pletion of A — see Remark 4.9) are isomorphic as multiplicative lattices. Thus, the ideal structure of $\bar{R}$ can be determined by purely lattice theoretical means from the lattice of ideals of R.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON
Houston, Texas

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Locally flat embeddings of Hilbert cubes are flat

by

T. A. Chapman * (Lexington, Ken.)

Abstract. In this paper it is shown that any locally flat embedding of the Hilbert cube Q into a Q-manifold is flat. The techniques employed in the proof of this result also imply that the group of homeomorphisms of $Q\times R^n$ onto itself which are fixed on $Q\times\{0\}$ has exactly two components.

1. Introduction. For topological spaces X and Y, an embedding $i\colon X\to Y$ is said to be *locally flat* (with codimension n) provided that each point of X has a neighborhood U and an open embedding $h\colon U\times R^n\to Y$ such that $h(x,0)=i(x)$, for all $x\in U$. We say that the embedding is *flat* if we can take $U=X$. We use R^n to denote euclidean n-space, Q to denote the Hilbert cube (i.e. the countable infinite product of closed intervals), and by a *Q-manifold* we mean a separable metric manifold modeled on Q. The following is the main result of this paper.

THEOREM 1. *If X is a Q-manifold and $i\colon Q\to X$ is a locally flat embedding, then i is flat.*

Of course this result is false if Q is replaced by a more complicated Q-manifold. For example let $X=M\times Q$, where M is the open Möbius band, let $i_1\colon S^1\to M$ be a homeomorphism of the 1-sphere onto the center circle, and let $i=i_1\times\mathrm{id}\colon S^1\times Q\to M\times Q$. Then i is a codimension 1 locally flat embedding, but i is not flat. (If i were flat, then arbitrarily small neighborhoods of $i_1(S^1)$ in M would be separated by $i_1(S^1)$.) A more general question would be to investigate when locally flat embeddings of Q-manifolds into Q-manifolds have normal bundles (see [2] and [4] for finite-dimensional results).

Let $\mathcal{H}_0(Q\times R^n)$ denote the space of all homeomorphisms of $Q\times R^n$ onto itself (with the CO-topology) which are the identity on $Q\times\{0\}$. The following result is a by-product of the proof of Theorem 1.

THEOREM 2. *$\pi_0(\mathcal{H}_0(Q\times R^n))=2$, for all $n\geqslant 1$. That is, $\mathcal{H}_0(Q\times R^n)$ has exactly two components.*

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We remark that Kirby's solution of the annulus conjecture [3] implies that $\pi_0(\mathcal{H}_0(R^n)) = 2$ for $n \neq 4$, where $\mathcal{H}_0(R^n)$ is the space of all homeomorphisms of R^n onto itself which are the identity on $\{0\}$.

The techniques we use for proving both Theorems 1 and 2 are infinite-dimensional. Theorem 2 is much easier to prove than Theorem 1. Its proof uses an infinite-dimensional version of the standard Alexander trick which is used to prove that any homeomorphism of an n-cell onto itself which fixes the cell's boundary is isotopic to the identity [1]. In § 3 we establish this infinite-dimensional Alexander isotopy and in § 4 we prove Theorem 2. In § 5 we establish some lemmas necessary for the proof of Theorem 1 and in § 6 we prove Theorem 1. The proofs of both Theorems 1 and 2 use some recent results concerning the triangulation and classification of Q-manifolds (see [7], [8], and [9]).

2. Definitions and notation. For each $r > 0$ we let

$$B_r^n = \{x \in R^n |\ \|x\| \leqslant r\},$$

the n-ball of radius r. S_r^{n-1} denotes its boundary and $\mathrm{Int}(B_r^n)$ denotes its interior. Throughout this paper we will use α: $R^n \to R^n$ to denote the orientation-reversing homeomorphism given by

$$\alpha(x_1, x_2, \ldots, x_n) = (-x_1, x_2, \ldots, x_n).$$

We use id to represent the identity mapping on any space. If f_t: $X \to Y$ is a homotopy, for $t \in [0, 1]$, such that for some $A \subset X$ we have $f_t = f_0$ on A for all t, then we write $f_0 \simeq f_1 \operatorname{rel} A$. If each f_t is a homeomorphism of X onto Y, then f_t is an isotopy and we write $f_0 \overset{\text{iso}}{\simeq} f_1$. A *proper mapping* is a mapping for which the pre-image of each compactum is compact. We will occasionally work in the proper category, and we will use such terms as proper homotopy, proper isotopy, etc.

We will use a considerable amount of infinite-dimensional machinery and a good source for some of the basic material is [6]. There is one technical result which is used throughout this paper that bears repeating. A closed subset A of a space X is said to be a *Z-set* in X provided that there exist arbitrarily "small" mappings f: $X \to X \setminus A$. This means that if $\mathcal{U}$ is any open cover of X, then there exists a mapping f: $X \to X \setminus A$ such that for each $x \in X$ there exists an element of $\mathcal{U}$ containing both x and $f(x)$. (We say that f is *limited by* $\mathcal{U}$). The following is the main technical tool concerning Z-sets [6].

