Note on the existence of convex iteration groups

by

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Abstract. Let f be a real function fulfilling the following conditions:

(H) $f(x)$ is convex, differentiable and strictly increasing in an interval (a, b), $-\infty \leqslant a < b \leqslant +\infty$, $a < f(x) < x$ in (a, b), $f'(x) > 0$ and $\lim_{x\to a+} f'(x) = s \neq 0$.

If a function f fulfils the hypothesis (H), then there exists the function

$$h(x) = \lim_{u\to 0} \frac{f^u(x) - x}{u}$$

where $\{f^u\}$ is the principal iteration group of f.

THEOREM. *Let a function f fulfil hypothesis* (H). *Then f has a convex iteration group if and only if the function h is convex in (a, b).*

In this paper we give a certain condition for the existence of the convex iteration group of a function f. (The definition of a convex iteration group may be found in [9].)

Let f be a real function fulfilling the following conditions:

(H) $f(x)$ is convex, differentiable and strictly increasing in an interval (a, b), $-\infty \leqslant a < b \leqslant +\infty$, $a < f(x) < x$ in (a, b),

(1) $$f'(x) > 0$$

and

(2) $$\lim_{x\to a+} f'(x) = s \neq 0 .$$

Let us note that the differentiability condition is not restrictive, for, as has been proved in [9], if a function f fulfils the remaining conditions of (H) and has a convex iteration group, then it must be differentiable in (a, b).

Conditions (H) imply that, in the case $0 < s < 1$, there exists the principal solution σ of the Schröder equation

(3) $$\sigma[f(x)] = s\sigma(x) .$$

This solution is given by the formula

$$\sigma(x) = \lim_{n\to\infty}\frac{f^n(x)-a}{f^n(x_0)-a} \quad \text{if} \quad a > -\infty ,$$

or

$$\sigma(x) = \lim_{n\to\infty}\frac{f^n(x)}{f^n(x_0)} \quad \text{if} \quad a = -\infty ,$$

where $x_0 \in (a, b)$ is fixed, and $\sigma(x)$ is a strictly increasing and convex function in (a, b) for $a > -\infty$ and strictly decreasing and concave in (a, b) for $a = -\infty$ (cf. [6] and [7]).

Similarly, if $s = 1$ (then necessarily $a = -\infty$), there exists the principal solution α of the Abel equation

(4) $$\alpha[f(x)] = \alpha(x)+1 .$$

This solution is given by the formula

$$\alpha(x) = \lim_{n\to\infty}\frac{f^n(x)-f^n(x_0)}{f^{n+1}(x_0)-f^n(x_0)} ,$$

where $x_0 \in (a, b)$ is fixed, and $\alpha(x)$ is a strictly decreasing and concave function (cf. [5]).

We shall prove that σ resp. α is differentiable in (a, b). We carry out the proof for σ; the proof for α is quite analogous.

The left derivative $\sigma_l'(x)$ and the right derivative $\sigma_r'(x)$ of $\sigma(x)$ are monotonic solutions of the functional equation

(5) $$\psi[f(x)] = \frac{s}{f'(x)}\psi(x)$$

in (a, b), where we have in view of (2)

$$\lim_{x\to a+} s/f'(x) = 1 .$$

Therefore (cf. [4]) there is a constant k such that

$$\sigma_l'(x) = k\sigma_r'(x) \quad \text{in} \quad (a, b) ,$$

and, since σ is differentiable almost everywhere in (a, b), the constant must be 1. This proves the differentiability of σ. Moreover, since σ is strictly monotonic and the sign of σ' is constant, and σ' is monotonic in (a, b), we must have

(6) $$\sigma'(x) \neq 0 \quad \text{in} \quad (a, b);$$

and a similar argument shows also that

(7) $$\alpha'(x) \neq 0 \quad \text{in} \quad (a, b) .$$

The principal solution of the Schröder equation (3) resp. Abel equation (4) generates the principal iteration group $\{f^u\}$ of f:

(8) $$f^u(x) = \sigma^{-1}(s^u\sigma(x)) ,$$

resp.

(9) $$f^u(x) = \alpha^{-1}(u+\alpha(x)) .$$

It follows from the differentiability of σ resp. α and from condition (6) resp. (7) that there exists the function (cf. [3])

(10) $$h(x) = \frac{\partial}{\partial u} f^u(x)|_{u=0} = \lim_{u\to 0}\frac{f^u(x)-x}{u} ,$$

where f^u is given by (8) resp. (9).

