian product $(X(n))^n$ is Lindelöf, while $(X(n))^{n+1}$ is paracompact but not Lindelöf (see [10], Example 1.4). Hence we cannot replace (b) of Corollary (3.22) by (b) of Corollary (3.20).

S. Willard proved that if a space X of the form $X = Y \times Z$ is paracompact, where Y is separable and Z has the Lindelöf property, then X is a Lindelöf space (see [14]).

From this fact and Corollary (3.22) it follows that in the part of Corollaries (3.20) and (3.21) out of the brackets it is sufficient to assume that $X_1, X_2, \ldots, X_{n-1}$ are hereditarily Lindelöf spaces and X_n is a Lindelöf space.

(3.23) Corollary. *If X_i is a GO_δ-space for $i = 1, 2, \ldots, n$ then the equivalence of* (a) *and* (b) *from Corollary* (3.22) *is independent of the usual axioms of set theory.*

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Added in proof. The notion of finite subparacompact spaces was introduced by T. R. Kramer under the name of *subnormal spaces* (see T. R. Kramer, *A note on countably subparacompact spaces*, Pacific J. Math. (1973), pp. 209–213.

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An insertion theorem for real functions

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Abstract. Characterizations of countably paracompact spaces and of normal countably paracompact spaces in terms of insertion of extended real-valued semi-continuous and continuous functions are given.

Dowker [1] and Katětov [2] proved that a topological space X is normal and countably paracompact if and only if for real-valued functions f and g defined on X such that f is lower semi-continuous and g is upper semi-continuous and $g(x) < f(x)$ for each x there exists a continuous real-valued function h on X such that $g(x) < h(x) < f(x)$ for each x. Mack [3] proved that a space is countably paracompact if and only if for each lower semi-continuous function g on X such that $g(x) > 0$ for each x there exist a lower semi-continuous function l and an upper semi-continuous function u such that $0 < l(x) \leqslant u(x) \leqslant g(x)$ for each x. This note generalizes these results by using extended real-valued functions.

The abbreviations lsc (resp. usc) for lower semi-continuous (resp. upper semi-continuous) are used, and we write $g \leqslant f$ (resp. $g < f$) in case $g(x) \leqslant f(x)$ (resp. $g(x) < f(x)$) for each x. Denote by L (resp. U) the set of extended real-valued lsc (resp. usc) functions defined on X. If f and g are extended real-valued functions defined on X, we write $g \ll f$ in case $g \leqslant f$ and if either $g(x)$ or $f(x)$ is a real number, then $g(x) < f(x)$.

THEOREM. *The following are equivalent:*

(α) *The space X is normal and countably paracompact.*

(β) *If $f \in L$, $g \in U$, and $g \ll f$, then there exist functions $f' \in L$ and $g' \in U$ such that $g \ll f' \leqslant g' \ll f$.*

(γ) *The space X is normal, and if $f \in L$, $g \in U$ and $g \ll f$, then there exists a function $h \in L$ such that $g \ll h \ll f$.*

(δ) *If $f \in L$, $g \in U$, and $g \ll f$, then there exists an extended real-valued continuous function b on X such that $g \ll b \ll f$.*

Proof. Observe that (δ) implies (β) trivially. The proof that (β) implies (α) is established as in the proof of Theorem 4 of [1]. In order to see that (γ) implies (δ), let $f \in L$, $g \in U$, and $g \ll f$. By (γ) there is a func-