ISOTOPY THEOREM. *Let X be a Q-manifold, A be a space, and let F: $A \times [0, 1] \to X$ be a proper mapping such that the levels F_0: $A \to X$, F_1: $A \to X$ are homeomorphisms of A onto Z-sets in X. Then the induced homeomorphism $F_1F_0^{-1}$ of $F_0(A)$ onto $F_1(A)$ can be extended to a manifold homeomorphism which is isotopic to the identity.*

We remark that if all the fibers $F(\{a\} \times [0, 1])$ are "short", then the manifold homeomorphism which extends $F_1F_0^{-1}$ can be chosen to be "close" to the identity. To be more precise this means that if $\mathcal{U}$ is an open cover of X and each fiber $F(\{a\} \times [0, 1])$ lies in some element of $\mathcal{U}$, then the homeomorphism extending $F_1F_0^{-1}$ can be chosen to be limited by $\mathrm{St}^2(\mathcal{U})$. Here $\mathrm{St}^2(\mathcal{U})$ is the open cover of X constructed by taking all sets of the form

$$U_1 \cup U_2 \cup U_{3'},$$

where $U_i \in \mathcal{U}$ and $U_1 \cap U_2 \neq \emptyset$, $U_2 \cap U_3 \neq \emptyset$.

This estimated version of the Isotopy theorem will be used in the proof of Lemma 5.1 to perform the modification of h_4.

3. An Alexander-type isotopy. We establish here a version of the Alexander trick which will be needed for the proof of Theorem 2.

LEMMA 3.1. *If h: $Q \times B_1^n \to Q \times B_1^n$ is a homeomorphism such that $h = \mathrm{id}$ on $Q \times (\{0\} \cup S_1^{n-1})$, then $h \overset{\text{iso}}{\simeq} \mathrm{id} \operatorname{rel} Q \times (\{0\} \cup S_1^{n-1})$.*

Proof. Let θ: $Q \times Q \times [0, 1] \to Q$ be a mapping such that θ_0 is projection off the second factor, θ_1 is projection off the first factor, and θ_t is a homeomorphism for all $t \neq 0, 1$. The construction of θ uses a standard coordinate-switching technique (see [6], Chapter III). Consider the function $\tilde{h}$: $Q \times B_1^n \to Q \times B_1^n$ defined by $\tilde{h}(q, x) = (\theta_t(q_1', q_2), x')$, where $\theta_{\|x\|/2}^{-1}(q) = (q_1, q_2)$, $h(q_1, x) = (q_1', x')$, and $t = \|x'\|/2$. It is easy to check that $\tilde{h}$ is a homeomorphism such that $\tilde{h} = \mathrm{id}$ on $Q \times (\{0\} \cup S_1^{n-1})$.

To show that $\tilde{h} \overset{\text{iso}}{\simeq} h \operatorname{rel} Q \times (\{0\} \cup S_1^{n-1})$ we define φ_t: $B_1^n \to [0, 1]$ by $\varphi_t(s) = (1-t) \cdot s/2 + t$ and for each $t \in [0, 1]$ let $\tilde{h}_t$: $Q \times B_1^n \to Q \times B_1^n$ be defined by $\tilde{h}_t(q, x) = (\theta_{t_1}(q_1', q_2), x')$, where $\theta_{\varphi_t(\|x\|)}^{-1}(q) = (q_1, q_2)$, $h(q_1, x) = (q_1', x')$, and $t_1 = \varphi_t(\|x'\|)$. It is clear that $\tilde{h}_t$ provides our required isotopy such that $\tilde{h}_0 = \tilde{h}$ and $\tilde{h}_1 = h$.

To show that $\tilde{h} \overset{\text{iso}}{\simeq} \mathrm{id} \operatorname{rel} Q \times (\{0\} \cup S_1^{n-1})$ we will define an isotopy g_t: $Q \times B_1^n \to Q \times B_1^n$ which fulfills our requirements. For $t = 0$ we define $g_0 = \mathrm{id}$. For $t \in (0, 1]$ define $g_t = \mathrm{id}$ on $(Q \times \{0\}) \cup Q \times (B_1^n \setminus \mathrm{Int}(B_t^n))$ and on $Q \times (B_t^n \setminus \{0\})$ we define $g_t(q, x) = (\theta_{t_1}(q_1', q_2), tx')$, where $\theta_{\|x\|/2}^{-1}(q) = (q_1, q_2)$, $h(q_1, x/t) = (q_1', x')$, and $t_1 = \|tx'\|/2$. Then g_t is our required isotopy.

4. Proof of Theorem 2. We will use Lemma 3.1 to prove Theorem 2. Choose any $h \in \mathcal{H}_0(Q \times R^n)$. We will prove that either $h \overset{\text{iso}}{\simeq} \mathrm{id} \operatorname{rel} Q \times \{0\}$ or $h \overset{\text{iso}}{\simeq} \mathrm{id} \times \alpha \operatorname{rel} Q \times \{0\}$. Choose $r > 0$ large enough so that $h(Q \times B_1^n) \subset Q \times \mathrm{Int}(B_r^n)$. Put $A = h(Q \times S_1^{n-1})$, $X = (Q \times B_r^n) \setminus h(Q \times \mathrm{Int}(B_1^n))$, and let i: $A \hookrightarrow X$ be the inclusion mapping.

