Hence we get by (13), (14), (15) and (18)

$$\frac{h(x_0)-h(z_0)}{x_0-z_0} \geqslant \frac{h(y_0)-h(z_0)}{y_0-z_0}+\frac{1}{u}\log k > \frac{h(y_0)-h(z_0)}{y_0-z_0}.$$

This contradicts the convexity of $h(x)$.

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Accepté par la Rédaction le 12. 10. 1973

Extensions of metrization theorems to higher cardinality

by

R. E. Hodel (Durham, North Carolina)

Abstract. Several well-known metrization theorems, stated in terms of the cardinal $\aleph_0$, are extended to higher cardinals.

1. Introduction. In recent years significant progress has been made in the area of cardinal functions. (A particularly notable achievement is Arhangel'skiĭ's solution [2] to Alexandroff's problem. See Comfort's paper [8] for an excellent survey of this and other results on cardinal invariants. The fundamental tract on cardinal functions is Jusász [13].) And, in spite of the brilliance of the Nagata–Smirnov–Bing solution of the "general metrization problem" in the early 1950's, metrization theory continues to be an active area of research. (See [12] for a survey of metrization theorems from 1950 to 1972.) In this paper we explore the connection between these two exciting areas of general topology. Specifically, we consider the problem of generalizing metrization theorems so that they can be stated in terms of cardinal functions. (For a result of this type, see [11].)

In § 2 we introduce a cardinal function, called the *metrizability degree*, which reflects in some sense how metrizable a space is. The definition is based on a metrization theorem due to Bing [5]. We then give several characterizations of the metrizability degree for the class of regular spaces. These characterizations are based on other well-known metrization theorems. In § 3 we note the relationship between metrizability degree, uniform weight, and ω_μ-metrizability. Finally, in § 4 we extend a recent metrization theorem of Nagata [25] to higher cardinality.

Throughout this paper $\mathfrak{m}$ and $\mathfrak{n}$ denote cardinal numbers, α, β, σ, τ, and ϱ denote ordinal numbers, and $|A|$ denotes the cardinality of a set A. The set of positive integers is denoted by N, and j and k denote elements of N. The reader is referred to p. 49 of Nagata's book [27] for a discussion of the various operations with covers used in this paper. (Note, however, that we use "st" instead of "S" when discussing the star of a set with respect to a cover.) We let w, L, d, c, χ, and ψ denote the following cardinal functions: weight, Lindelöf degree, density, cellularity, character, and pseudo-character. (For definitions, see Juhász [13].) Unless otherwise stated, no separation axioms are assumed. However, paracompact and compact spaces are always Hausdorff and p-spaces are always Tychonoff spaces.

2. Metrizability degree. The *metrizability degree* of a space X, denoted $m(X)$, is $\aleph_0\cdot\mathfrak{m}$, where $\mathfrak{m}$ is the smallest cardinal such that there is a base for X which is the union of $\mathfrak{m}$ discrete collections. It is clear that $m(X) \leqslant w(X)$ for any space X, and that a regular T_1 space X is metrizable if and only if $m(X)=\aleph_0$. (See [5].) Moreover, the following basic result is easily proved.

THEOREM 2.1. *For any space* X, $w(X)=m(X)\cdot d(X)=m(X)\cdot c(X) = m(X)\cdot L(X)$.

Next we give several characterizations of the metrizability degree for the class of regular spaces. These characterizations are based on well-known metrization theorems. First we establish the following result.

THEOREM 2.2. *Let X be a regular space, let $\mathfrak{m}$ be an infinite cardinal. The following six conditions are equivalent.*

B$(\mathfrak{m})$: *X has a base which is the union of $\leqslant\mathfrak{m}$ discrete collections; i.e. $m(X)\leqslant\mathfrak{m}$.*

NS$(\mathfrak{m})$: *X has a base which is the union of $\leqslant\mathfrak{m}$ locally finite collections.* (See [26], [31].)

AS$(\mathfrak{m})$: *X has a collection $\{\mathcal{G}_t\colon t \text{ in } A\}$ of open covers with $|A|\leqslant\mathfrak{m}$ such that for each point p in X and each neighborhood R of p, there is a neighborhood V of p and some t in A such that* $\mathrm{st}(V,\mathcal{G}_t)\subseteq R$. (See [4], [33].)

MM$(\mathfrak{m})$: *X has a collection $\{\mathcal{G}_t\colon t \text{ in } A\}$ of open covers with $|A|\leqslant\mathfrak{m}$ such that for each point p in X, $\{\mathrm{st}^2(p,\mathcal{G}_t)\colon t \text{ in } A\}$ is a fundamental system of neighborhoods of p.* (See [19], [20].)

M$(\mathfrak{m})$: *X has a collection $\{\mathcal{F}_t)\colon t \text{ in } A\}$ of locally finite closed covers with $|A|\leqslant\mathfrak{m}$ such that for each point p in X, $\{\mathrm{st}(p,\mathcal{F}_t)\colon t \text{ in } A\}$ is a "base" for p in the sense that given any neighborhood R of p, there is some t in A such that $\mathrm{st}(p,\mathcal{F}_t)\subseteq R$.* (See [21].)

N$(\mathfrak{m})$: *X has a collection $\{\mathcal{F}_t\colon t \text{ in } A\}$ of closure preserving closed covers with $|A|\leqslant\mathfrak{m}$ such that for each point p in X, $\{\mathrm{st}(p,\mathcal{F}_t)\colon t \text{ in } A\}$ is a "base" for p.* (See [24].)

