Hence we get by (13), (14), (15) and (18)

\[ h(x_n) - h(x_0) \geq h(y_n) - h(y_0) + \frac{1}{m \log k} \frac{h(y_n) - h(y_0)}{y_n - y_0}. \]

This contradicts the convexity of \( h(x) \).

References


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Extensions of metrization theorems to higher cardinality

by

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Abstract. Several well-known metrization theorems, stated in terms of the cardinal \( \kappa \), are extended to higher cardinals.

1. Introduction. In recent years significant progress has been made in the area of cardinal functions. A particularly notable achievement is Arhangel’ski’s solution [2] to Alexandroff’s problem. See Comfort’s paper [3] for an excellent survey of this and other results on cardinal invariants. The fundamental tract on cardinal functions is Juskič [13].

And, in spite of the brilliance of the Nagata–Smirnov–Bing solution of the “general metrization problem” in the early 1960’s, metrization theory continues to be an active area of research. (See [12] for a survey of metrization theorems from 1950 to 1972.) In this paper we explore the connection between these two exciting areas of general topology. Specifically, we consider the problem of generalizing metrization theorems so that they can be stated in terms of cardinal functions. (For a result of this type, see [11].)

In § 2 we introduce a cardinal function, called the metrizability degree, which reflects in some sense how metrizable a space is. The definition is based on a metrization theorem due to Bing [5]. We then give several characterizations of the metrizability degree for the class of regular spaces. These characterizations are based on other well-known metrization theorems. In § 3 we note the relationship between metrizability degree, uniform weight, and \( \omega_* \)-metrizability. Finally, in § 4 we extend a recent metrization theorem of Nagata [25] to higher cardinality.

Throughout this paper \( m \) and \( n \) denote cardinal numbers, \( a, \beta, \sigma, \tau, \) and \( \theta \) denote ordinal numbers, and \( |A| \) denotes the cardinality of a set \( A \). The set of positive integers is denoted by \( \mathbb{N} \), and \( j \) and \( k \) denote elements of \( \mathbb{N} \). The reader is referred to p. 49 of Nagata’s book [27] for a discussion of the various operations with covers used in this paper. (Note, however, that we use “st” instead of “S” when discussing the star of a set with
a base for $X$ such that $\mathcal{B} = \bigcup \{\mathcal{B}_t; t \in A\}$, where each $\mathcal{B}_t$ is a locally finite collection and $|\mathcal{B}| \leq m$. We may assume that $X \times \mathcal{B}_t$ for all $t \in A$.

For $p$ in $X$, $t$ in $A$ let

$$V_d(p) = (p \cup (\bigcup \{B; p \in \mathcal{B}_t\}) \cap (X - \bigcup \{B; B \in \mathcal{B}_t, t \notin E\})$$

Clearly $V_d(p)$ is an open set and $p \in V_d(p)$. For $t, s \in A$ let

$$\mathcal{F}_{ts} = (V_d(p) \cap V_d(p); p \in X).$$

Then $\{\mathcal{F}_{ts}; s, t \in A\}$ is the desired collection of open covers of $X$. For, let $p \in X$, let $R$ be a neighborhood of $p$. Choose $t \in A$ and $B$ in $\mathcal{B}$ such that $p \in B \subseteq R$, and then choose $s$ in $A$ and $U$ in $\mathcal{B}_s$ such that $p \in U \subseteq U \subseteq R$. Then $st(V_d(p), \mathcal{B}_s) \subseteq R$. For, let $V_d(p) \cap [V_d(q) \cap V_d(q)] \neq \emptyset$, and let us show that $[V_d(q) \cap V_d(q)] \subseteq R$. First note that $q \in B$. If $q \in B$, then $q \in U$ so $V_d(q) \subseteq X - B$. But $V_d(p) \subseteq U$, so $V_d(p) \cap V_d(q) = B$, a contradiction. Since $q \in B$, $V_d(q) \subseteq B$ and so $[V_d(q) \cap V_d(q)] \subseteq R$.