We will first show that i is a homotopy equivalence. To do this we prove that i induces an isomorphism on all homotopy groups. First assume that $n=1$. Then all we have to do is use the fact that if (Y, Y_0) is an ANR pair such that Y_0 is closed, bicollared, and the inclusion $Y_0 \hookrightarrow Y$ is a homotopy equivalence, then Y_0 separates Y into two disjoint open sets and the inclusion of Y into the closure of each of these components is a homotopy equivalence. For details see [5].

We now treat the case $n=2$. Note that X is a strong deformation retract of $Q\times(B^n_r\setminus\{0\})$, thus $\pi_m(X)=0$ for all $m\geqslant 2$. All we need to do is prove that i induces an isomorphism on π_1. Using singular homology with integral coefficients consider the commutative diagram

$$\begin{array}{ccc} \pi_1(A) & \xrightarrow{\;i_*\;} & \pi_1(X) \\ \big\downarrow & & \big\downarrow \\ H_1(A) & \xrightarrow[\;i_\#\;]{} & H_1(X), \end{array}$$

where i_* and $i_\#$ are induced by i and the vertical arrows are Hurewicz homomorphisms. Since $\pi_1(A)$ and $\pi_1(X)$ are abelian, these vertical arrows are isomorphisms. So all we have to do is prove that $i^\#$ is an isomorphism. Using the homology exact sequence for the pair (X, A) all we have to do is prove that $H_1(X, A)=0$ and $H_2(X, A)=0$. By excision we have

$$H_1(X, A)\cong H_1\big(Q\times B^n_r, h(Q\times B^n_1)\big)$$

and the exact sequence for the pair $\big(Q\times B^n_r, h(Q\times B^n_r)\big)$ gives us $H_1\big(Q\times B^n_r, h(Q\times B^n_1)\big)=0$. Thus $H_1(X, A)=0$. We can similarly prove that $H_2(X, A)=0$.

We now treat the case $n\geqslant 3$. In this case A and X are both simply connected, therefore all we need to do is prove that i induces an isomorphism on all homology groups. But this easily follows from the Mayer-Vietoris sequence for the triad $\big(Q\times B^n_r, X, h(Q\times B^n_1)\big)$.

Our next step is to show that there exists a homeomorphism $h_1\colon Q\times B^n_r\to Q\times B^n_r$ which agrees with h on $Q\times B^n_1$. Let $f\colon Q\times\big(B^n_r\setminus\mathrm{Int}(B^n_1)\big)\to h^{-1}(A)$ be a homotopy equivalence and note that $if\colon Q\times\big(B^n_r\setminus\mathrm{Int}(B^n_1)\big)\to X$ is a homotopy equivalence. Since $\pi_1(X)$ is "nice", the Whitehead group $\mathrm{Wh}(X)$ vanishes and therefore if is a simple homotopy equivalence. This implies that if is homotopic to a homeomorphism $g\colon Q\times\big(B^n_r\setminus\mathrm{Int}(B^n_1)\big)\to X$ (see [9] for references). Using the Isotopy theorem we can correct g to get $g=h$ on $Q\times S^{n-1}_1$. Then g extends to our required homeomorphism h_1.

Now consider the mapping $\varphi\colon S^{n-1}_r\to S^{n-1}_r$ given by $\varphi(x)=\gamma p h_1(q_0, x)$, where $q_0\in Q$, $p\colon Q\times B^n_r\to B^n_r$ is the projection mapping, and $\gamma\colon B^n_r\setminus\{0\}\to S^{n-1}_r$ is a radial projection. Then φ is a homotopy equivalence, so it has degree ± 1. If $\deg\varphi=+1$, then $\varphi\simeq\mathrm{id}$, hence the restriction $h_1|Q\times S^{n-1}_r\colon Q\times S^{n-1}_r\to X$ is homotopic to the inclusion $Q\times S^{n-1}_r\hookrightarrow X$. Using the Isotopy theorem we can correct h_1 so that we have $h_1=\mathrm{id}$ on $Q\times S^{n-1}_r$.

Extend h_1 by the identity to a homeomorphism $\tilde h_1\colon Q\times R^n\to Q\times R^n$ and note that Lemma 3.1 implies that $h_1\overset{\mathrm{iso}}{\simeq}\mathrm{id}\,\mathrm{rel}\,Q\times\{0\}$. Thus $h_2=\tilde h_1^{-1}h\overset{\mathrm{iso}}{\simeq}h$ rel $Q\times\{0\}$. But $h_2=\mathrm{id}$ on $Q\times B^n_1$. Then it is easy to prove that $h_2\overset{\mathrm{iso}}{\simeq}\mathrm{id}\,\mathrm{rel}\,Q\times B^n_1$ by using the standard Alexander trick. This takes care of the case in which $\deg\varphi=+1$.