Proof. We shall show that $\mathrm{B}(\mathfrak{m})\Rightarrow \mathrm{NS}(\mathfrak{m})\Rightarrow \mathrm{AS}(\mathfrak{m})\Rightarrow \mathrm{MM}(\mathfrak{m})\Rightarrow \mathrm{B}(\mathfrak{m})\Rightarrow \mathrm{M}(\mathfrak{m})\Rightarrow \mathrm{N}(\mathfrak{m})\Rightarrow \mathrm{AS}(\mathfrak{m})$. Note that the implications $\mathrm{B}(\mathfrak{m})\Rightarrow \mathrm{NS}(\mathfrak{m})$ and $\mathrm{M}(\mathfrak{m})\Rightarrow \mathrm{N}(\mathfrak{m})$ are obvious.

$\mathrm{NS}(\mathfrak{m})\Rightarrow \mathrm{AS}(\mathfrak{m})$: The technique used here is a combination of ideas due to Nagata (see p. 195 of [27]) and Morita (see p. 35 of [23]). Let $\mathcal{B}$ be a base for X such that $\mathcal{B}=\bigcup\{\mathcal{B}_t\colon t \text{ in } A\}$, where each $\mathcal{B}_t$ is a locally finite collection and $|A|\leqslant\mathfrak{m}$. We may assume that $X\in\mathcal{B}_t$ for all t in A. For p in X, t in A let

$$V_t(p)=\left(\bigcap\{B\colon p\in B\in\mathcal{B}_t\}\right)\cap\left(X-\bigcup\{\bar{B}\colon B\in\mathcal{B}_t,\ p\notin\bar{B}\}\right).$$

Clearly $V_t(p)$ is an open set and $p\in V_t(p)$. For s,t in A let

$$\mathcal{G}_{st}=\{V_s(p)\cap V_t(p)\colon p\in X\}.$$

Then $\{\mathcal{G}_{st}\colon s,t \text{ in } A\}$ is the desired collection of open covers of X. For, let $p\in X$, let R be a neighborhood of p. Choose t in A and B in $\mathcal{B}_t$ such that $p\in B\subseteq R$, and then choose s in A and U in $\mathcal{B}_s$ such that $p\in U\subseteq\bar{U}\subseteq B$. Then $\mathrm{st}(V_s(p),\mathcal{G}_{st})\subseteq R$. For, let $V_s(p)\cap[V_s(q)\cap V_t(q)]\neq\emptyset$, and let us show that $[V_s(q)\cap V_t(q)]\subseteq R$. First note that $q\in B$. (If $q\notin B$, then $q\notin\bar{U}$ so $V_s(q)\subseteq X-\bar{U}$. But $V_s(p)\subseteq U$, so $V_s(p)\cap V_s(q)=\emptyset$, a contradiction.) Since $q\in B$, $V_t(q)\subseteq B$ and so $[V_s(q)\cap V_t(q)]\subseteq R$.

$\mathrm{AS}(\mathfrak{m})\Rightarrow\mathrm{MM}(\mathfrak{m})$: Let $\{\mathcal{G}_t\colon t \text{ in } A\}$ be a collection of open covers of X with $|A|\leqslant\mathfrak{m}$ satisfying the condition stated for $\mathrm{AS}(\mathfrak{m})$. For s,t in A let $\mathcal{H}_{st}=\mathcal{G}_s\wedge\mathcal{G}_t$. Then $\{\mathcal{H}_{st}\colon s,t \text{ in } A\}$ is a collection of open covers of X such that for any p in X, $\{\mathrm{st}^2(p,\mathcal{H}_{st})\colon s,t \text{ in } A\}$ is a fundamental system of neighborhoods of p. Indeed, given p in X and a neighborhood R of p, choose t in A and a neighborhood V of p such that $\mathrm{st}(V,\mathcal{G}_t)\subseteq R$, and then choose s in A such that $\mathrm{st}(p,\mathcal{G}_s)\subseteq V$. Then $\mathrm{st}^2(p,\mathcal{H}_{st})\subseteq R$.

$\mathrm{MM}(\mathfrak{m})\Rightarrow\mathrm{B}(\mathfrak{m})$: Let $\{\mathcal{G}_t\colon t \text{ in } A\}$ be a collection of open covers of X with $|A|\leqslant\mathfrak{m}$ such that for each p in X, $\{\mathrm{st}^2(p,\mathcal{G}_t)\colon t \text{ in } A\}$ is a fundamental system of neighborhoods of p. We may assume that for each p in X, $\{\mathrm{st}^3(p,\mathcal{G}_t)\colon t \text{ in } A\}$ is a fundamental system of neighborhoods of p. (If necessary, one could replace the covers $\{\mathcal{G}_t\colon t \text{ in } A\}$ with the covers $\{\mathcal{G}_s\wedge\mathcal{G}_t\colon s,t \text{ in } A\}$.) To prove that X satisfies $B(\mathfrak{m})$, it suffices to show that every open cover of X has an open refinement which is the union of $\leqslant\mathfrak{m}$ discrete collections. (One then obtains a base for X which is the union of $\leqslant\mathfrak{m}$ discrete collections in the same way that Bing [5] shows that every strongly screenable developable space has a σ-discrete base.) So let $\mathcal{U}=\{U_\alpha\colon 0\leqslant\alpha<\mathfrak{n}\}$ be an arbitrary open cover of X. For $\alpha<\mathfrak{n}$ and t in A let $F_{\alpha t}=\{p \text{ in } X\colon p\notin\bigcup_{\beta<\alpha}U_\beta,\ \mathrm{st}^3(p,\mathcal{G}_t)\subseteq U_\alpha\}$, let $W_{\alpha t}=\bigcup\{G \text{ in } \mathcal{G}_t\colon G\cap F_{\alpha t}\neq\emptyset\}$, let $\mathcal{W}_t=\{W_{\alpha t}\colon 0\leqslant\alpha<\mathfrak{n}\}$, and let $\mathcal{W}=\bigcup\{\mathcal{W}_t\colon t \text{ in } A\}$. It is clear that $\mathcal{W}$ is an open refinement of $\mathcal{U}$. To see that $\mathcal{W}_t$ is discrete, let $p\in X$. Choose G in $\mathcal{G}_t$ such that $p\in G$. Then G is a neighborhood of p which intersects at most one element of $\mathcal{W}_t$.

