Finite dimensional completions in Noether lattices

by

Johnny A. Johnson (Houston, Tex.)

Abstract. This paper is concerned with the completion of Noetherian lattice modules. It is shown that under relatively general conditions the completion of a Noetherian Λ-module Ω is Noetherian as a lattice module over the completion of Λ.

§ 1. Introduction. In [3] R. P. Dilworth began a study of the ideal theory of commutative rings with identity from a lattice theoretic viewpoint and introduced Noether lattices. In a natural manner this theory can be generalized to include the study of module theory as a branch of lattice theory and there are reasons to do so (for example see [7]). Let R be a Noetherian ring and let M be a Noetherian R-module. Denote the lattice of ideals of R by $L(R)$ and the lattice of submodules of M by $L(M)$. Then, under the canonical multiplication between elements of $L(R)$ and elements of $L(M)$, it is easily verified that $L(R)$ is a Noether lattice (see [3]) and that $L(M)$ is a Noetherian $L(R)$-module (the definition of a lattice module is given in Section 2). We note here that there exist Noether lattices Λ and Noetherian Λ-modules Ω which are not lattice isomorphic to the lattice of submodules of any Noetherian module over a Noetherian ring (see [1] and [2] for some interesting examples in Noether lattices) and thus the class of Noetherian lattice modules derived from Noetherian rings and modules does not exhaust the class of all Noetherian lattice modules. In this paper we are concerned with the completion (see Section 4) of a Noetherian Λ-module Ω. This concept is closely related to the a-adic completion of Noetherian rings and modules (see Remark 4.10).

In Section 2 we introduce some notation and establish some useful results concerning dimensions of certain sublattices. If Λ is a Noether lattice, Ω is a Noetherian Λ-module, and b is an element of Λ, we show (Theorem 2.2) that the sublattice $[bB, B]$ of Ω is finite dimensional, for each B in Ω, provided that the sublattice $[b, I]$ of Λ is finite dimensional.

Assume that a is an element of Λ such that the a-adic pseudometric on Ω is a metric. Section 3 is devoted to determining the nature of certain Cauchy sequences (Cauchy relative to the a-adic metric on Ω) of elements of Ω which are needed in the subsequent section. Let $\langle E_i\rangle$, $i=1,2,\ldots$, be a Cauchy sequence of elements of Ω satisfying the condition that: given any natural number n, it follows that $E_{i+1}\leqslant E_i\vee a^nM$, for all integers $i\geqslant n$. It is shown (Theorem 3.5) that the sequence $\langle D_n\rangle$, where $D_n=\bigwedge_{i\geqslant n}(E_i\vee a^nM)$, $n=1,2,\ldots$, is the completely regular representative (see definition in Section 3) of the equivalence class determined by the Cauchy sequence $\langle E_i\rangle$. Thus the unique completely regular representative of a Cauchy sequence satisfying the above condition is completely determined. This result is needed in Section 4 to determine the completely regular representatives for meet and residuation of elements in the completion of Ω.

Let a be an element of Λ such that the a-adic pseudometric on Λ and Ω is a metric (see Section 4) and $[a, I]$ is finite dimensional. Let Λ^* and Ω^* be the a-adic completions of Λ and Ω, respectively (see Section 4 for definitions). Theorems 4.1 and 4.2 determine the structure of the completely regular representatives for meet and join, respectively, of elements of Ω^*. It is shown in Theorem 4.3 that Ω^* is modular and the completely regular representative for residuation is determined in Theorem 4.4. Theorem 4.6 provides a way to obtain principal elements in Ω^* from principal elements in Ω and this result is used to show (Theorem 4.7) that the Λ^*-module Ω^* is principally generated and satisfies the ascending chain condition. Our last result is the following (Theorem 4.8):

Let Λ be a Noether lattice, let Ω be a Noetherian Λ-module, let c be an element of Λ such that the c-adic pseudometric on Λ and Ω is a metric, and let Λ^* and Ω^* be the c-adic completions of Λ and Ω, respectively. If there exists a natural number n such that $[c^n, I]$ is finite dimensional, then

(1.1) Λ^* is a Noether lattice, and
(1.2) Ω^* is a Noetherian Λ^* module.

Finally, an application of Theorem 4.8 to the special case of local and semi-local Noether lattices is given in Remark 4.9.

§ 2. Notation and preliminary results. A lattice is said to be *multiplicative* in case it is a complete lattice on which there is defined a commutative, associate, join distributive multiplication such that the unit element of the lattice is an identity for the multiplication. Let Ω be a complete lattice and let Λ be a multiplicative lattice. Elements of Ω will be denoted by $A, B, C, \ldots$ The null element and unit element of Ω will be denoted by 0 and M, respectively. Elements of Λ will be denoted by $a, b, c, \ldots$ The null element and the unit element of Λ will be denoted by 0 and I, respectively. Ω is said to be an *Λ-module* in case there is a multiplication between elements of Λ and elements of Ω, denoted aA, for a in Λ and A in Ω, such that (i) $(ab)A=a(bA)$; (ii) $(\bigvee_\alpha b_\alpha)(\bigvee_\beta B_\beta)=\bigvee_{\alpha,\beta} b_\alpha B_\beta$; (iii) $IA=A$; and (iv) $0A=0$; for all a, b_α, b in Λ and for all A, B_β in Ω.

