The Borsuk homotopy extension theorem without the binormality condition
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Abstract. Borsuk's homotopy extension theorem states that if $C$ is a closed subset of a binormal space $X$, then a continuous function $\varphi$ from $C \times I$ into any absolute neighborhood retract can be extended to all of $X \times I$. In this paper we prove that the binormality condition on $X$ can be weakened to just normality in the case where the range of $\varphi$ is a compact, Hausdorff or metrizable absolute neighborhood retract.

In 1937 Borsuk [1] proved a homotopy extension theorem, one generalization of which is the following: If $C$ is a closed subset of a binormal space $X$, $A$ is a compact Hausdorff absolute neighborhood retract, and $\varphi: C \times I \cup X \times \{0\} \rightarrow A$ is continuous, then there exists a continuous function $\Phi: X \times I \rightarrow A$ extending $\varphi$. A binormal space is a normal space whose product with an interval is also normal. For many years it was hoped that all normal, Hausdorff spaces were binormal. All hope finally had to be abandoned, however, when Mary Ellen Rudin [3], [4] produced examples of normal, Hausdorff spaces which are not binormal. In this paper we show that Borsuk's homotopy extension theorem nevertheless remains true without the binormality condition.

I am indebted to Mary Ellen Rudin for her mathematical help and her great encouragement.

Borsuk's theorem follows from the extension theorem below:

THEOREM 1. If $C$ is a closed subset of a normal space $X$ and $\varphi: C \times I \cup X \times \{0\} \rightarrow [-1, 1]$ is a continuous function, then there exists a continuous function $\Phi: X \times I \rightarrow [-1, 1]$ extending $\varphi$.

Proof. We imitate a proof of the Tietze extension theorem by producing a sequence of continuous functions, $(g_k)_{k=0}^\infty$, with

$g_k: X \times I \rightarrow \left(-\frac{1}{k}p^n, \frac{1}{k}p^n\right)$

such that

$\left(\varphi - \sum_{k=0}^\infty g_k\right)^{-1}\left((-\frac{1}{k}p^n, -\frac{1}{k}p^n)\right) \subseteq g_k^{-1}\left(-\frac{1}{k}p^n\right)$
and

\[
\{ g - \sum_{\gamma \in \mathcal{C}} g_{\gamma} \} \leq \{ g_{\gamma} \} \subset \mathcal{C} \leq \{ g_{\gamma} \}.
\]

Then \( \Phi = \sum_{\gamma} g_{\gamma} \) is continuous and extends \( \varphi \).

The theorem follows after inductively constructing the \( g_{\gamma} \)'s. After appropriate modifications of the range, each \( g_{\gamma} \) is obtained by an application of the following lemma to \( \varphi - \sum_{\gamma} g_{\gamma} \).

**Lemma.** If \( C \) is a closed subset of a normal space \( X \), and \( f : C \times I \to \mathbb{R} \) is continuous, then there is a continuous extension \( g : X \times I \to \mathbb{R} \) such that \( f^{-1}([-1, 1]) \subset g^{-1}(0) \) and \( f^{-1}([-1, 1]) \subset g^{-1}(1) \).

The lemma is proved in three steps.

**Step 1.** If \( H \) and \( K \) are subsets of \( X \times I \) with disjoint closures such that for every point \( x \in X \), \( \{ x \} \times I \cap H \) and \( \{ x \} \times I \cap K \) are relatively open in \( \{ x \} \times I \), then \( H \) and \( K \) are disjoint open sets in \( X \times I \).

**Proof of Step 1.** We produce a countable cover of \( H \) (resp. \( K \)) by open sets whose closures each miss \( K \) (resp. \( H \)). Let \( (B_{\gamma})_{\gamma} \) be a basis for \( I \). Let \( H_t := \{ (x, t) \in H \mid t \in B_t \} \) and \( K_t := \{ (x, t) \in K \mid t \in B_t \} \). Let \( H^*_t \) and \( K^*_t \) be the projections of \( H_t \) and \( K_t \) respectively into the \( x \)-coordinate. Note that \( H^*_t \) and \( K^*_t \) are closed subsets of \( X \).

Since \( B_t \) is a basis and \( H \cap K = \emptyset \), for each \( x \neq H^*_t \) and \( y \neq K^*_t \), there exist open subsets of \( X \), \( U_x \) and \( V_y \), such that \( x \in U_x \), \( y \in V_y \), \( x \times I \cap H \), \( x \times I \cap K \) are relatively open in \( \{ x \} \times I \), and \( U_x \cup V_y \cup T_x \cap I \cap B_t = \emptyset \). Let \( U_t := \bigcup U_x \) and \( V_t := \bigcup V_y \).

By Step 1, we can find open sets \( U' \) and \( V' \) such that \( f^{-1}([-1, 1]) \cap \{ x \} \times I \cap U' \cap V' = \emptyset \) and \( f^{-1}([-1, 1]) \cap \{ x \} \times I \cap U' \cap V' = \emptyset \). We may assume that \( U' \cup f^{-1}(1, 1) = \emptyset \) and \( V' \cup f^{-1}(-1, -1) = \emptyset \). Note that \( U' \cap V' \subseteq I \times I \).

Since \( U' \cap V' \) is closed and in the complement of \( C \times I \), there is an open set \( W \) containing \( C \) such that \( W \times I \cap (U' \cap V') = \emptyset \). If \( C \subseteq W \), then the construction is normal, there is an open set \( W_1 \) such that \( C \subseteq I 

Let \( U' = U' \cap (W_1 \times I) \) and \( V' = V' \cap (W_1 \times I) \). Then \( U' \cap V' \cap f^{-1}([-1, 1]) = \emptyset \times I \) from \( f^{-1}(1, 1) \cap \emptyset \times I \) in the desired manner.