$\mathrm{B}(\mathfrak{m})\Rightarrow\mathrm{M}(\mathfrak{m})$: Let $\mathcal{B}$ be a base for X such that $\mathcal{B}=\bigcup\{\mathcal{B}_t\colon t \text{ in } A\}$, where each $\mathcal{B}_t$ is discrete and $|A|\leqslant\mathfrak{m}$. For each t in A let $\mathcal{F}_t=\{\bar{B}\colon B \text{ in } \mathcal{B}_t\}\cup\{(X-\bigcup\mathcal{B}_t)\}$. Clearly $\mathcal{F}_t$ is a locally finite closed cover of X, and it is easy to check that for each p in X, $\{\mathrm{st}(p, \mathcal{F}_t)\colon t \text{ in } A\}$ is a "base" for p.

$\mathrm{N}(\mathfrak{m}) \Rightarrow \mathrm{AS}(\mathfrak{m})$: Let $\{\mathcal{F}_t\colon t \text{ in } A\}$ be a collection of closure preserving closed covers of X with $|A| \leqslant \mathfrak{m}$ such that for each p in X, $\{\mathrm{st}(p, \mathcal{F}_t)\colon t \text{ in } A\}$ is a "base" for p. For each p in X, t in A let

$$V_t(p) = X - \bigcup \{F \in \mathcal{F}_t\colon p \notin F\}.$$

Note that $V_t(p)$ is an open set containing p. Moreover, the following three facts are easy to verify:

(1) if $V_t(p) \cap V_t(q) \neq \emptyset$, then $q \in \mathrm{st}(p, \mathcal{F}_t)$;

(2) $V_t(p) \subseteq \mathrm{st}(p, \mathcal{F}_t)$;

(3) if $q \in V_t(p)$, then $V_t(q) \subseteq V_t(p)$.

For s, t in A let $\mathcal{G}_{st} = \{V_s(p) \cap V_t(p)\colon p \text{ in } X\}$. Then $\{\mathcal{G}_{st}\colon s, t \text{ in } A\}$ is the desired collection of open covers of X. For, let $p \in X$ and let R be a neighborhood of p. Choose t in A such that $\mathrm{st}(p, \mathcal{F}_t) \subseteq R$, and then choose s in A such that $\mathrm{st}(p, \mathcal{F}_s) \subseteq V_t(p)$. Then $\mathrm{st}(V_s(p), \mathcal{G}_{st}) \subseteq R$. For, let $V_s(p) \cap [V_s(q) \cap V_t(q)] \neq \emptyset$, and let us show that $[V_s(q) \cap V_t(q)] \subseteq R$. Now $V_s(p) \cap V_s(q) \neq \emptyset$, and so by (1) $q \in \mathrm{st}(p, \mathcal{F}_s)$. Hence $q \in V_t(p)$, and so by (3) $V_t(q) \subseteq V_t(p)$. By (2), $V_t(p) \subseteq \mathrm{st}(p, \mathcal{F}_t)$, and so $V_t(q) \subseteq \mathrm{st}(p, \mathcal{F}_t)$. Hence $[V_s(q) \cap V_t(q)] \subseteq R$.

As a consequence of Theorem 2.2 we have the following characterizations of the metrizability degree.

THEOREM 2.3. *Let X be a regular space. Then*

$$\begin{aligned} m(X) &= \aleph_0 \cdot \min\{\mathfrak{m}\colon X \textit{ satisfies } \mathrm{NS}(\mathfrak{m})\} \\ &= \aleph_0 \cdot \min\{\mathfrak{m}\colon X \textit{ satisfies } \mathrm{AS}(\mathfrak{m})\} \\ &= \aleph_0 \cdot \min\{\mathfrak{m}\colon X \textit{ satisfies } \mathrm{MM}(\mathfrak{m})\} \\ &= \aleph_0 \cdot \min\{\mathfrak{m}\colon X \textit{ satisfies } \mathrm{M}(\mathfrak{m})\} \\ &= \aleph_0 \cdot \min\{\mathfrak{m}\colon X \textit{ satisfies } \mathrm{N}(\mathfrak{m})\}. \end{aligned}$$

3. Uniform weight and ω_μ-metrizability. In this section we note the relationship between metrizability degree, uniform weight, and ω_μ-metrizability. (Recall that the *uniform weight* of a completely regular space X, denoted $u(X)$, is $\aleph_0 \cdot \mathfrak{m}$, where $\mathfrak{m}$ is the smallest cardinal which arises as the cardinality of a base for a uniformity which is compatible with the topology of X.) In [22] (see also [30]), Mrówka essentially proves the following two results for a completely regular space X: (1) if $u(X) \leqslant \mathfrak{m}$, then X has a base which is the union of $\leqslant \mathfrak{m}$ locally finite collections; (2) if X is normal and has a base which is the union of $\leqslant \mathfrak{m}$ locally finite collections, then $u(X) \leqslant \aleph_0 \cdot \mathfrak{m}$. As a consequence of Theorem 2.2, these two results can be restated as follows.