Let Ω be an Λ-module. For A and B in Ω, $A{:}B$ will denote the largest b in Λ such that $bB\leqslant A$. An element A in Ω is said to be *meet principal* in case

$$\big(b\wedge(B{:}A)\big)A=bA\wedge B\,,$$

for all b in Λ and for all B in Ω; A is said to be *join principal* in case

$$b\vee(B{:}A)=(bA\vee B){:}A\,,$$

for all b in Λ and for all B in Ω; and A is said to be *principal* in case A is both meet and join principal. Ω is called *principally generated* if each element of Ω is the join (finite or infinite) of principal elements of Ω. If Ω is principally generated, modular, and satisfies the ascending chain condition, Ω is said to be a *Noetherian Λ-module*. If Λ is a Noetherian Λ-module, Λ is called a *Noether lattice*. For other properties and general definitions related to Noether lattices and Noetherian lattice modules the reader should consult the references.

The following lemma is needed in the proof of Theorem 2.2. The reader is referred to Lemma 2.5 of [4] for a proof.

LEMMA 2.1. *Let Ω be an Λ-module, let A be a principal element of Ω, let a and b be elements of Λ such that $a\leqslant b$, let A be modular, and let $(0{:}A)$, $b\leqslant a$. Then the map $\varphi\colon [a,b]\to[aA,bA]$ defined by $\varphi(x)=xA$ is a lattice isomorphism of $[a,b]$ onto $[aA,bA]$.*

We now establish the following results on dimensions which will be required later.

THEOREM 2.2. *Let Λ be a Noether lattice, let Ω be a Noetherian Λ-module, and let b be an element of Λ. If $[b, I]$ is finite dimensional, then $[bB, B]$ is finite dimensional for all elements B in Ω.*

Proof. Assume $[b, I]$ is finite dimensional and let B be an arbitrary element of Ω. Since Ω is Noetherian there are principal elements $B_1, B_2, \ldots, B_k$ in Ω such that $B=B_1\vee\ldots\vee B_k$. Define $T_0=bB$, and define $T_{j+1}=T_j\vee B_{j+1}$, for each j, $0\leqslant j\leqslant k-1$. Thus,

$$T_j=bB\vee B_1\vee\ldots\vee B_j\quad\text{ for }\quad 1\leqslant j\leqslant k\,.$$

Since each B_m, $1\leqslant m\leqslant k$, is principal we obtain

$$(2.1)\qquad [T_j,\,T_{j+1}]=[T_j,\,T_j\vee B_{j+1}]\cong[T_j\wedge B_{j+1},\,B_{j+1}]$$
$$=[(T_j{:}B_{j+1})B_{j+1},\,B_{j+1}]\,,$$

for each j, $0 \leqslant j \leqslant k-1$, by the isomorphism theorems. Also, for each j, $0 \leqslant j \leqslant k-1$, we have

$$(0:B_{j+1}) \wedge I = (0:B_{j+1}) \leqslant T_j:B_{j+1},$$

so that

$$(2.2)\qquad [T_j:B_{j+1}, I] \cong [(T_j:B_{j+1})B_{j+1}, B_{j+1}],$$

for each j, $0 \leqslant j \leqslant k-1$, by Lemma 2.1. From (2.1) and (2.2) it now follows that

$$(2.3)\qquad [T_j, T_{j+1}] \cong [T_j:B_{j+1}, I]$$

for each j, $0 \leqslant j \leqslant k-1$. Now, observe also that

$$bB_{j+1} \leqslant bB \vee B_1 \vee \ldots \vee B_j = T_j,$$

so that

$$b \leqslant (T_j:B_{j+1})$$

for each j, $0 \leqslant j \leqslant k-1$. Consequently, $[T_j:B_{j+1}, I]$ is finite dimensional, and hence $[T_j, T_{j+1}]$ is finite dimensional from (2.3), for each j, $0 \leqslant j \leqslant k-1$. Hence, since

$$bB = T_0 \leqslant T_1 \leqslant \ldots \leqslant T_k = bB \vee B_1 \vee \ldots \vee B_k = B,$$

it follows that $[bB, B]$ is finite dimensional which completes the proof.

As a useful corollary we obtain the following result.

Corollary 2.3. *Let Λ be a Noether lattice and let Ω be a Noetherian Λ-module. Let b be an element of Λ and let B be an element of Ω. If $[b, I]$ is finite dimensional, then for each number n, $[b^nB, B]$ is finite dimensional.*

Proof. Assume that $[b, I]$ is finite dimensional and let n be a natural number. We have that

$$b^nB \leqslant b^{n-1}B \leqslant \ldots \leqslant bB \leqslant B.$$

By Theorem 2.2, each quotient $[b^kB, b^{k-1}B]$ is finite dimensional, for $1 \leqslant k \leqslant n$. Since Ω is modular it follows that $[b^nB, B]$ is finite dimensional as claimed.