$\mathcal{S}(m) \Rightarrow \mathcal{M}(m)$: Let $\{\mathcal{F}_t; t \in A\}$ be a collection of open covers of $X$ with $|\mathcal{F}| \leq m$ such that for each $p \in X$ and each neighborhood $U$ of $p$, there is a neighborhood $V$ of $p$ such that $|st(p, \mathcal{F}_t) \subseteq V| < m$. Then $\mathcal{F}_t \subseteq \mathcal{F}$.

$\mathcal{M}(m) \Rightarrow B(m)$: Let $\{\mathcal{F}_t; t \in A\}$ be a collection of open covers of $X$ with $|\mathcal{F}| \leq m$ such that for each $p \in X$ and each $t \in A$, $\mathcal{F}_t \subseteq \mathcal{F}$.

Proof. We shall show that $B(m) \Rightarrow \mathcal{N}(m) \Rightarrow \mathcal{A}(m) \Rightarrow \mathcal{MM}(m) \Rightarrow B(m) \Rightarrow \mathcal{M}(m)$.

Note that the implications $B(m) \Rightarrow \mathcal{N}(m)$ and $\mathcal{M}(m) \Rightarrow \mathcal{N}(m)$ are obvious.

$\mathcal{S}(m) \Rightarrow \mathcal{M}(m)$: The technique used here is a combination of ideas due to Nagata (see p. 185 of [21]) and Morita (see p. 35 of [23]). Let $\mathcal{S}$ be

respect to a cover. We let $\nu, \mu, \xi, \zeta$ denote the following cardinal functions: weight, Lindelöf degree, density, cardinality, character, and pseudo-character. For definitions, see Juhász [13]. Unless otherwise stated, no separation axioms are assumed. However, paracompact and Hausdorff and $\pi$-spaces are always Hausdorff and $\pi$-spaces are always Tychonoff spaces.

2. Metrizability degree. The metrizability degree of a space $X$, denoted $m(X)$, is $\kappa$, $m$, if $\kappa$ is the smallest cardinal such that there is a base for $X$ which is the union of $\kappa$ discrete collections. It is clear that $m(X) \leq \omega(X)$ for any space $X$, and that a regular $\tau_1$ space $X$ is metrizable if and only if $m(X) = \kappa_1$. (See [5].) Moreover, the following basic result is easily proved.

Theorem 2.1. For any space $X$, $\omega(X) = m(X) - d(X) = m(X) - c(X) = m(X) - L(X)$.

Next we give several characterizations of the metrizability degree for the class of regular spaces. These characterizations are based on well-known metrization theorems. First we establish the following result.

Theorem 2.2. Let $X$ be a regular space, let $m$ be an infinite cardinal. The following six conditions are equivalent:

1. $X$ has a base which is the union of $\leq m$ discrete collections;
2. $m(X) \leq m$;
3. $\omega(X) \leq m$;
4. $\aleph(X) \leq m$;
5. $\mathcal{N}(m)$;
6. $\mathcal{M}(m)$.

Theorem 2.3. Let $X$ be a regular space, let $m$ be an infinite cardinal. The following six conditions are equivalent:

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6. $\mathcal{M}(m)$.

Theorem 2.4. Let $X$ be a regular space, let $m$ be an infinite cardinal. The following six conditions are equivalent:

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3. $\omega(X) \leq m$;
4. $\aleph(X) \leq m$;
5. $\mathcal{N}(m)$;
6. $\mathcal{M}(m)$.
of \( X \), and it is easy to check that for each \( p \in X \), \( \{ \text{st}(p, \mathcal{F}_t) : t \in \mathcal{A} \} \) is a "base" for \( p \).