THEOREM 3.1 (Mrówka). *If X is completely regular, then $m(X) \leqslant u(X)$. If X is completely regular and normal, then $m(X) = u(X)$.*

Remark. H. M. Shaerf [29] has proved that $w(X) \leqslant u(X) \cdot c(X)$ for any completely regular space X. This theorem can be obtained from Theorem 3.1 as follows: $w(X) \leqslant m(X) \cdot c(X) \leqslant u(X) \cdot c(X)$.

In [30] Shu-tang proves that a regular T_1 space X is ω_μ-metrizable if and only if it is ω_μ-additive and has a base which is the union of $\leqslant \aleph_\mu$ locally finite collections. In light of Theorem 2.2, this result can be restated as follows.

THEOREM 3.2 (Shu-tang). *Let X be a regular T_1 space. Then X is ω_μ-metrizable if and only if X is ω_μ-additive and $m(X) \leqslant \aleph_\mu$.*

4. Nagata's theorem. In 1970 Nagata [25] proved that every paracompact p-space with a point-countable separating open cover is metrizable. (Relevant definitions will be stated below.) This theorem has an interesting history. In 1962 Miščenko [18] proved that every compact space with a point-countable base has a countable base, and in 1968 Filippov [9] generalized this result by proving that every paracompact p-space with a point-countable base is metrizable. Parallel to these results, Šneĭder [32], in 1945, proved that every compact space with a G_δ-diagonal has a countable base, and around 1965 Borges [6] and Okuyama [28] independently generalized this result by proving that every paracompact p-space with a G_δ-diagonal is metrizable. Nagata's theorem generalizes the results of Filippov and Borges-Okuyama. In summary, Nagata's theorem, together with the results by Šneĭder, Miščenko, Borges-Okuyama, and Filippov which proceed it, represent a highlight in metrization theory.

In this section we extend Nagata's theorem to higher cardinality. We begin with some definitions.

Let X be a set and let $\mathcal{G}$ be a cover of X. The cover $\mathcal{G}$ is said to be *separating* if given distinct points p and q in X, there is some G in $\mathcal{G}$ such that $p \in G$, $q \notin G$. For p in X, the *order of p with respect to* $\mathcal{G}$, denoted $\mathrm{ord}(p, \mathcal{G})$, is the cardinality of the set $\{G \text{ in } \mathcal{G}\colon p \text{ in } G\}$. Now suppose X is a T_1 topological space. Let $\mathfrak{m}$ be the smallest cardinal such that X has a separating open cover $\mathcal{S}$ with $\mathrm{ord}(p, \mathcal{S}) \leqslant \mathfrak{m}$ for all p in X. (Since X is T_1, it is easy to see that $\mathfrak{m}$ exists and $\mathfrak{m} \leqslant m(X)$.) The *point separating weight of* X, denoted $\mathrm{psw}(X)$, is $\aleph_0 \cdot \mathfrak{m}$. Note that for a T_1 space X, $\mathrm{psw}(X) = \aleph_0$ if and only if X has a point-countable separating open cover.

Next we introduce a cardinal function which extends the notion of paracompactness to higher cardinality. Of the numerous characterizations of paracompactness upon which the definition could be based (see, e.g., [15], [16], [17]), the following seems most natural. The *paracompactness degree* of a space X, denoted $\mathrm{pa}(X)$, is $\aleph_0 \cdot \mathfrak{m}$, where $\mathfrak{m}$ is the smallest cardinal such that every open cover of X has an open refinement which is the union of $\leqslant \mathfrak{m}$ locally finite collections. Note that $\mathrm{pa}(X) \leqslant L(X)$, $\mathrm{pa}(X) \leqslant m(X)$, and that a regular T_1 space X is paracompact if and only if $\mathrm{pa}(X) = \aleph_0$. (See [15].)

In [11] the concept of a p-space [3] was extended to higher cardinals as follows. A collection $\{\mathcal{G}_t\colon t \text{ in } A\}$ of open covers of a space X is a *pluming* for X if the following holds: if $p \in G_t \in \mathcal{G}_t$ for all t in A, then (a) $C(p) = \bigcap_{t\in A} \bar{G}_t$ is compact; (b) $\{\bigcap_{t\in F} \bar{G}_t\colon F$ a finite subset of $A\}$ is a "base" for $C(p)$ in the sense that given any open set R containing $C(p)$, there is a finite subset F of A such that $\bigcap_{t\in F} \bar{G}_t \subseteq R$. See [11] for a proof that every regular space has a pluming. For a regular space X, the *pluming degree* of X, denoted $\mathrm{p}(X)$, is $\aleph_0\cdot\mathfrak{m}$, where $\mathfrak{m}$ is the smallest cardinal such that X has a pluming $\{\mathcal{G}_t\colon t \text{ in } A\}$ with $|A| = \mathfrak{m}$. It follows from Theorem 2.2 that $p(X) \leqslant m(X)$ for any regular T_1 space X. (Use $\mathrm{AS}(\mathfrak{m})$.)

The definition of a pluming for X is based on an internal characterization of p-spaces given by Burke [7]. From Burke's theorem it follows that a Tychonoff space X is a p-space if and only if $\mathrm{p}(X) = \aleph_0$. Moreover, Burke's technique can be used to prove the following result. (Note that (2) is Arhangel'skiĭ's original definition of a p-space extended to higher cardinals.)