On the other hand assume that $\deg\varphi=-1$. Then φ is homotopic to a restriction of a, therefore $h_1|Q\times S^{n-1}_r\colon Q\times S^{n-1}_r\to X$ is homotopic to the inclusion $Q\times S^{n-1}_r\hookrightarrow X$ followed by $\mathrm{id}\times a$. The Isotopy theorem implies that h_1 can be corrected so that we additionally have $h_1=\mathrm{id}\times a$ on $Q\times S^{n-1}_r$. Extend h_1 by $\mathrm{id}\times a$ to a homeomorphism $\tilde h_1$ of $Q\times R^n$ onto itself. Using Lemma 3.1 it follows that

$$(\mathrm{id}\times a)h_1\overset{\mathrm{iso}}{\simeq}\mathrm{id}\,\mathrm{rel}\,Q\times\{0\}.$$

Therefore $h_1\overset{\mathrm{iso}}{\simeq}\mathrm{id}\times a\,\mathrm{rel}\,Q\times\{0\}$, which implies that

$$h_2=\tilde h_1^{-1}h\overset{\mathrm{iso}}{\simeq}(\mathrm{id}\times a)h\,\mathrm{rel}\,Q\times\{0\}.$$

But once again we have $h_2\overset{\mathrm{iso}}{\simeq}\mathrm{id}\,\mathrm{rel}\,Q\times\{0\}$. Therefore $h\overset{\mathrm{iso}}{\simeq}\mathrm{id}\times a\,\mathrm{rel}\,Q\times\{0\}$.

5. Some lemmas for Theorem 1. In this section we prove two results which will be needed in the proof of Theorem 1.

LEMMA 5.1. *Let $h\colon Q\times R\times R^n\to Q\times R\times R^n$ be an open embedding such that $h=\mathrm{id}$ on $Q\times R\times\{0\}$. Then there exists a homeomorphism f of $Q\times R\times R^n$ onto itself with compact support such that $f=\mathrm{id}$ on $Q\times R\times\{0\}$ and either $fh=\mathrm{id}$ or $fh=\mathrm{id}\times a$ on $Q\times[-1,1]\times B^n_1$.*

Proof. Choose $r>0$ such that

$$h(Q\times[-1,1]\times B^n_1)\subset Q\times(-r, r)\times\mathrm{Int}(B^n_r).$$

We will construct a homeomorphism g of $Q\times[-r, r]\times B^n_r$ onto itself such that $g=\mathrm{id}$ on $Q\times[-r, r]\times\{0\}$, $g=h$ on $Q\times[-1,1]\times B^n_1$, and either $g=\mathrm{id}$ or $g=\mathrm{id}\times a$ on $\mathrm{Bd}(Q\times[-r, r]\times B^n_r)$. This will clearly fulfill our requirements. Our first goal will be to work our way through the accompanying diagram of spaces and maps. The homeomorphism h_4 at the top of the diagram will be used to obtain g.

I. Construction of h_1. We want h_1 to be a proper embedding such that $h_1\big(Q\times(-r, r)\times B^n_1\big)\subset Q\times(-r, r)\times\mathrm{Int}(B^n_r)$, $h_1=\mathrm{id}$ on $Q\times(-r, r)\times\times\{0\}$, and $h_1=h$ on $Q\times[-1,1]\times B^n_1$. The details of the construction of h_1 are routine.

II. Construction of h_2. We want h_2 to be a homeomorphism which extends h_1. Let $A=h_1\big(Q\times(-r, r)\times S^{n-1}_1\big)$ and let

$$X=\big(Q\times(-r, r)\times B^n_r\big)\setminus h_1\big(Q\times(-r, r)\times\mathrm{Int}(B^n_1)\big).$$

Just as in the proof of Theorem 2 we can show that the inclusion $i\colon A \subset_{\to} X$ is a homotopy equivalence. We will show that the inclusion $j\colon Q\times(-r, r)\times S_r^{n-1} \subset_{\to} X$ is a proper homotopy equivalence.

$$\begin{array}{ccc}
Q\times[-r, r]\times B_r^n & \xrightarrow{h_4} & Q\times[-r, r]\times B_r^n \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\
X & \xrightarrow{h_3} & X \\
\cup\uparrow & & \cup\uparrow \\
Q\times(-r, r)\times B_r^n & \xrightarrow{h_2} & Q\times(-r, r)\times B_r^n \\
\cup\uparrow & & \cup\uparrow \\
Q\times(-r, r)\times B_1^n & \xrightarrow{h_1} & Q\times(-r, r)\times B_r^n \\
\cap\downarrow & & \cap\downarrow \\
Q\times R\times R^n & \xrightarrow{h} & Q\times R\times R^n
\end{array}$$

Let $\tilde{u}\colon X \subset_{\to} Q\times(-r, r)\times(B_r^n\setminus\{0\})$ be the inclusion mapping. Let $v\colon B_r^n\setminus\{0\}\to S_r^{n-1}$ be the mapping given by radial projection and let $\tilde{v}\colon Q\times \times(-r, r)\times(B_r^n\setminus\{0\})\to Q\times(-r, r)\times S_r^{n-1}$ be given by $\tilde{v} = \mathrm{id}\times v$. Then it is easy to see that $\tilde{v}\tilde{u}\colon X\to Q\times(-r, r)\times S_r^{n-1}$ is a proper mapping. We will prove that it is a proper homotopy inverse of j. This means that we must prove that $\tilde{v}\tilde{u}$ is proper homotopic to id (in X).