\( N(m) = AS(m) \): Let \( \{ \mathcal{F}_t : t \in \mathcal{A} \} \) be a collection of closure preserving closed covers of \( X \) with \( |\mathcal{A}| \leq m \) such that for each \( p \in X \), \( \{ \text{st}(p, \mathcal{F}_t) : t \in \mathcal{A} \} \) is a "base" for \( p \). For each \( p \in X \), \( \mathcal{A} \) let

\[
V_p(p) = X - \bigcup \{ \mathcal{F} \in \mathcal{F}_t : p \notin \mathcal{F} \}.
\]

Note that \( V_p(p) \) is an open set containing \( p \). Moreover, the following three facts are easy to verify:

1. If \( V_p(p) \cap V_q(p) \neq \emptyset \), then \( q \in \text{st}(p, \mathcal{F}_t) \).
2. If \( V_p(p) \subseteq \text{st}(p, \mathcal{F}_t) \).
3. If \( q \in V_p(p) \), then \( V_q(p) \subseteq V_p(p) \).

For \( s, t \in \mathcal{A} \) let \( \mathcal{G}_{st} = \{ V_{p}(p) \cap V_{q}(p) : p \in X \} \). Then \( (\mathcal{G}_{st}, s, t \in \mathcal{A} \) is the desired collection of open covers of \( X \). For, let \( p \in X \) and let \( \mathcal{E} \) be a neighborhood of \( p \). Choose \( t \in \mathcal{A} \) such that \( \text{st}(p, \mathcal{F}_t) \subseteq \mathcal{E} \), and then choose \( s \in \mathcal{A} \) such that \( \text{st}(p, \mathcal{F}_s) \subseteq V_p(p) \). Then \( \text{st}(V_q(p), \mathcal{G}_{st}) \subseteq V_q(p) \). For, let \( V_p(p) \cap \text{st}(V_q(p), \mathcal{G}_{st}) \neq \emptyset \), and let us show that \( \text{st}(V_q(p), \mathcal{G}_{st}) \subseteq V_q(p) \). Now \( V_q(p) \cap V_q(p) \neq \emptyset \), and so by (1) \( q \in \text{st}(p, \mathcal{F}_t) \). Hence \( q \in V_p(p) \), and so by (2) \( V_q(p) \subseteq V_p(p) \). By (2) \( V_p(p) \subseteq \text{st}(p, \mathcal{F}_t) \), and so \( \mathcal{G}_{st} \subseteq \text{st}(p, \mathcal{F}_t) \). Hence \( \text{st}(V_q(p), \mathcal{G}_{st}) \subseteq V_q(p) \).

As a consequence of Theorem 2.2 we have the following characterizations of the metrizability degree.

**Theorem 2.3.** Let \( X \) be a regular space. Then

\[
\mu(X) = \kappa_{\mu} \min (m: X \text{ satisfies } NS(m))
\]

\[
= \kappa_{\mu} \min (m: X \text{ satisfies } AS(m))
\]

\[
= \kappa_{\mu} \min (m: X \text{ satisfies } MM(m))
\]

\[
= \kappa_{\mu} \min (m: X \text{ satisfies } M(m))
\]

\[
= \kappa_{\mu} \min (m: X \text{ satisfies } N(m)).
\]

3. Uniform weight and \( \omega_{s} \)-metrizability. In this section we note the relationship between metrizability degree, uniform weight, and \( \omega_{s} \)-metrizability. (Recall that the uniform weight of a completely regular space \( X \), denoted \( u(X) \), is \( \kappa_{\mu} m \), where \( m \) is the smallest cardinal which arises as the cardinality of a base for a uniformity which is compatible with the topology of \( X \).) In [39] (see also [30]), Mrówka essentially proves the following two results for a completely regular space \( X \): (1) if \( X \) is completely regular, then \( X \) has a base which is the union of \( \leq m \) locally finite collections; (2) if \( X \) is normal and has a base which is the union of \( \leq m \) locally finite collections, then \( u(X) \leq \kappa_{\mu} m \). As a consequence of Theorem 2.2, these two results can be restated as follows.