The purpose of the present note is to prove the following

THEOREM. *Let a function f fulfil hypothesis* (H). *Then f has a convex iteration group if and only if the function h given by* (10) *is convex in* (a, b).

Proof. 1. Necessity. Let the function f have a convex iteration group $\{f^u(x)\}$. As has been proved in [8], $\{f^u(x)\}$ must be then the principal iteration group of f, i.e., it must be given by formula (8) (if $0 < s < 1$), or (9) (if $s = 1$). Hence, in view of (10),

$$h(x) = \lim_{n\to\infty}\frac{f^{1/n}(x)-x}{1/n} ,$$

and h as the limit of a sequence of convex functions, is convex.

2. Sufficiency. We shall show that the principal iteration group $\{f^u(x)\}$ of f is convex. For an indirect proof suppose that this is not true, i.e., there exists a $u > 0$ such that $f^u(x)$ is not convex in (a, b). Then there exist a constant $k > 1$ and points $x_1, y_1, z_1 \in (a, b)$, $x_1 < y_1 \neq z_1 \neq x_1$, such that

(11) $$\frac{f^u(x_1)-f^u(z_1)}{x_1-z_1} > k\frac{f^u(y_1)-f^u(z_1)}{y_1-z_1}$$

(cf. [2]). Since

$$\frac{f^u(x)-f^u(z_1)}{x-z_1} = \frac{f^{u/2}[f^{u/2}(x)]-f^{u/2}[f^{u/2}(z_1)]}{f^{u/2}(x)-f^{u/2}(z_1)}\cdot\frac{f^{u/2}(x)-f^{u/2}(z_1)}{x-z_1} ,$$

we obtain by (11)

$$\frac{f^{u/2}(x_2)-f^{u/2}(z_2)}{x_2-z_2} > k^{1/2}\frac{f^{u/2}(y_2)-f^{u/2}(z_2)}{y_2-z_2},$$

where either $x_2=x_1$, $y_2=y_1$, $z_2=z_1$, or $x_2=f^{u/2}(x_1)$, $y_2=f^{u/2}(y_1)$, $z_2=f^{u/2}(z_1)$.

Continuing this procedure we arrive at a sequence

$$u_n = u\sum_{i=1}^{n} 2^{-i}c_i,$$

where $c_i=0$ or 1 (in particular, $c_1=0$), such that

$$\frac{f^{2^{-n}u}(x_n)-f^{2^{-n}u}(z_n)}{x_n-z_n} > k^{2^{-n}}\frac{f^{2^{-n}u}(y_n)-f^{2^{-n}u}(z_n)}{y_n-z_n} \tag{12}$$

for

$$x_n=f^{u_n}(x_1), \quad y_n=f^{u_n}(y_1), \quad z_n=f^{u_n}(z_1).$$

The sequence u_n converges to the limit

$$u_0 = u\sum_{i=1}^{\infty} 2^{-i}c_i,$$

and since $f^u(x)$ is a continuous and strictly increasing function of u ([1], [9]; cf. also formulae (8) and (9)), there exist the limits

$$x_0=\lim_{n\to\infty}x_n=f^{u_0}(x_1), \quad y_0=\lim_{n\to\infty}y_n=f^{u_0}(y_1), \quad z_0=\lim_{n\to\infty}z_n=f^{u_0}(z_1). \tag{13}$$

Moreover, $f^{u_0}(x)$ is a strictly increasing function of x, whence

$$x_0=f^{u_0}(x_1)<f^{u_0}(y_1)=y_0 \tag{14}$$

and

$$x_0=f^{u_0}(x_1)\neq f^{u_0}(z_1)=z_0, \quad y_0=f^{u_0}(y_1)\neq f^{u_0}(z_1)=z_0. \tag{15}$$

Now we must distinguish two cases according to the value of s.

If $0<s<1$, the iteration group $f^u(x)$ is given by (8), whence

$$\frac{f^{2^{-n}u}(x)-x}{2^{-n}u} = \psi_n(x)\,\sigma(x)\,\frac{s^{2^{-n}u}-1}{2^{-n}u}, \tag{16}$$

where

$$\psi_n(x) = \frac{\sigma^{-1}[s^{2^{-n}u}\sigma(x)]-\sigma^{-1}[\sigma(x)]}{\sigma(x)(s^{2^{-n}u}-1)}.$$

Since $\sigma^{-1}(x)$ is a concave function, $\sigma(x)>0$ and $0<s<1$, the sequence $\psi_n(x)$ is decreasing. This sequence converges to the continuous function

$$(\sigma^{-1})'[\sigma(x)] = \frac{1}{\sigma'(x)} = \frac{h(x)}{\sigma(x)}\cdot\frac{1}{\ln s}.$$

The function h is continuous, since it is convex. Also all the functions $\psi_n(x)$ are continuous. By Dini's theorem the convergence of the sequence ψ_n is uniform on every compact subset of (a, b). Since the sequence $(s^{2^{-n}u}-1)/2^{-n}u$ tends to a negative limit, we infer by (16) that the sequence

$$\frac{f^{2^{-n}u}(x)-x}{2^{-n}u} \tag{17}$$

is uniformly convergent on every compact subset of (a, b).