Let $w\colon B_1^n\setminus\{0\}\to S_1^{n-1}$ be the radial projection and define $\tilde{w}\colon Q\times \times(-r, r)\times(B_r^n\setminus\{0\})\to X$ by setting $\tilde{w} = \mathrm{id}$ on X and $\tilde{w} = h_1(\mathrm{id}\times w)$ on $h_1(Q\times(-r, r)\times B_1^n)$. Let $v_t\colon B_r^n\setminus\{0\}\to B_r^n\setminus\{0\}$ be a radially defined homotopy such that $v_0 = \mathrm{id}$ and $v_1 = v$. Let $\tilde{v}_t\colon Q\times(-r, r)\times(B_r^n\setminus\{0\})\to Q\times \times(-r, r)\times(B_r^n\setminus\{0\})$ be defined by $\tilde{v}_t = \mathrm{id}\times v_t$. Consider the homotopy $\gamma_t\colon X\to X$ defined by $\gamma_t = \tilde{w}\tilde{v}_t\tilde{u}$. It is easy to see that γ_t is a proper homotopy such that $\gamma_0 = \mathrm{id}$ and $\gamma_1 = \tilde{v}\tilde{u}$.

Now for the construction of h_2. Since j is a proper homotopy equivalence it must be an infinite simple homotopy equivalence (see [9] for references). Therefore we can find a homeomorphism

$$\theta\colon Q\times(-r, r)\times S_r^{n-1}\times[0, 1]\to X$$

such that $\theta(x, 0) = x$, for all $x\in Q\times(-r, r)\times S_r^{n-1}$. Choose $r_0\in(0, r)$ so that

$$h_1(Q\times[r_0, r)\times S_1^{n-1})\subset\theta(Q\times(1, r)\times S_r^{n-1}\times[0, 1]),$$
$$h_1(Q\times(-r, -r_0]\times S_1^{n-1})\subset\theta(Q\times(-r, -1)\times S_r^{n-1}\times[0, 1]).$$

It is clear that the inclusions

$$h_1(Q\times[r_0, r)\times S_1^{n-1})\subset_{\to}\theta(Q\times[1, r)\times S_r^{n-1}\times[0, 1]),$$
$$h_1(Q\times(-r, -r_0]\times S_1^{n-1})\subset_{\to}\theta(Q\times(-r, -1]\times S_r^{n-1}\times[0, 1])$$

are homotopy equivalences, and because of the presence of the half-open interval factors they are easily seen to be proper homotopy equivalences. Thus we can find homeomorphisms

$$\varphi_1\colon h_1(Q\times[r_0, r)\times S_1^{n-1})\times[0, 1]\to\theta(Q\times[1, r)\times S_r^{n-1}\times[0, 1]),$$
$$\varphi_2\colon h_1(Q\times(-r, -r_0]\times S_r^{n-1})\times[0, 1]\to\theta(Q\times(-r, -1]\times S_r^{n-1}\times[0, 1])$$

such that $\varphi_1(x, 0) = x$ and $\varphi_2(x, 0) = x$. Choose $r_1\in(r_0, r)$ close enough to r so that

$$\varphi_1(h_1(Q\times[r_1, r)\times S_1^{n-1})\times[0, 1])\subset\theta(Q\times(1, r)\times S_r^{n-1}\times[0, 1]),$$
$$\varphi_2(h_1(Q\times(-r, -r_1]\times S_1^{n-1})\times[0, 1])\subset\theta(Q\times(-r, -1)\times S_r^{n-1}\times[0, 1]).$$

Now let $A_0 = h_1(Q\times[-r_1, r_1]\times S_1^{n-1})$ and let

$$X_0 = X\setminus\big[\varphi_1(h_1(Q\times(r_1, r)\times S_1^{n-1})\times[0, 1])\cup \cup\big[\varphi_2(h_1(Q\times(-r, -r_1)\times S_1^{n-1})\times[0, 1])\big].$$

It is clear that the inclusion $A_0\subset_{\to}X_0$ is a homotopy equivalence and therefore a simple homotopy equivalence. Thus there exists a homeomorphism $\varphi_0\colon A_0\times[0, 1]\to X_0$ such that $\varphi_0(x, 0) = x$. If we knew that φ_0 agreed with φ_1 and φ_2 on

$$\varphi_1(h_1(Q\times\{r_1\}\times S_1^{n-1})\times[0, 1]) \quad\text{and}\quad \varphi_2(h_1(Q\times\{-r_1\}\times S_1^{n-1})\times[0, 1]),$$

respectively, then we could piece them together to obtain a homeomorphism $\varphi\colon A\times[0, 1]\to X$ satisfying $\varphi(x, 0) = x$. This would clearly imply the existence of our required h_2. The manipulation of φ_0 to satisfy these requirements is just a simple application of the Isotopy theorem.