**Theorem 3.1.** (Mrówka). If \( X \) is completely regular, then \( u(X) \leq \kappa_{\mu} m \).

**Remark.** H. M. Shafar [39] has proved that \( u(X) \leq \kappa_{\mu} m \) for any completely regular space \( X \). This theorem can be obtained from Theorem 3.1 as follows: \( u(X) \leq \kappa_{\mu} m \).

In [30] Shu-tang proves that a regular \( T_{1} \) space \( X \) is \( \omega_{s} \)-metrizable if and only if it is \( \omega_{s} \)-additive and has a base which is the union of \( \leq \kappa_{\mu} m \) locally finite collections. In light of Theorem 2.2, this result can be restated as follows.

**Theorem 3.2.** (Shu-tang). Let \( X \) be a regular \( T_{1} \) space. Then \( X \) is \( \omega_{s} \)-metrizable if and only if \( X \) is \( \omega_{s} \)-additive and \( u(X) \leq \kappa_{\mu} m \).

4. Nagata's theorem. In 1970 Nagata [32] proved that every paracompact space with a point-countable separating open cover is metrizable. (Relevant definitions will be stated below.) This theorem has an interesting history. In 1982 Mićenko [18] proved that every compact space with a point-countable base has a countable base, and in 1988 Filipov [9] generalized this result by proving that every paracompact \( p \)-space with a point-countable base is metrizable. Parallel to these results, Snědker [32], in 1945, proved that every compact space with a \( G_{\delta} \)-diagonal has a countable base, and around 1965 Borges [6] and Okuyama [38] independently generalized this result by proving that every paracompact \( p \)-space with a \( G_{\delta} \)-diagonal is metrizable. Nagata's theorem generalizes the results of Filipov and Borges-Okuyama. In summary, Nagata's theorem, together with the results by Snědker, Mićenko, Borges-Okuyama, and Filipov which proceed it, represent a high-light in metrization theory.

In this section we extend Nagata's theorem to higher cardinality.

We begin with some definitions.

Let \( X \) be a set and let \( \mathcal{G} \) be a cover of \( X \). The cover \( \mathcal{G} \) is said to be separating if given distinct points \( p \) and \( q \) in \( X \), there is some \( G \in \mathcal{G} \) such that \( p \in G \) and \( q \notin G \). For \( p \in X \), the order of \( p \) with respect to \( \mathcal{G} \), denoted \( \text{ord}(p, \mathcal{G}) \), is the cardinality of the set \( \{ G \in \mathcal{G} : p \in G \} \). Suppose \( X \), and let \( \mathcal{G} \) be the smallest cardinal such that \( X \) has a separating open cover \( \mathcal{G} \) with \( \text{ord}(p, \mathcal{G}) \leq m \) for all \( p \in X \). (Since \( \mathcal{G} \) is \( T_{1} \), it is easy to see that \( m \) exists and \( m \leq \mu(X) \).) The **point separating weight** of \( X \), denoted \( \mu_{\text{psw}}(X) \), is \( m \). Note that for a \( T_{1} \) space \( X \), \( \mu_{\text{psw}}(X) = \kappa_{\mu} m \) if and only if \( X \) has a point-countable separating open cover.

Next we introduce a cardinal function which extends the notion of paracompactness to higher cardinality. Of the numerous characterizations of paracompactness upon which the definition could be based (see, e.g., [15], [16], [17]), the following seems most natural. The **paracompactness degree** of a space \( X \), denoted \( \mu_{\text{ps}}(X) \), is \( \kappa_{\mu} m \), where \( m \) is the...
smallest cardinal such that every open cover of $X$ has an open refinement which is the union of $< \alpha$ locally finite collections. Note that $p | \alpha(X) \leq L(X)$, $p | \alpha(X) \leq m(X)$, and that a regular $T_1$ space $X$ is paracompact if and only if $p | \alpha(X) = \aleph_1$ (see (15)).

