The Borsuk homotopy extension theorem without the binormality condition

by

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Abstract. Borsuk's homotopy extension theorem states that if C is a closed subset of a binormal space X, then a continuous function φ from $C\times I\cup X\times\{0\}$ into any absolute neighborhood retract can be extended to all of $X\times I$. In this paper we prove that the binormality condition on X can be weakened to just normality in the case where the range of φ is a compact, Hausdorff or metrizable absolute neighborhood retract.

In 1937 Borsuk [1] proved a homotopy extension theorem, one generalization of which is the following: If C is a closed subset of a binormal space X, A is a compact Hausdorff absolute neighborhood retract, and $\varphi\colon C\times I\cup X\times\{0\}\to A$ is continuous, then there exists a continuous function $\Phi\colon X\times I\to A$ extending φ. A binormal space is a normal space whose product with an interval is also normal. For many years it was hoped that all normal, Hausdorff spaces were binormal. All hope finally had to be abandoned, however, when Mary Ellen Rudin [3], [4] produced examples of normal, Hausdorff spaces which are not binormal. In this paper we show that Borsuk's homotopy extension theorem nevertheless remains true without the binormality condition.

I am indebted to Mary Ellen Rudin for her mathematical help and her great encouragement.

Borsuk's theorem follows from the extension theorem below.

THEOREM 1. *If C is a closed subset of a normal space X and $\varphi\colon C\times I\cup X\times\{0\}\to[-1,1]$ is a continuous function, then there exists a continuous function $\Phi\colon X\times I\to[-1,1]$ extending φ.*

Proof. We imitate a proof of the Tietze extension theorem by producing a sequence of continuous functions, $\{g_n\}_{n=0}^{\infty}$, with

$$g_n\colon X\times I\to[-\tfrac{1}{3}(\tfrac{2}{3})^n,\tfrac{1}{3}(\tfrac{2}{3})^n]$$

such that

$$\Big(\varphi-\sum_{i<n}g_i\Big)^{-1}\big([-(\tfrac{2}{3})^n,-\tfrac{1}{3}(\tfrac{2}{3})^n]\big)\subset g_n^{-1}\big(-\tfrac{1}{3}(\tfrac{2}{3})^n\big)$$

and

$$\Big(\varphi-\sum_{i<n} g_i\Big)^{-1}\big([\tfrac{1}{3}(\tfrac{2}{3})^n, (\tfrac{2}{3})^n]\big) \subset g_n^{-1}\big(\tfrac{1}{3}(\tfrac{2}{3})^n\big).$$

Then $\Phi = \sum_{i=0}^{\infty} g_i$ is continuous and extends φ.

The theorem follows after inductively constructing the g_n's. After appropriate modifications of the range, each g_n is obtained by an application of the following lemma to $\varphi-\sum_{i<n} g_i$.

LEMMA. *If C is a closed subset of a normal space X, and $f\colon C\times I\cup X\times\{0\}\to[-1,1]$ is continuous, then there a continuous function $g\colon X\times I\to[0,1]$ such that $f^{-1}([-1,-\frac{1}{3}])\subset g^{-1}(0)$ and $f^{-1}([\frac{1}{3},1])\subset g^{-1}(1)$.*

The lemma is proved in three steps.

Step 1. *If H and K are subsets of $X\times I$ with disjoint closures such that for every point $x\in X$, $(\{x\}\times I)\cap H$ and $(\{x\}\times I)\cap K$ are relatively open in $\{x\}\times I$, then H and K can be separated by disjoint open sets in $X\times I$.*

Proof of Step 1. We produce a countable cover of H (resp. K) by open sets whose closures each miss K (resp. H). Let $\{B_i\}_{i\in\omega_0}$ be a basis for I. Let $H_i=\{(x,t)\in\bar H|\ t\in\bar B_i$ and $\{x\}\times\bar B_i\subset\bar H\}$ and $K_i=\{(x,t)\in\bar K|\ t\in\bar B_i$ and $\{x\}\times\bar B_i\subset\bar K\}$. Let H_i^* and K_i^* be the projections of H_i and K_i respectively into the X coordinate. Note that H_i^* and K_i^* are closed subsets of X.

Since $\bar B_i$ is compact and $\bar H\cap\bar K=\emptyset$, for each $x\in H_i^*$ and $y\in K_i^*$, there exist open subsets of X, U_x and V_y, such that $x\in U_x$, $y\in V_y$, $U_x\times\bar B_i\cap\bar K=\emptyset$, and $V_y\times\bar B_i\cap\bar H=\emptyset$. Let $U_i=\bigcup_{x\in H_i^*}U_x$ and $V_i=\bigcup_{y\in K_i^*}V_y$.

H_i^* and K_i^* are closed subsets of U_i and V_i respectively, hence by normality of X, there are open sets S_i and T_i such that $H_i^*\subset S_i\subset\bar S_i\subset U_i$ and $K_i^*\subset T_i\subset\bar T_i\subset V_i$.