III. Construction of h_3 and X. Define

$$X = (Q\times(-r, r)\times B_r^n)\cup\{r\}\cup\{-r\},$$

where X is the compact metric space obtained from $Q\times(-r, r)\times B_r^n$ by compactifying the two ends. We choose notation so that $(Q\times[0, r)\times B_r^n)\cup \cup\{r\}$ and $(Q\times(-r, 0]\times B_r^n)\cup\{-r\}$ are compact. Then h_3 is defined to agree with h_2 on $Q\times(-r, r)\times B_r^n$ and to be the identity on $\{-r\}\cup\{r\}$.

IV. Construction of δ and h_4. We want δ to be a homeomorphism such that $\delta = \mathrm{id}$ on $Q\times[-r_1, r_1]\times B_r^n$ and

$$\delta(Q\times[-r, r]\times\{0\}) = (Q\times(-r, r)\times\{0\})\cup\{r\}\cup\{-r\},$$

where r_1 is chosen so that $0 < r_1 < r$ and

$$h_3(Q\times[-1, 1]\times B_1^n)\subset Q\times[-r_1, r_1]\times B_{r_1}^n.$$

Then h_4 will be defined to make the appropriate rectangle commute.

It will suffice to produce a mapping δ' of $Q\times[-r,r]\times B^n_r$ onto itself such that

(1) $\delta'(Q\times\{r\}\times B^n_r)=\{0\}\times\{r\}\times\{0\}$ and

$\delta'(Q\times\{-r\}\times B^n_r)=\{0\}\times\{-r\}\times\{0\}$,

(2) δ' restricts to a homeomorphism of $Q\times(-r,r)\times B^n_r$ onto

$$\big(Q\times[-r,r]\times B^n_r\big)\setminus\big((\{0\}\times\{r\}\times\{0\})\cup(\{0\}\times\{-r\}\times\{0\})\big),$$

(3) $\delta'\big(Q\times[-r,r]\times\{0\}\big)=Q\times[-r,r]\times\{0\}$,

(4) $\delta'=\mathrm{id}$ on $Q\times[-r_1,r_1]\times B^n_r$.

Once δ' is obtained we just define $\delta=q(\delta')^{-1}$, where $q\colon Q\times[-r,r]\times B^n_r\to X$ is the quotient mapping.

Using the fact that Q is homeomorphic to its own cone [6], there exists a mapping δ_1 of $Q\times[-r,r]\times B^n_r$ onto itself such that

(1) $\delta_1\big(Q\times\{r\}\times\{x\}\big)=\{0\}\times\{r\}\times\{x\}$ and

$\delta_1\big(Q\times\{-r\}\times\{x\}\big)=\{0\}\times\{-r\}\times\{x\}$ for each $x\in B^n_r$,

(2) δ_1 restricts to a homeomorphism of $Q\times(-r,r)\times\{x\}$ onto

$$\big(Q\times[-r,r]\times\{x\}\big)\setminus\big((\{0\}\times\{r\}\times\{x\})\cup(\{0\}\times\{-r\}\times\{x\})\big)\text{ for each } x,$$

(3) $\delta_1=\mathrm{id}$ on $Q\times[-r_1,r_1]\times B^n_r$.

Now let θ be a mapping of $[-r,r]\times B^n_r$ onto itself such that

(1) $\theta(\{r\}\times B^n_r)=\{r\}\times\{0\}$ and $\theta(\{-r\}\times B^n_r)=\{-r\}\times\{0\}$,

(2) θ restricts to a homeomorphism of $(-r,r)\times B^n_r$ onto

$$\big([-r,r]\times B^n_r\big)\setminus\big((\{r\}\times\{0\})\cup(\{-r\}\times\{0\})\big),$$

(3) $$\theta|\big([-r,r]\times\{0\}\big)\cup\big([-r_1,r_1]\times B^n_r\big)=\mathrm{id}.$$

This gives a mapping θ_0 of $\{0\}\times[-r,r]\times B^n_r$ onto itself defined by $\theta_0(0,x,y)=(0,x',y')$, where $(x',y')=\theta(x,y)$. For each $q\in Q$, $q\neq 0$, we can clearly define a homeomorphism θ_q of $\{q\}\times[-r,r]\times B^n_r$ onto itself which is the identity on

$$(\{q\}\times[-r,r]\times\{0\})\cup(\{q\}\times[-r_1,r_1]\times B^n_r)$$

and such that the θ_q's continuously fit together to define a mapping δ_2 of $Q\times[-r,r]\times B^n_r$ onto itself by setting $\delta_2=\theta_q$ on $\{q\}\times[-r,r]\times B^n_r$. This follows since θ is a uniform limit of homeomorphisms. Then $\delta'=\delta_2\delta_1$ fulfills our requirements. We note that h_4 is a homeomorphism such that $h_4=\mathrm{id}$ on $Q\times[-r,r]\times\{0\}$ and $h_4=h$ on $Q\times[-1,1]\times B^n_1$.