Now, let \( H \) be a closed subset of \( X \) such that \( H \times \{ 0 \} = f^{-1}([-1, 1]) \cap \emptyset \times I \) and \( H \times \{ 1 \} = f^{-1}([-1, 1]) \cap \emptyset \times I \). Note that \( H \cap \emptyset \subseteq X \times \emptyset \).

We can separate \( H \) from \( f^{-1}([-1, 1]) \cap \emptyset \times \{ 0 \} \) by \( C \) and \( K \) from \( f^{-1}([-1, 1]) \cap \emptyset \times \{ 1 \} \) by \( C \) and obtain open subsets \( S_t \), \( T_t \), \( T \times I \cap (U' \cap V') = \emptyset \). Let \( U = U' \cap (S_t \times I) \cap T \times I \cap V = V' \cap (S_t \times I) \cap T \times I \cap V \cap f^{-1}([-1, 1]) \cap \emptyset \times I \).

**Proof of Step 2.**. Then there is a continuous function \( g : X \times I \to \mathbb{R} \) with \( g(0) = 0 \) and \( g(1) = 1 \).

Let \( (B_{\gamma})_{\gamma} \) be the collection of all finite unions of basic open sets of \( I \). We define two open covers of \( X \).

**Step 3.** If \( H \) and \( K \) are closed sets and \( U \) and \( V \) are open sets in \( X \times I \) with \( H \subseteq U \subseteq K \times I \subseteq V \), and \( U \cap V = \emptyset \), then there is a continuous function \( g : X \times I \to \{ 0, 1 \} \) with \( g(0) = 0 \) and \( g(1) = 1 \).

**Proof of Step 3:**. The proof of this step involves the definition of a countable cover for \( X \) which has a locally finite refinement and uses some ideas of countable paracompactness found in [3].
Recall that $T_i \subset W_{k_i}$ for some pair $(k_i, 1)$. By normality of $I$, there exists a continuous function $h_i : I \to [0, 1)$ such that $B_\theta \cup H(i) = (0, 1)$ and $B_\theta \cup H(i) = (0, 1)$. Let $h_i : X \times I \to [0, 1)$ be defined by $h_i(x, y) = h_i(x, y)$. Note that $h_i$ is continuous, $h_i(x, y) \in H(i)$, and $X \times I \setminus (T_i \times (I - B_\theta)) \subset h_i(x, y) = 1$. We are now prepared to define the function $g : X \times I \to [0, 1)$ required in Step 3. Let $g(x, y) = \min \{ \sum_{i \in E} h_i(x, y), 1 \}$.

Since the $h_i$'s had locally finite supports, $g$ is continuous. For each $i$, $K \subset L_i = 1$; hence $K \subset g^{-1}(1)$. For every point $(x, y) \in H(i)$ there is an $i$ such that $x \in E_i$, hence $h_i(x, y) = 1$. So $H \subset g^{-1}(1)$. Step 3 is therefore proved.

Steps 2 and 3 prove the lemma which in turn proves the theorem.

A consequence of this extension theorem is Borsuk's homotopy extension theorem without the binormality condition. This follows:

**Theorem 2.** If $C$ is a closed subset of a normal space $X$, $A$ is any compact Hausdorff absolute neighborhood retract, and $\phi : C \times I \to X \times (0, 1)$ is continuous, then there exists a continuous function $\Phi : X \times I \to A$ extending $\phi$.

**Proof of Theorem 2.** $A$ can be embedded as a closed subset of a product of intervals which is a retract of an open set in the product. By Theorem 1, $\phi$ extends to a continuous function from $X \times I$ into this product of intervals. Hence $\phi$ can be extended to a continuous function $\Phi : W \times I \to X \times (0, 1)$ where $W$ is open in $X$ and $C \subset W$. By normality of $X$ there exists a continuous function $g : X \to [0, 1]$ such that $X - W \subset g^{-1}(0)$ and $C \subset g^{-1}(1)$. We define the desired $\Phi$ by $\Phi(x, y) = \Phi(x, g(x))$.

It should be noted that we have actually proved somewhat stronger versions of Theorems 1 and 2, namely:

**Theorem 1'.** If $C$ is a closed subset of a normal space $X$, $A$ is any compact metric space, and $\phi : C \times M \to X \times (0, 1)$ is continuous, then there exists a continuous function $\Phi : X \times M \to [0, 1]$ extending $\phi$.

**Theorem 2'.** If $C$ is a closed subset of a compact space $X$, $A$ is any retract of an open subset of a product of intervals, and $\phi : C \times I \to X \times (0, 1)$ is continuous, then there exists a continuous function $\Phi : X \times I \to A$ extending $\phi$.

**Theorem 3.** If $C$ is a closed subset of a normal space $X$, $A$ is any metrizable absolute neighborhood retract, and $\phi : C \times I \to X \times (0, 1)$ is continuous, then there exists a continuous function $\Phi : X \times I \to A$ extending $\phi$.

**Proof.** Note that by Theorem 3', Theorem 1 remains true when $[0, 1)$ is replaced by $(-1, 1)$. By theorems of E. Michael (see 5, Theorem 3, and Lemma 4.1), $A$ can be embedded as a closed subset of a countable product of open intervals. Extend $\phi$ to a map from $X \times I$ into that product, note that $A$ is a retract of an open subset of that product, and then use the same proof as that of Theorem 2 to complete the proof of Theorem 3.

The question that remains is whether the above theorem is true when $A$ is an arbitrary absolute neighborhood retract.

**References**


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