In [11] the concept of a $p$-space [3] was extended to higher cardinals as follows. A collection $\{ \mathcal{O}_t : t \in A \}$ of open covers of a space $X$ is a paracompact for $X$ if the following holds: if $p \in G \subseteq \mathcal{O}_t$ for all $t \in A$, then (a) $C(p) = \bigcap G_t$ is compact; (b) $\{ \bigcap G_t : F \in \text{a finite subset of } A \}$ is a "base" for $C(p)$ in the sense that given any open set $\mathcal{O}$ containing $C(p)$, there is a finite subset $\mathcal{F}$ of $A$ such that $\bigcap G_{t} \subseteq \mathcal{F}$ (see 11) for a proof that every regular space has a paracompact for. For a regular space $X$, the paracompact degree of $X$, denoted $p(X)$, is $\aleph_0$, where $\aleph$ is the smallest cardinal such that $X$ has a paracompact for $X$ with $|A| = \aleph_0$. It follows from Theorem 2.2 that $p(X) \leq m(X)$ for any regular $T_1$ space $X$. (Use AS(m).)

The definition of a paracompact for $X$ is based on an internal characterization of $p$-spaces given by Burke [7]. From Burke's theorem it follows that a Tychonoff space $X$ is a $p$-space if and only if $p(X) = \aleph_0$. Moreover, Burke's technique can be used to prove the following result. (Note that (2) is Arhangel'skiǐ's original definition of a $p$-space extended to higher cardinals.)

**Theorem 4.3.** The following fact about paracompactness will be used in the proof of the main theorem.

**Proposition 4.4.** Let $X$ be a topological space, let $\{ \mathcal{O}_t : t \in A \}$ be a collection of open covers of $X$ with $|A| = \aleph_0$, and let $\mathcal{O}(p) = \bigcap G_t$. Then $C(p)$ is compact and $\bigcap G_t \subseteq C(p)$ is a "base" for $C(p)$.

**Proof.** Let $C(p) = \bigcap G_t$. Recall that $C(p)$ is compact and $\bigcap G_t$ is a "base" for $G(p)$. Since $\mathcal{O}(p)$ is closed and $\mathcal{O}(p) \subseteq C(p)$, it follows that $\mathcal{O}(p)$ is compact. Let $\mathcal{R}$ be an open such that $\mathcal{O}(p) \subseteq \mathcal{R}$. Now $E = G(p) \subseteq \mathcal{R}$ is compact and $\bigcap G_t \subseteq \mathcal{R}$ so there is a finite subset $\mathcal{F}$ of $A$ such that $\bigcap G_t \subseteq \mathcal{F}$. Let $W = \mathcal{R} \cup (\bigcup \mathcal{F}) = \mathcal{F}$. Then $W$ is open and $C(p) \subseteq W$ so there is a finite subset $\mathcal{F}$ of $A$ such that $\bigcap G_t \subseteq \mathcal{F}$. Let $C(p) \subseteq \mathcal{F}_t$.

**Corollary 4.5.** Let $X$ be a regular $T_1$ space. Then $\chi(X) = \gamma(X) = \rho(X)$.

**Proof.** Let $\mathcal{O}(X) : p(X) = m(X)$. Let $\{ \mathcal{O}_t : t \in A \}$ be a collection of open sets such that $p \in \mathcal{O}_t$ for all $t \in A$, then (a) $C(p) = \bigcap G_t$ is compact; (b) $\{ \bigcap G_t : F \in \text{a finite subset of } A \}$ is a "base" for $C(p)$ in the sense that given any open set $\mathcal{O}$ containing $C(p)$, there is a finite subset $\mathcal{F}$ of $A$ such that $\bigcap G_{t} \subseteq \mathcal{F}$.

Finally, we need a set-theoretic result due to Miščenko [18]. This result was abstracted by Filippov [9] from Miščenko's proof that every compact space with a point-countable base has a countable base. It plays an important role in the proofs of the above mentioned metrization theorems of Filippov and Nagata.