Just as in the proof of Theorem 2 we can correct h_4 so that we have $h_4=\mathrm{id}$ or $h_4=\mathrm{id}\times\alpha$ on $Q\times[-r,r]\times S^{n-1}_r$. For the remainder of the argument we assume that $h_4=\mathrm{id}$ on $Q\times[-r,r]\times S^{n-1}_r$. The treatment of the case $h_4=\mathrm{id}\times\alpha$ on $Q\times[-r,r]\times S^{n-1}_r$ is similar. Note that if we had $h_4=\mathrm{id}$ on $Q\times\{-r,r\}\times B^n_r$, then we would be done. The remainder of the proof of Lemma 5.1 takes steps to modify h_4 to get this extra condition.

Using Lemma 3.1 there exists an isotopy $f_t\colon Q\times[-r,r]\times B^n_r\to Q\times[-r,r]\times B^n_r\ \mathrm{rel}\ Q\times[-r,r]\times(\{0\}\cup S^{n-1}_r)$ such that $f_0=\mathrm{id}$ and $f_1=h_4$. Choose ε, $0<\varepsilon<r$, such that $f_t(Q\times\{-r,r\}\times B^n_r)$ does not intersect $Q\times[-r_1,r_1]\times B^n_\varepsilon$, for all $t\in[0,1]$. The restriction of f_t to $Q\times\{-r,r\}\times B^n_r$ gives us a homotopy

$$g_t\colon Q\times\{-r,r\}\times B^n_r\to Q\times[-r,r]\times B^n_r$$

such that $g_0=\mathrm{id}$, $g_1=h_4$ on $Q\times\{-r,r\}\times B^n_r$, $g_t=\mathrm{id}$ on $Q\times\{-r,r\}\times(\{0\}\cup S^{n-1}_r)$ for all t, and $g_t(Q\times\{-r,r\}\times\mathrm{Int}(B^n_r)\setminus\{0\})$ does not intersect $Q\times[-r,r]\times(\{0\}\cup S^{n-1}_r)$ for all t. It is clear that we may adjust g_t so that additionally $g_t(Q\times\{-r,r\}\times B^n_r)$ does not intersect $Q\times[-r_1,r_1]\times B^n_{r_1}$, for all t. Applying the Isotopy theorem to the manifold

$$[Q\times[-r,r]\times(\mathrm{Int}(B^n_r)\setminus\{0\})]\setminus(Q\times[-r_1,r_1]\times B^n_{r_1})$$

we can easily find a homeomorphism $\tau\colon Q\times[-r,r]\times B^n_r\to Q\times[-r,r]\times B^n_r$ such that $\tau=h_4$ on $Q\times\{-r,r\}\times B^n_r$, $\tau=\mathrm{id}$ on $Q\times[-r,r]\times(\{0\}\cup S^{n-1}_r)$, and $\tau=\mathrm{id}$ on $Q\times[-r_1,r_1]\times B^n_{r_1}$. Then $\tau^{-1}h_4$ gives our desired modification of h_4.

LEMMA 5.2. *Let $h\colon Q\times[0,1)\times R^n\to Q\times[0,1)\times R^n$ be an open embedding such that $h=\mathrm{id}$ on $Q\times[0,1)\times\{0\}$. Then there exists a homeomorphism f of $Q\times[0,1]\times R^n$ onto itself with compact support such that $f=\mathrm{id}$ on $Q\times[0,1)\times\{0\}$ and either $fh=\mathrm{id}$ or $fh=\mathrm{id}\times\alpha$ on $Q\times[0,\tfrac{1}{2}]\times B^n_1$.*

Proof. Similar to the proof of Lemma 5.1.

6. Proof of Theorem 1. We are given a Q-manifold X and a locally flat embedding $i\colon Q\to X$. We will represent Q by $\prod_{j=1}^{\infty}I_j$, where each I_j is the closed interval $[0,1]$. For each m let

$$Q_m=I_1\times I_2\times\ldots\times I_{m-1}\times I_{m+1}\times I_{m+2}\times\ldots$$

and for $A\subset Q_m$, $B\subset I_m$ let

$$A*B=\{(q_j)\in Q|\ q_m\in B\text{ and }(q_1,\ldots,q_{m-1},q_{m+1},\ldots)\in A\}.$$

We can choose m large enough so that there exists an open cover $\mathfrak{U}$ of Q_m satisfying the property that for each $U\in\mathfrak{U}$ there exists an open embedding $h_U\colon(U*I_m)\times R^n\to X$ such that $h_U(x,0)=i(x)$.