$H\subset\bigcup_{i\in\omega_0}(S_i\times B_i)$, $K\subset\bigcup_{i\in\omega_0}(T_i\times B_i)$, and for each i, $\bar S_i\times\bar B_i\cap\bar K=\emptyset$, and $\bar T_i\times\bar B_i\cap\bar H=\emptyset$. Consequently,

$$\bigcup_{i\in\omega_0}\Big(S_i\times B_i-\bigcup_{j\leqslant i}(\bar T_j\times\bar B_j)\Big)$$

and

$$\bigcup_{i\in\omega_0}\Big(T_i\times B_i-\bigcup_{j\leqslant i}(\bar S_j\times\bar B_j)\Big)$$

are disjoint open sets containing H and K respectively.

Step 2. *There exist open sets U and V in $X\times I$ such that $f^{-1}([-1,-\frac{1}{3}])\subset U$, $f^{-1}([\frac{1}{3},1])\subset V$ and $\bar U\cap\bar V=\emptyset$.*

Proof of Step 2. We will first separate $f^{-1}([-1,-\frac{1}{3}])\cap C\times I$ from $f^{-1}([\frac{1}{3},1])\cap C\times I$ in the desired manner and later deal with the rest.

By Step 1, we can find open sets U' and V' such that $f^{-1}([-1,-\frac{1}{3}])\cap C\times I\subset U'$, $(f^{-1}([0,1])\cap C\times I)\cap\bar U'=\emptyset$, $f^{-1}([\frac{1}{3},1])\cap C\times I\subset V'$, and $(f^{-1}([-1,0])\cap C\times I)\cap\bar V'=\emptyset$. We may assume that $U'\cap f^{-1}([\frac{1}{3},1])=\emptyset$ and $V'\cap f^{-1}([-1,-\frac{1}{3}])=\emptyset$. Note that $\bar U'\cap\bar V'\subset X\times I-C\times I$.

Since $\bar U'\cap\bar V'$ is closed and in the complement of $C\times I$, there is an open set W of X containing C such that $W\times I\cap(\bar U'\cap\bar V')=\emptyset$. Since $C\subset W$ and X is normal, there is an open set W_1 such that $C\subset W_1\subset\bar W_1\subset W$. Let $U''=U'\cap(W_1\times I)$ and $V''=V'\cap(W_1\times I)$.

U'' and V'' separate $f^{-1}([-1,-\frac{1}{3}])\cap C\times I$ from $f^{-1}([\frac{1}{3},1])\cap C\times I$ in the desired manner.

Now let H and K be the closed subsets of X such that $H\times\{0\}=f^{-1}([-1,-\frac{1}{3}])-(U''\cup V'')$ and $K\times\{0\}=f^{-1}([\frac{1}{3},1])-(U''\cup V'')$. Note that $H\cup K\subset X-C$.

We can separate H from $(f^{-1}([\frac{1}{3},1])\cap X\times\{0\})\cup C$ and K from $(f^{-1}([-1,-\frac{1}{3}])\cap X\times\{0\})\cup C$ and obtain open subsets S_1, S, T_1, and T of X with $H\subset S_1\subset\bar S_1\subset S$, $K\subset T_1\subset\bar T_1\subset T$, $\bar S\times I\cap(f^{-1}([\frac{1}{3},1])\cup C\times I\cup\bar T\times I)=\emptyset$, and $\bar T\times I\cap(f^{-1}([-1,-\frac{1}{3}])\cup C\times I\cup\bar S\times I)=\emptyset$.

Let $U=U''\cup S_1\times I-\bar T\times I$ and $V=V''\cup T_1\times I-\bar S\times I$. U and V meet the requirements of Step 2.

Step 3. *If H and K are closed sets and U and V are open sets in $X\times I$ with $H\subset U$, $K\subset V$, and $\bar U\cap\bar V=\emptyset$, then there exists a continuous function $g\colon X\times I\to[0,1]$ with $H\subset g^{-1}(1)$ and $K\subset g^{-1}(0)$.*

Proof of Step 3. The proof of this step involves the definition of a countable cover for X which has a locally finite refinement and uses some ideas of countable paracompactness found in [2].

Let $\{B_i\}_{i\in\omega_0}$ be the collection of all finite unions of basic open sets of I. We define two open covers of X.

For each pair (B_i,B_j) with $\bar B_i\cap\bar B_j=\emptyset$, let $W_{i,j}=\{x\in X|$ the pair (B_i,B_j) has the properties that $(\{x\}\times I)\cap H\subset\{x\}\times B_i$ and $(\{x\}\times I)\cap K\subset\{x\}\times B_j\}$ and let $W'_{i,j}=\{x\in X|$ the pair (B_i,B_j) has the properties that $(\{x\}\times I)\cap\bar U\subset\{x\}\times B_i$ and $(\{x\}\times I)\cap\bar V\subset\{x\}\times B_j\}$. Note that $W_{i,j}$ and $W'_{i,j}$ are open subsets of X and that the properties of U and V guarantee us that $\bar W'_{i,j}\subset W_{i,j}$.