THEOREM 4.1. *The following are equivalent for a Tychonoff space X and an infinite cardinal $\mathfrak{m}$.*

(1) *X has a pluming $\{\mathcal{G}_t\colon t \text{ in } A\}$ with $|A| \leqslant \mathfrak{m}$; i.e., $\mathrm{p}(X) \leqslant \mathfrak{m}$.*

(2) *In the Stone-Ĉech compactification $\beta(X)$ of X there is a collection $\{\mathcal{G}_t\colon t \text{ in } A\}$ of open covers of X with $|A| \leqslant \mathfrak{m}$ such that for each p in X, $\bigcap_{t\in A} \mathrm{st}(p, \mathcal{G}_t) \subseteq X$.*

The following fact about plumings will be used in the proof of the main theorem.

PROPOSITION 4.2. *Let X be a topological space, let $\{\mathcal{G}_t\colon t \text{ in } A\}$ be a pluming for X, let $p \in X$, for each t in A let $p \in H_t \subseteq G_t \in \mathcal{G}_t$, and let $C^*(p) = \bigcap_{t\in A} \bar{H}_t$. Then $C^*(p)$ is compact and $\{\bigcap_{t\in F} \bar{H}_t\colon F$ a finite subset of $A\}$ is a "base" for $C^*(p)$.*

Proof. Let $C(p) = \bigcap_{t\in A} \bar{G}_t$. Recall that $C(p)$ is compact and $\{\bigcap_{t\in F} \bar{G}_t\colon F$ a finite subset of $A\}$ is a "base" for $C(p)$. Since $C^*(p)$ is closed and $C^*(p) \subseteq C(p)$, it follows that $C^*(p)$ is compact. Let R be an open set such that $C^*(p) \subseteq R$. Now $Z = C(p) - R$ is compact and $\{X - \bar{H}_t\colon t \text{ in } A\}$ covers Z so there is a finite subset F_0 of A such that $\{X - \bar{H}_t\colon t \in F_0\}$ covers Z. Let $W = R \cup (\bigcup_{t\in F_0} (X - \bar{H}_t))$. Then W is open and $C(p) \subseteq W$ so there is a finite subset F_1 of A such that $\bigcap_{t\in F_1} \bar{G}_t \subseteq W$. Let $F = F_0 \cup F_1$. Then $\bigcap_{t\in F} \bar{H}_t \subseteq R$.

COROLLARY 4.3. *Let X be a regular T_1 space. Then $\chi(X) \leqslant \psi(X)\cdot \mathrm{p}(X)$.*

Proof. Let $\psi(X)\cdot\mathrm{p}(X) = \mathfrak{m}$. Let $\{\mathcal{G}_\alpha\colon 0 \leqslant \alpha < \mathfrak{m}\}$ be a pluming for X. (Repeatedly count a cover if necessary.) Fix p in X, and let $\{W_\alpha\colon 0 \leqslant \alpha < \mathfrak{m}\}$ be a collection of open sets such that $p \in W_\alpha$ for all $\alpha < \mathfrak{m}$ and $\bigcap_{\alpha<\mathfrak{m}} \bar{W}_\alpha = \{p\}$. For each $\alpha < \mathfrak{m}$ choose G_α in $\mathcal{G}_\alpha$ such that $p \in G_\alpha$, and let $H_\alpha = G_\alpha \cap W_\alpha$. By Proposition 4.2, $C(p) = \bigcap_{\alpha<\mathfrak{m}} \bar{H}_\alpha = \{p\}$ is compact and $\{\bigcap_{\alpha\in F} \bar{H}_\alpha\colon F \subseteq \mathfrak{m}, F \text{ finite}\}$ is a "base" for p. Thus $\{\bigcap_{\alpha\in F} H_\alpha\colon F \subseteq \mathfrak{m}, F \text{ finite}\}$ is a fundamental system of neighborhoods of p of cardinality $\leqslant \mathfrak{m}$.

Finally, we need a set-theoretic result due to Miščenko [18]. This result was abstracted by Filippov [9] from Miščenko's proof that every compact space with a point-countable base has a countable base. It plays an important rọle in the proofs of the above mentioned metrization theorems of Filippov and Nagata.

MIŠČENKO'S LEMMA. *Let X be a set, let $\mathfrak{m}$ be an infinite cardinal, let $\mathcal{S}$ be a collection of subsets of X such that $\mathrm{ord}(p, \mathcal{S}) \leqslant \mathfrak{m}$ for all p in X, and let H be a subset of X. Then the cardinality of the set of all finite minimal covers of H by elements of $\mathcal{S}$ does not exceed $\mathfrak{m}$.*

THEOREM 4.4. *Let X be a regular T_1 space. Then $m(X) = \mathrm{p}(X)\cdot\mathrm{pa}(X)\cdot\mathrm{psw}(X)$.*

Proof. Clearly $\mathrm{p}(X)\cdot\mathrm{pa}(X)\cdot\mathrm{psw}(X) \leqslant m(X)$. Suppose, then, that $\mathrm{p}(X)\cdot\mathrm{pa}(X)\cdot\mathrm{psw}(X) = \mathfrak{m}$, and let us construct a base $\mathcal{B}$ for X which is the union of $\leqslant \mathfrak{m}$ locally finite collections. Let $\mathcal{F}(\mathfrak{m})$ be all finite subsets of $\mathfrak{m}$. Let $\mathcal{S}$ be a separating open cover of X such that $\mathrm{ord}(p, \mathcal{S}) \leqslant \mathfrak{m}$ for all p in X. We may assume that $X \in \mathcal{S}$, and hence for any subset H of X there is at least one finite minimal cover of H by elements of $\mathcal{S}$, namely $\{X\}$.