**Miščenko's Lemma.** Let $X$ be a set, let $m$ be an infinite cardinal, and let $\mathcal{F}$ be a collection of subsets of $X$ such that $\mathcal{F}(X) = m(X)$ and $\mathcal{F}$ is a subfamily of $\mathcal{F}$.

**Theorem 4.6.** Let $\mathcal{F}$ be a regular $T_1$ space. Then $m(X) = p(X) : p(x) = \rho(X)$.

**Proof.** Clearly $p(X) : p(x) = \rho(X) \leq m(X)$. Suppose, then, that $p(X) = \rho(X) = m(X)$, and let us construct a base $\mathcal{F}$ for $X$ which is the union of $< \alpha$ locally finite collections. Let $\mathcal{F}(m)$ be all finite subfamilies of $\mathcal{F}$ for $\mathcal{F}(m)$. Let $\mathcal{F}$ be a separating open cover of $X$ such that $\mathcal{F}(X) = m(X)$ for all $x \in X$. We may assume that $X \in S$, and hence for any subset $\mathcal{H}$ of $X$ there is at least one finite minimal cover of $\mathcal{H}$ by elements of $S$, namely $X$.

Let $\{ \mathcal{O}_t : t \in A \}$ be a collection of open sets such that $\mathcal{O}(X) = m(X)$, and for each $t \in A$ let $\mathcal{F}_t$ be an open refinement of $\mathcal{O}_t$ such that $\mathcal{X}_t = \bigcup \mathcal{O}( \mathcal{X}_t , \mathcal{F}_t )$. Let $X = \bigcup \mathcal{O}(X)$, where each $\mathcal{X}_t, \mathcal{F}_t$ is a locally finite collection. Let $\mathcal{E}$ be all finite subfamilies of $\mathcal{F}$, and for each $t \in A$, $\mathcal{X}_t \in S$. Note that $X_t$ is a locally finite open collection in $X$.

**Observation.** Let $\mathcal{O} \in T_1$. For each $a < \alpha$ choose $a_\alpha < a$ and $\mathcal{H}_a$ in $\mathcal{X}(a, \mathcal{F}_t)$ such that $p \in \mathcal{H}_a$. By Proposition 4.5, $\mathcal{H}(a) = \bigcap \mathcal{H}_a$ is compact and $\bigcap \mathcal{H}_a$ is a "base" for $\mathcal{O}(p)$.

The construction of the required base for $X$ is accomplished in four steps.

**Step 1.** $X$ has a separating closed cover which is the union of $< \alpha$ locally finite collections. Fix $\gamma$ in $\mathcal{F}$. For each $\mathcal{K}_\gamma$ let $\{ b(\gamma , \mathcal{K}_\gamma , \gamma )$.
0 ≤ σ < m) be all finite minimal covers of \( K \) by elements of \( S \) (use Mīshenko's lemma), and for each \( σ < m \) let \( (K, K') = (K, K' : K \in \mathcal{K}, S = \emptyset \) or \( S \in \mathcal{S}(K, K') \). Clearly \( (v, o) \) is a locally finite closed collection in \( X \).