Our first goal will be to prove that there exists an open embedding $f\colon Q_m * [0, \frac{3}{4}) \times R^n \to X$ such that $f(x, 0) = i(x)$. There exists an integer l so that for each r, $1 \leqslant r \leqslant l$, there exists an open cover $\{U_{r,k}\}_{k=1}^{l}$ of Q_m refining $\mathfrak{U}$ such that

(1) each $U_{r,k}$ is contractible,

(2) $U_{r,k+1} \cap (U_{r,1} \cup U_{r,2} \cup \ldots \cup U_{r,k})$ is contractible for all r and $1 \leqslant k \leqslant l-1$,

(3) $\overline{U}_{r+1,k} \subset U_{r,k}$ for all k and $1 \leqslant r \leqslant l-1$.

The construction of the open covers $\{U_{r,k}\}_{k=1}^{l}$ is routine. Choose numbers t_r, $1 \leqslant r \leqslant l$, so that

$$\tfrac{3}{4} = t_l < t_{l-1} < \ldots < t_2 < t_1 = 1\,.$$

We will inductively prove that for each k, $1 \leqslant k \leqslant l$, there exists an open embedding

$$f_k\colon (U_{k,1} \cup \ldots \cup U_{k,k}) * [0, t_k) \times R^n \to X$$

such that $f_k(x, 0) = i(x)$. This will imply the existence of our desired open embedding f. The statement is clearly true for $k = 1$ so we assume it to be true for some $k \leqslant l-1$. We will prove that f_{k+1} exists.

Let $\theta\colon U_{k,k+1} * [0, t_k) \times R^n \to X$ be an open embedding such that $\theta(x, 0) = i(x)$. Let $G = U_{k,1} \cup \ldots \cup U_{k,k}$ and let $C \subset G \cap U_{k,k+1}$ be the intersection of $\overline{U}_{k+1,1} \cup \ldots \cup \overline{U}_{k+1,k}$ and $\overline{U}_{k+1,k+1}$. It is clear that θ can be modified so that we additionally have

$$\theta\big(C * [0, t_{k+1}] \times R^n\big) \subset f_k\big(G * [0, t_k) \times R^n\big)\,.$$

Using θ and f_k we can easily construct an open embedding

$$\varphi\colon (G \cap U_{k,k+1}) * [0, t_k) \times R^n \to (G \cap U_{k,k+1}) * [0, t_k) \times R^n$$

such that $\varphi = \mathrm{id}$ on $(G \cap U_{k,k+1}) * [0, t_k) \times \{0\}$ and $\varphi = f_k^{-1}\theta$ on $C * [0, t_{k+1}] \times R^n$. But $G \cap U_{k,k+1}$ is a contractible Q-manifold and therefore $(G \cap U_{k,k+1}) * [0, t_k)$ is homeomorphic to $Q \times [0, 1)$ [5]. Using Lemma 5.2 we can find a homeomorphism

$$h\colon (G \cap U_{k,k+1}) * [0, t_k) \times R^n \to (G \cap U_{k,k+1}) * [0, t_k) \times R^n$$

with compact support such that $h = \mathrm{id}$ on $(G \cap U_{k,k+1}) * [0, t_k) \times \{0\}$ and either $h\varphi = \mathrm{id}$ or $h\varphi = \mathrm{id} \times \alpha$ on $C * [0, t_{k+1}] \times B_1^n$.

Define a homeomorphism $\tilde{h}\colon X \to X$ by $\tilde{h} = f_k h f_k^{-1}$ on

$$f_k\big((G \cap U_{k,k+1}) * [0, t_k) \times R^n\big)$$

and $\tilde{h} = \mathrm{id}$ otherwise. Then let $\tilde{\theta} = \tilde{h}\theta$. Note that $\tilde{\theta}\colon U_{k,k+1} * [0, t_k) \times R^n \to X$ is an open embedding such that $\tilde{\theta}(x, 0) = i(x)$ and either $\tilde{\theta} = f_k$ or $\tilde{\theta} = f_k(\mathrm{id} \times \alpha)$ on $C * [0, t_{k+1}] \times B_1^n$. If $\tilde{\theta} = f_k$ on $C * [0, t_{k+1}] \times B_1^n$, then we can piece together f_k on $(U_{k+1,1} \cup \ldots \cup U_{k+1,k}) * [0, t_{k+1}) \times \mathrm{Int}(B_1^n)$ and $\tilde{\theta}$ on $U_{k+1,k+1} * [0, t_{k+1}) \times \mathrm{Int}(B_1^n)$ to get our required f_{k+1}. If $\tilde{\theta} = f(\mathrm{id} \times \alpha)$ on $C * [0, t_{k+1}] \times B_1^n$, then we replace $\tilde{\theta}$ by $\tilde{\theta}(\mathrm{id} \times \alpha)$ and proceed as above.

Thus we have constructed an open embedding $f\colon Q_m * [0, \frac{3}{4}) \times R^n \to X$ such that $f(x, 0) = i(x)$. Similarly we can construct an open embedding $g\colon Q_m * (\frac{1}{4}, 1] \times R^n \to X$ such that $g(x, 0) = i(x)$. Just as we used Lemma 5.2 to construct f_{k+1} above, we can use Lemma 5.1 to piece together f and g to obtain our required open embedding of $Q_m * [0, 1] \times R^n = Q \times R^n$ into X.

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UNIVERSITY OF KENTUCKY
Lexington, Kentucky

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