If $s=1$, then $f^u(x)$ is given by (9), whence

$$\frac{f^{2^{-n}u}(x)-x}{2^{-n}u} = \frac{\alpha^{-1}[2^{-n}u+\alpha(x)]-\alpha^{-1}[\alpha(x)]}{2^{-n}u}.$$

Since α^{-1} is concave, the above sequence is an increasing sequence of continuous functions. This sequence tends to the continuous limit $h(x)$. Therefore sequence (17) is uniformly convergent on every compact subset of (a, b).

We have established the uniform convergence of sequence (17) on compact subsets of (a, b), independently of the value of s. This implies that

$$\begin{cases} \lim\limits_{n\to\infty}\dfrac{f^{2^{-n}u}(x_n)-x_n}{2^{-n}u} = h(x_0), \\ \lim\limits_{n\to\infty}\dfrac{f^{2^{-n}u}(y_n)-y_n}{2^{-n}u} = h(y_0), \\ \lim\limits_{n\to\infty}\dfrac{f^{2^{-n}u}(z_n)-z_n}{2^{-n}u} = h(z_0). \end{cases} \tag{18}$$

Inequality (12) may be written as

$$\frac{\dfrac{f^{2^{-n}u}(x_n)-x_n}{2^{-n}u}-\dfrac{f^{2^{-n}u}(z_n)-z_n}{2^{-n}u}}{x_n-z_n}$$

$$< k^{2^{-n}}\frac{\dfrac{f^{2^{-n}u}(y_n)-y_n}{2^{-n}u}-\dfrac{f^{2^{-n}u}(z_n)-z_n}{2^{-n}u}}{y_n-z_n}+\frac{k^{2^{-n}}-1}{2^{-n}}\cdot\frac{1}{u}.$$

Hence we get by (13), (14), (15) and (18)

$$\frac{h(x_0)-h(z_0)}{x_0-z_0} \geqslant \frac{h(y_0)-h(z_0)}{y_0-z_0}+\frac{1}{u}\log k > \frac{h(y_0)-h(z_0)}{y_0-z_0}.$$

This contradicts the convexity of $h(x)$.

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Extensions of metrization theorems to higher cardinality

by

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Abstract. Several well-known metrization theorems, stated in terms of the cardinal $\aleph_0$, are extended to higher cardinals.

1. Introduction. In recent years significant progress has been made in the area of cardinal functions. (A particularly notable achievement is Arhangel'skiĭ's solution [2] to Alexandroff's problem. See Comfort's paper [8] for an excellent survey of this and other results on cardinal invariants. The fundamental tract on cardinal functions is Jusász [13].) And, in spite of the brilliance of the Nagata–Smirnov–Bing solution of the "general metrization problem" in the early 1950's, metrization theory continues to be an active area of research. (See [12] for a survey of metrization theorems from 1950 to 1972.) In this paper we explore the connection between these two exciting areas of general topology. Specifically, we consider the problem of generalizing metrization theorems so that they can be stated in terms of cardinal functions. (For a result of this type, see [11].)

In § 2 we introduce a cardinal function, called the *metrizability degree*, which reflects in some sense how metrizable a space is. The definition is based on a metrization theorem due to Bing [5]. We then give several characterizations of the metrizability degree for the class of regular spaces. These characterizations are based on other well-known metrization theorems. In § 3 we note the relationship between metrizability degree, uniform weight, and ω_μ-metrizability. Finally, in § 4 we extend a recent metrization theorem of Nagata [25] to higher cardinality.

Throughout this paper $\mathfrak{m}$ and $\mathfrak{n}$ denote cardinal numbers, α, β, σ, τ, and ϱ denote ordinal numbers, and $|A|$ denotes the cardinality of a set A. The set of positive integers is denoted by $\boldsymbol{N}$, and j and k denote elements of $\boldsymbol{N}$. The reader is referred to p. 49 of Nagata's book [27] for a discussion of the various operations with covers used in this paper. (Note, however, that we use "st" instead of "S" when discussing the star of a set with