Let $\{\mathcal{G}_\alpha\colon 0 \leqslant \alpha < \mathfrak{m}\}$ be a pluming for X (repeatedly count a cover if necessary), and for each $\alpha < \mathfrak{m}$ let $\mathcal{H}_\alpha$ be an open refinement of $\mathcal{G}_\alpha$ such that $\mathcal{H}_\alpha = \bigcup\{\mathcal{H}(\alpha, \beta)\colon 0 \leqslant \beta < \mathfrak{m}\}$, where each $\mathcal{H}(\alpha, \beta)$ is a locally finite collection. Let Γ be all finite subsets of $\mathfrak{m}\times\mathfrak{m}$, and for each γ in Γ let $\mathcal{K}_\gamma = \bigwedge\{\mathcal{H}(\alpha, \beta)\colon (\alpha, \beta) \in \gamma\}$. Note that $\mathcal{K}_\gamma$ is a locally finite open collection in X.

Observation. Let $p \in X$. For each $\alpha < \mathfrak{m}$ choose $\beta_\alpha < \mathfrak{m}$ and H_α in $\mathcal{H}(\alpha, \beta_\alpha)$ such that $p \in H_\alpha$. By Proposition 4.2, $C(p) = \bigcap_{\alpha<\mathfrak{m}} \bar{H}_\alpha$ is compact and $\{\bigcap_{\alpha\in F} \bar{H}_\alpha\colon F \text{ in } \mathcal{F}(\mathfrak{m})\}$ is a "base" for $C(p)$.

The construction of the required base for X is accomplished in four steps.

Step 1. *X has a separating closed cover which is the union of $\leqslant \mathfrak{m}$ locally finite collections.* Fix γ in Γ. For each K in $\mathcal{K}_\gamma$ let $\{\mathcal{S}(\gamma, K, \sigma)$:

$0 \leq \sigma < \mathfrak{m}\}$ be all finite minimal covers of $\bar{K}$ by elements of $\mathcal{S}$ (use Miščenko's lemma), and for each $\sigma < \mathfrak{m}$ let $\mathcal{L}(\gamma, \sigma) = \{\bar{K} - S\colon K \in \mathcal{K}_\gamma,\ S = \emptyset$ or $S \in \mathcal{S}(\gamma, K, \sigma)\}$. Clearly $\mathcal{L}(\gamma, \sigma)$ is a locally finite closed collection in X.

Now let $\mathcal{L} = \bigcup\{\mathcal{L}(\gamma,\sigma)\colon \gamma \in \Gamma,\ 0 \leq \sigma < \mathfrak{m}\}$. Then $\mathcal{L}$ is the union of $\leq \mathfrak{m}$ locally finite closed collections, and so to complete the proof it remains to show $\mathcal{L}$ separating. Let p and q be distinct points of X. For each $\alpha < \mathfrak{m}$ choose $\beta_\alpha < \mathfrak{m}$ and H_α in $\mathcal{H}(\alpha, \beta_\alpha)$ such that $p \in H_\alpha$, and let $C(p) = \bigcap_{\alpha<\mathfrak{m}} \bar{H}_\alpha$. Choose S_0 in $\mathcal{S}$ such that $q \in S_0$, $p \notin S_0$, and then let $\mathcal{S}_0 = \{S_0, S_1, \ldots, S_k\}$ be a finite subcollection of $\mathcal{S}$ which covers $C(p)$ such that $q \notin \bigcup_{j=1}^{k} S_j$. Choose F in $\mathcal{F}(\mathfrak{m})$ such that $\bigcap_{\alpha\in F} \bar{H}_\alpha \subseteq \bigcup \mathcal{S}_0$, let $\gamma = \{(\alpha, \beta_\alpha)\colon \alpha \text{ in } F\}$, and let $K = \bigcap_{\alpha\in F} H_\alpha$. Note that $K \in \mathcal{K}_\gamma$. Now $\mathcal{S}_0$ covers $\bar{K}$, so some finite minimal subcollection of $\mathcal{S}_0$ covers $\bar{K}$, say $\mathcal{S}(\gamma, K, \sigma)$. If $q \notin \bar{K}$, then $\bar{K}$ is an element of $\mathcal{L}$ which contains p and not q. Assume, then, that $q \in \bar{K}$. Then $S_0 \in \mathcal{S}(\gamma, K, \sigma)$, and so $\bar{K} - S_0$ is an element of $\mathcal{L}(\gamma, \sigma)$ which contains p and not q.