Now let \( \mathcal{C} = \{ (\mathcal{C}(v, o) : v \neq \emptyset, 0 ≤ σ < m \}. \) Then \( \mathcal{C} \) is the union of \( (v, o) \) locally finite closed collections, and so to complete the proof it remains to show \( \mathcal{C} \) separating. Let \( p \) and \( q \) be distinct points of \( X \). For each \( K \in \mathcal{K} \) choose \( p_{\mathcal{K} \in \mathcal{K}} \) and \( H_{\mathcal{K} \in \mathcal{K}} \) such that \( p \in H_{\mathcal{K} \in \mathcal{K}} \) and let \( O(p) = \bigcap H_{\mathcal{K} \in \mathcal{K}} \). Choose \( S_{\mathcal{K} \in \mathcal{K}} \) in \( S \) such that \( q \in S_{\mathcal{K} \in \mathcal{K}} \), then \( S_{\mathcal{K} \in \mathcal{K}} \) be a finite subcollection of \( S \) which covers \( O(p) \) such that \( q \neq S_{\mathcal{K} \in \mathcal{K}} \). Replace \( F \) in \( \mathcal{F}(m) \) such that \( \bigcap H_{\mathcal{K} \in \mathcal{K}} \subseteq S_{\mathcal{K} \in \mathcal{K}} \), and let \( K = \bigcap H_{\mathcal{K} \in \mathcal{K}} \). Note that \( K \in \mathcal{K} \). Now \( S_{\mathcal{K} \in \mathcal{K}} \) covers \( K \), so some finite minimal subcollection of \( S_{\mathcal{K} \in \mathcal{K}} \) covers \( K \), say \( (\gamma, K, τ) \). If \( q \neq K \), then \( S_{\mathcal{K} \in \mathcal{K}} \) is also a covering of \( K \) which contains \( p \) and not \( q \). Assume, then, that \( q = K \). Then \( S_{\mathcal{K} \in \mathcal{K}} \) is an element of \( (\gamma, o) \) which contains \( p \) and not \( q \).

Step 2. Every open subset of \( X \) is the union of \( \leq m \) closed sets. Clearly it suffices to show that \( X \) has a net \( N \) which is the union of \( \leq m \) locally finite closed collections. (Recall that \( N \) is a net for \( X \) if given any point \( p \) in \( X \) and any neighborhood \( E \) of \( p \), there is some \( N \) in \( N \) such that \( p \in N \subseteq E \). See (1)). Let \( \mathcal{C} \) be a separating closed cover of \( X \) such that \( \mathcal{C} = \{ (\mathcal{C}(v, o) : v \neq \emptyset, 0 ≤ σ < m \} \). Each \( \mathcal{C}(v, o) \) is a locally finite collection. We may assume that each \( \mathcal{C}(v, o) \) is closed under finite intersections. For each \( E \in \mathcal{F}(m) \) let \( \mathcal{W}(E) = \bigcap (\mathcal{C}(v, o) : v \neq \emptyset) \). \( E \) is a locally finite collection. For each \( v \in \mathcal{V} \), \( v \in \mathcal{E}(m) \) let \( \mathcal{W}(v) \subseteq \mathcal{W}(E) \) and let \( \mathcal{N}(v, o) \) = \( \mathcal{W}(v) \cap \mathcal{E}(m) \). \( \mathcal{N}(v, o) \) is the union of \( \leq m \) locally finite closed collections. To see that \( \mathcal{N}(v, o) \) is a net, let \( p \) be a point in \( X \) and let \( E \) be an open neighborhood of \( p \). For each \( e \in \mathcal{E}(m) \) choose \( \mathcal{C}(v, o) \) and \( H_{\mathcal{C}(v, o)} \) such that \( p \in H_{\mathcal{C}(v, o)} \) and let \( O(p) = \bigcap H_{\mathcal{C}(v, o)} \). Then \( \mathcal{C}(v, o) \in \mathcal{N}(v, o) \), and so \( \mathcal{N}(v, o) \subseteq \mathcal{E}(m) \) is an element of \( \mathcal{N}(v, o) \) such that \( p \in H_{\mathcal{C}(v, o)} \).