Index the $W_{i,j}$'s and the corresponding $W'_{i,j}$'s by a single indexing sequence. So $\{W_{i,j}\}_{i,j\in\omega_0}=\{W_n\}_{n\in\omega_0}$ and $\{W'_{i,j}\}_{i,j\in\omega_0}=\{W'_n\}_{n\in\omega_0}$.

Let $Q_n=\bigcup_{i\leqslant n}W_i$ and $Q'_n=Q_n-\bigcup_{i<n}\bar W'_i$. The Q'_n's form a locally finite cover of X. Divide each Q'_n into n parts namely $Q'_n\cap W_1, Q'_n\cap W_2, \ldots, Q'_n\cap W_n$ to obtain a countable locally finite refinement, $\{T_i\}_{i\in\omega_0}$, of $\{W_n\}_{n\in\omega_0}$.

By normality of X we shrink each T_i to obtain an open cover $\{S_i\}_{i\in\omega_0}$ with $S_i\subset\bar S_i\subset T_i$ for each i. Again by normality of X, there exists a Urysohn function $h'_i\colon X\to[0,1]$ such that $\bar S_i\subset(h'_i)^{-1}(1)$ and $X-T_i\subset(h'_i)^{-1}(0)$.

Recall that $T_i \subset W_{k,1}$ for some pair $(k, 1)$. By normality of I, there exists a continuous function $h_i''\colon I \to [0,1]$ such that $\bar{B}_k \subset (h_i'')^{-1}(1)$ and $\bar{B}_1 \subset (h_i'')^{-1}(0)$.

Let $h_i\colon X \times I \to [0,1]$ be defined by $h_i(x, y) = h_i'(x) \cdot h_i''(y)$. Note that h_i is continuous, $\bar{S}_i \times \bar{B}_k \subset h_i^{-1}(1)$, and $X \times I - \big(T_i \times (I - B_1)\big) \subset h_i^{-1}(0)$.

We are now prepared to define the function $g\colon X \times I \to [0,1]$ required in Step 3. Let $g(x, y) = \min\{\sum_{i \in \omega_0} h_i(x, y), 1\}$.

Since the h_i's had locally finite supports, g is continuous. For each i, $K \subset h_i^{-1}(0)$; hence $K \subset g^{-1}(0)$. For every point $(x, y) \in H$ there was an i such that $x \in S_i$, hence $h_i(x, y) = 1$. So $H \subset g^{-1}(1)$. Step 3 is therefore proved.

Steps 2 and 3 prove the lemma which in turn proves the theorem.

A consequence of this theorem is Borsuk's homotopy extension theorem without the binormality condition. That is,

THEOREM 2. *If C is a closed subset of a normal space X, A is any compact Hausdorff absolute neighborhood retract, and $\varphi\colon C \times I \cup X \times \{0\} \to A$ is continuous, then there exists a continuous function $\Phi\colon X \times I \to A$ extending φ.*

Proof of Theorem 2. A can be embedded as a closed subset of a product of intervals which is a retract of an open set in the product. By Theorem 1, φ extends to a continuous function from $X \times I$ into this product of intervals. Hence φ can be extended to a continuous function $\Phi'\colon W \times I \cup X \times \{0\} \to A$ where W is open in X and $C \subset W$. By normality of X there exists a continuous function $g\colon X \to [0,1]$ such that $X - W \subset g^{-1}(0)$ and $C \subset g^{-1}(1)$. We define the desired Φ by $\Phi(x, t) = \Phi'\big(x, g(x) \cdot t\big)$.

It should be noted that we have actually proved somewhat stronger versions of Theorems 1 and 2, namely:

THEOREM 1'. *If C is a closed subset of a normal space X, M is any compact metric space, $p \in M$, and $\varphi\colon C \times M \cup X \times \{p\} \to [-1, 1]$ is continuous, then there exists a continuous function $\Phi\colon X \times M \to [-1, 1]$ extending φ.*

THEOREM 2'. *If C is a closed subset of a normal space X, A is any retract of an open subset of a product of intervals, and $\varphi\colon C \times I \cup X \times \{0\} \to A$ is continuous, then there exists a continuous function $\Phi\colon X \times I \to A$ extending φ.*

THEOREM 3. *If C is a closed subset of a normal space X, A is any metrizable absolute neighborhood retract, and $\varphi\colon C \times I \cup X \times \{0\} \to A$ is continuous, then there exists a continuous function $\Phi\colon X \times I \to A$ extending φ.*

Proof. Note that by Theorem 2', Theorem 1 remains true when $[-1, 1]$ is replaced by $(-1, 1)$. By theorems of E. Michael [see 5, Theorem 3.1 and Lemma 4.1], A can be embedded as a closed subset of a countable product of open intervals. Extend φ to a map from $X \times I$ into that product, note that A is a retract of an open subset of that product, and then use the same proof as that of Theorem 2 to complete the proof of Theorem 3.

The question that remains is whether the above theorem is true when A is an arbitrary absolute neighborhood retract.

References

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