Step 2. *Every open subset of X is the union of $\leq \mathfrak{m}$ closed sets.* Clearly it suffices to show that X has a net $\mathcal{N}$ which is the union of $\leq \mathfrak{m}$ locally finite closed collections. (Recall that $\mathcal{N}$ is a net for X if given any point p in X and any neighborhood R of p, there is some N in $\mathcal{N}$ such that $p \in N \subseteq R$. See [1].) Let $\mathcal{L}$ be a separating closed cover of X such that $\mathcal{L} = \bigcup\{\mathcal{L}_\sigma\colon 0 \leq \sigma < \mathfrak{m}\}$, where each $\mathcal{L}_\sigma$ is a locally finite collection. We may assume that each $\mathcal{L}_\sigma$ is closed under finite intersections. For each F in $\mathcal{F}(\mathfrak{m})$ let $\mathcal{W}_F = \bigwedge\{\mathcal{L}_\sigma\colon \sigma \text{ in } F\}$. Note that $\mathcal{W}_F$ is a locally finite closed collection in X. For γ in Γ, F in $\mathcal{F}(\mathfrak{m})$ let $\mathcal{N}(\gamma, F) = \{\bar{K} \cap W\colon K \in \mathcal{K}_\gamma,\ W = X$ or $W \in \mathcal{W}_F\}$, and let $\mathcal{N} = \bigcup\{\mathcal{N}(\gamma, F)\colon \gamma \text{ in } \Gamma, F \text{ in } \mathcal{F}(\mathfrak{m})\}$. Clearly $\mathcal{N}$ is the union of $\leq \mathfrak{m}$ locally finite closed collections. To see that $\mathcal{N}$ is a net, let p be a point in X, let R be an open neighborhood of p. For each $\alpha < \mathfrak{m}$ choose $\beta_\alpha < \mathfrak{m}$ and H_α in $\mathcal{H}(\alpha, \beta_\alpha)$ such that $p \in H_\alpha$, and let $C(p) = \bigcap_{\alpha<\mathfrak{m}} \bar{H}_\alpha$. Let $Z = C(p) - R$, and assume that $Z \neq \emptyset$. (The case $Z = \emptyset$ is easy.) Now Z is compact, $\mathcal{L}$ is a separating closed cover, and each $\mathcal{L}_\sigma$ is closed under finite intersections, so there exists F in $\mathcal{F}(\mathfrak{m})$ and L_σ in $\mathcal{L}_\sigma$ for each σ in F such that $p \in \bigcap_{\sigma\in F} L_\sigma$ and $Z \subseteq \bigcup_{\sigma\in F} (X - L_\sigma) = V$. Note that $\bigcap_{\sigma\in F} L_\sigma = W$ belongs to $\mathcal{W}_F$. Now $C(p) \subseteq R \cup V$, so there exists F^* in $\mathcal{F}(\mathfrak{m})$ such that $\bigcap_{\alpha\in F^*} \bar{H}_\alpha \subseteq R \cup V$. Let $\gamma = \{(\alpha, \beta_\alpha)\colon \alpha \text{ in } F^*\}$ and let $K = \bigcap_{\alpha\in F^*} H_\alpha$. Then $K \in \mathcal{K}_\gamma$, and so $N = \bar{K} \cap W$ is an element of $\mathcal{N}(\gamma, F)$ such that $p \in N \subseteq R$.

Step 3. *There is an open cover $\mathcal{W}$ of X which is the union of $\leq \mathfrak{m}$ locally finite open collection such that if p and q are any two distinct points of X, then there exists W in $\mathcal{W}$ such that $p \in W$, $q \notin \bar{W}$.* Fix γ in Γ. For each K in $\mathcal{K}_\gamma$ let $\{\mathcal{S}(\gamma, K, \sigma)\colon 0 \leq \sigma < \mathfrak{m}\}$ be all finite minimal covers of K by elements of $\mathcal{S}$ (use Miščenko's lemma). For each $\sigma < \mathfrak{m}$ let $\mathcal{L}(\gamma, \sigma) = \{K \cap S\colon K \in \mathcal{K}_\gamma,\ S \in \mathcal{S}(\gamma, K, \sigma)\}$, and note that $\mathcal{L}(\gamma, \sigma)$ is a point finite open collection in X. For each k in N let $\mathcal{U}(\gamma, \sigma, k) = \{U\colon U$ is the intersection of exactly k distinct elements of $\mathcal{L}(\gamma, \sigma)\}$, and let $V(\gamma, \sigma, k) = \bigcup \mathcal{U}(\gamma, \sigma, k)$. Now $V(\gamma, \sigma, k)$ is an open set so by Step 2, $V(\gamma, \sigma, k) = \bigcup\{D(\gamma, \sigma, k, \tau)\colon 0 \leq \tau < \mathfrak{m}\}$, where each $D(\gamma, \sigma, k, \tau)$ is a closed set. Let $\mathcal{V}(\gamma, \sigma, k, \tau) = \mathcal{U}(\gamma, \sigma, k) \cup \{X - D(\gamma, \sigma, k, \tau)\}$. Now $\mathcal{V}(\gamma, \sigma, k, \tau)$ is an open cover of X, so there is an open cover $\mathcal{W}(\gamma, \sigma, k, \tau)$ of X such that the closure of each element of $\mathcal{W}(\gamma, \sigma, k, \tau)$ is contained in some element of $\mathcal{V}(\gamma, \sigma, k, \tau)$ and $\mathcal{W}(\gamma, \sigma, k, \tau) = \bigcup\{\mathcal{W}(\gamma, \sigma, k, \tau, \varrho)\colon 0 \leq \varrho < \mathfrak{m}\}$, where each $\mathcal{W}(\gamma, \sigma, k, \tau, \varrho)$ is a locally finite collection.