Step 3. There is an open cover \( \mathcal{W} \) of \( X \) which is the union of \( \leq m \) locally finite open collection such that if \( p \) and \( q \) are any two distinct points of \( X \), then there exists \( W \in \mathcal{W} \) such that \( p \in W, q \in W \). Fix \( y \) in \( Y \). For each \( K \in \mathcal{K} \), let \( (\gamma, K, o) = (K, K' : K \in \mathcal{K}, S = \emptyset \) or \( S \in \mathcal{S}(K, K') \). Clearly \( (v, o) \) is a point finite open collection in \( X \). For each \( K \in \mathcal{K} \) let \( \mathcal{W}(v, o, k) = (K, K', V : V = \bigcap (v, o, k) \subseteq \mathcal{E}(m) \) is the intersection of exactly \( k \) distinct elements of \( \gamma, o \), and let \( \mathcal{W}(v, o, k) = \bigcup (\mathcal{C}(v, o, k) \subseteq \mathcal{E}(m) \) is a closed subset. Let \( (\gamma, K, o) = (K, K', V : V = \bigcap (v, o, k) \subseteq \mathcal{E}(m) \) be an open cover of \( X \), so there is an open cover \( \mathcal{W}(v, o, k, r) \subseteq \mathcal{E}(m) \) such that the closure of each element of \( \mathcal{W}(v, o, k, r) \subseteq \mathcal{E}(m) \) is contained in some element of \( \mathcal{W}(v, o, k, r) \subseteq \mathcal{E}(m) \) and \( \mathcal{W}(v, o, k, r) = \bigcup (\mathcal{W}(v, o, k, r) \subseteq \mathcal{E}(m) \) where each \( \mathcal{W}(v, o, k, r) \subseteq \mathcal{E}(m) \) is a locally finite collection.

Now let \( \mathcal{W} = \bigcup (\mathcal{W}(v, o, k, r) \subseteq \mathcal{E}(m) \) be an open collection in \( X \), which is the union of \( \leq m \) locally finite collections. Let \( p \) and \( q \) be distinct points of \( X \). For each \( \sigma < m \) pick \( \beta_{\sigma} < m \) and \( H_{\sigma} \) in \( \mathcal{X}(\sigma, \beta_{\sigma}) \) such that \( p \in H_{\sigma} \), and let \( \mathcal{P}(p) = \bigcap H_{\sigma} \). Choose \( S_{\sigma} \in S \) such that \( p \in S_{\sigma} \), let \( S_{\sigma} \subseteq S_{\sigma} \), and let \( S_{\sigma} = (S_{\sigma}, S_{\sigma} \subseteq S_{\sigma}, S_{\sigma} \subseteq S_{\sigma}) \) be a finite subcollection of \( S \) which covers \( \mathcal{P}(p) \) such that \( q \neq S_{\sigma} \). Clearly \( \mathcal{W} \) is an open collection in \( X \) which is the union of \( \leq m \) locally finite collections.
For each $\alpha < m$ choose $\beta < m$ and $H_\alpha$ in $\mathfrak{B}(\alpha, \beta)$ such that $p \in H_\alpha$, and let $N(p) = \bigcap H_\alpha$. Let $Z = N(p) - R$ and assume $Z \neq \emptyset$. (The case $Z = \emptyset$ is easy.) Then there exists $F$ in $\mathfrak{B}(m)$, $W_\alpha$ in $\mathfrak{B}$, for each $\alpha < m$, and an open set $U$ such that $p \in \bigcap W_\alpha \subseteq U$ and $U \cap F = Z$. Choose $y = \{y_\alpha : \alpha < \delta\}$, and let $X = \bigcap H_\alpha$. Then $B = X \cap F$ is an element of $\mathfrak{B}(y, F)$ such that $p \in B \subseteq R$.

**Corollary 4.5** (Hodel [11]). Let $X$ be a regular $T_1$ space. Then $w(X) = X(\{X\}, p(x), \text{paw}(X))$.

**Corollary 4.6** (Nagata [20]). Let $X$ be a regular $T_1$ space. Then $X$ is metrizable if and only if it is a paracompact $T_1$-space with a point-countable separating open cover.

**Corollary 4.7** (Filippov [9]). Let $m$ be an infinite cardinal, let $L$ be a paracompact $T_1$-space having a base $\mathfrak{B}$ such that $|\text{ord}(\mathfrak{B}, p)| < m$ for all $p$ in $L$. Then $m(X) < m$.

**References**


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