Now let $\mathcal{W} = \bigcup\{\mathcal{W}(\gamma, \sigma, k, \tau, \varrho)\colon \gamma \in \Gamma,\ \sigma, \tau, \varrho \in \mathfrak{m},\ k \in N\}$. Clearly $\mathcal{W}$ is an open collection in X which is the union of $\leq \mathfrak{m}$ locally finite collections. Let p and q be distinct points of X. For each $\alpha < \mathfrak{m}$ pick $\beta_\alpha < \mathfrak{m}$ and H_α in $\mathcal{H}(\alpha, \beta_\alpha)$ such that $p \in H_\alpha$, and let $C(p) = \bigcap_{\alpha<\mathfrak{m}} \bar{H}_\alpha$. Choose S_0 in $\mathcal{S}$ such that $p \in S_0$, $q \notin S_0$, and let $\mathcal{S}_0 = \{S_0, S_1, \ldots, S_k\}$ be a finite subcollection of $\mathcal{S}$ which covers C_p such that $p \notin \bigcup_{j=1}^{k} S_j$. Choose F in $\mathcal{F}(\mathfrak{m})$ such that $\bigcap_{\alpha\in F} \bar{H}_\alpha \subseteq \bigcup_{j=0}^{k} S_j$. Let $\gamma = \{(\alpha, \beta_\alpha)\colon \alpha \text{ in } F\}$, let $K = \bigcap_{\alpha\in F} H_\alpha$, and note that $K \in \mathcal{K}_\gamma$. Let $\mathcal{S}_1$ be a finite minimal subcollection of $\mathcal{S}_0$ which covers K, and note that $S_0 \in \mathcal{S}_1$. Now $\mathcal{S}_1 = \mathcal{S}(\gamma, K, \sigma)$ for some $\sigma < \mathfrak{m}$, and so $K \cap S_0$ is an element of $\mathcal{L}(\gamma, \sigma)$ which contains p and not q. Now let $L_1, \ldots, L_k$ be the distinct elements of $\mathcal{L}(\gamma, \sigma)$ which contain p, and let $U = \bigcap_{j=1}^{k} L_j$. Then $p \in U$, $q \notin U$, and U is the only element of $\mathcal{U}(\gamma, \sigma, k)$ which contains p. Now $p \in V(\gamma, \sigma, k)$, so $p \in D(\gamma, \sigma, k, \tau)$ for some $\tau < \mathfrak{m}$. Thus U is the only element of $\mathcal{V}(\gamma, \sigma, k, \tau)$ which contains p. Finally, there exists $\varrho < \mathfrak{m}$ and W in $\mathcal{W}(\gamma, \sigma, k, \tau, \varrho)$ such that $p \in W$. Now $\bar{W} \subseteq U$, so W is an element of $\mathcal{W}$ such that $p \in W$, $q \notin \bar{W}$.

Step 4. *X has a base which is the union of $\leq \mathfrak{m}$ locally finite collections.* Let $\mathcal{W}$ be an open cover of X having the properties stated in Step 3. Thus, $\mathcal{W} = \bigcup\{\mathcal{W}_\sigma\colon 0 \leq \sigma < \mathfrak{m})$, where each $\mathcal{W}_\sigma$ is a locally finite open collection. We may assume each $\mathcal{W}_\sigma$ is closed under finite intersections. For each F in $\mathcal{F}(\mathfrak{m})$ let $\mathcal{V}_F = \bigwedge\{\mathcal{W}_\sigma\colon \sigma \text{ in } F\}$. For γ in Γ, F in $\mathcal{F}(\mathfrak{m})$ let $\mathcal{B}(\gamma, F) = \{K \cap V\colon K \in \mathcal{K}_\gamma,\ V = X$ or $V \in \mathcal{V}_F\}$, and let

$$\mathcal{B} = \bigcup\{\mathcal{B}(\gamma, F)\colon \gamma \in \Gamma,\ F \in \mathcal{F}(\mathfrak{m})\}.$$

Clearly $\mathcal{B}$ is the union of $\leq \mathfrak{m}$ locally finite collections. To see that $\mathcal{B}$ is a base, let p be a point of X and let R be an open neighborhood of p.

For each $\alpha < \mathfrak{m}$ choose $\beta_\alpha < \mathfrak{m}$ and H_α in $\mathcal{K}(\alpha, \beta_\alpha)$ such that $p \in H_\alpha$, and let $C(p) = \bigcap_{\alpha<\mathfrak{m}} \bar{H}_\alpha$. Let $Z = C(p) - R$, and assume $Z \neq \emptyset$. (The case $Z = \emptyset$ is easy.) Then there exists F in $\mathcal{F}(\mathfrak{m})$, W_σ in $\mathcal{W}_\sigma$ for each σ in F, and an open set U such that $p \in \bigcap_{\sigma \in F} W_\sigma = V$, $Z \subseteq U$, and $U \cap V = \emptyset$. Choose F^* in $\mathcal{F}(\mathfrak{m})$ such that $\bigcap_{\alpha \in F^*} \bar{H}_\alpha \subseteq R \cup U$, let $\gamma = \{(\alpha, \beta_\alpha)\colon \alpha \in F^*\}$, and let $K = \bigcap_{\alpha \in F^*} H_\alpha$. Then $B = K \cap V$ is an element of $\mathcal{B}(\gamma, F)$ such that $p \in B \subseteq R$.

COROLLARY 4.5 (Hodel [11]). *Let X be a regular T_1 space. Then* $w(X) = L(X)\cdot \mathrm{p}(X)\cdot \mathrm{psw}(X)$.

COROLLARY 4.6 (Nagata [25]). *Let X be a regular T_1 space. Then X is metrizable if and only if it is a paracompact p-space with a point-countable separating open cover.*

COROLLARY 4.7 (Filippov [9]). *Let $\mathfrak{m}$ be an infinite cardinal, let X be a paracompact p-space having a base $\mathcal{B}$ such that* $\mathrm{ord}(p, \mathcal{B}) \leqslant \mathfrak{m}$ *for all p in X. Then $m(X) \leqslant \mathfrak{m}$.*

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DUKE UNIVERSITY

Accepté par la Rédaction le 23. 10. 1973