Corollary 2.4. *Let Λ be a Noether lattice and let b be an element of Λ. If $[b, I]$ is finite dimensional, then $[b^n, I]$ is finite dimensional, for each natural number n.*

Proof. Apply Corollary 2.3 to the Noetherian Λ-module Λ.

Corollary 2.5. *Let Λ be a Noether lattice and let b be an element of Λ. Then the following three conditions are equivalent:*

(2.4) *$[b, I]$ is finite dimensional.*

(2.5) *$[b^n, I]$ is finite dimensional for each natural number n.*

(2.6) *$[b^n, I]$ is finite dimensional for some natural number n.*

Proof. This follows from the modularity of Λ and Corollary 2.4.

§ 3. Representatives. Throughout this section Λ is a Noether lattice and Ω is a Noetherian Λ-module. We assume that a is an element of Λ such that the a-adic pseudometric (see [6], § 3) on Ω is a metric and that the quotient lattice $[a, I]$ is finite dimensional (see [6], § 2).

In this section we will establish a few results concerning certain Cauchy sequences of elements of Ω which will be needed in later sections of this paper. We begin with the following.

Theorem 3.1. *Let $\langle E_i\rangle$, $i = 1, 2, \ldots$, be a sequence of elements of Ω such that, given a natural number n, it follows that*

$$(3.1)\qquad E_{i+1} \leqslant E_i \vee a^nM \quad \text{for all integers } i \geqslant n.$$

Then the sequence $\langle E_i\rangle$, $i = 1, 2, \ldots$, is Cauchy.

Proof. Let $\varepsilon > 0$. Choose n to be the least natural number k such that $2^{-k} < \varepsilon$. Consider the sequence

$$\langle E_i \vee a^nM\rangle \quad \text{for} \quad i = 1, 2, \ldots$$

From (3.1) it follows that this sequence is decreasing in i, for $i \geqslant n$. Also note that

$$a^nM \leqslant E_i \vee a^nM \quad \text{for} \quad i = 1, 2, \ldots$$

Consequently, since $[a^nM, M]$ is finite dimensional by Corollary 2.3, there exists a natural number s such that

$$E_i \vee a^nM = E_j \vee a^nM \quad \text{for all integers } i, j \geqslant s.$$

It follows that

$$d_a(E_i, E_j) \leqslant 2^{-n} < \varepsilon \quad \text{for all integers } i, j \geqslant s.$$

Hence the sequence $\langle E_i\rangle$, $i = 1, 2, \ldots$, is Cauchy and the proof is complete.

Since any decreasing sequence of elements of Ω satisfies the conditions of the above theorem, we obtain the following.

Corollary 3.2. *Let $\langle A_i\rangle$, $i = 1, 2, \ldots$, be a decreasing sequence of elements of Ω. Then $\langle A_i\rangle$ is a Cauchy sequence.*

Recall ([6], Definition 4.7) that a Cauchy sequence $\langle B_i\rangle$ of elements of Ω is said to be *regular* in case

$$B_i \vee a^iM = B_{i+1} \vee a^iM,$$

for all positive integers i, and completely regular in case

$$B_i = B_{i+1} \vee a^iM,$$

for all positive integers i.

THEOREM 3.3. *Let $\langle E_i \rangle$, $i=1,2,\ldots$, be a sequence of elements of Ω such that given a natural number n, it follows that*

$$E_{i+1} \leqslant E_i \vee a^n M \quad \text{for all integers } i \geqslant n \,. \tag{3.2}$$

For each natural number n, define D_n by

$$D_n = \bigwedge_{i \geqslant n} (E_i \vee a^n M)\,.$$

Then the sequence $\langle D_i \rangle$, $i=1,2,\ldots$, is a completely regular Cauchy sequence.

Proof. In order to establish that the sequence $\langle D_i \rangle$, $i=1,2,\ldots$, is a completely regular Cauchy sequence we must show that it is Cauchy and that

$$D_n = D_{n+1} \vee a^n M \quad \text{for all integers } n \geqslant 1\,. \tag{3.3}$$

Thus, let $n \geqslant 1$. Since, from (3.2), the sequence

$$\langle E_i \vee a^n M \rangle \quad \text{for} \quad i=1,2,\ldots$$

is decreasing in i, for $i \geqslant n$, and since

$$a^n M \leqslant E_i \vee a^n M \quad \text{for} \quad i=1,2,\ldots,$$

it follows from Corollary 2.3 that there exists a natural number j such that

$$\bigwedge_{i \geqslant n} (E_i \vee a^n M) = E_j \vee a^n M = E_k \vee a^n M \quad \text{for all integers } k \geqslant j\,. \tag{3.4}$$

Similarly, for the sequence

$$\langle E_i \vee a^{n+1} M \rangle \quad \text{for} \quad i=1,2,\ldots,$$

there exists a natural number p such that

$$\bigwedge_{i \geqslant n+1} (E_i \vee a^{n+1} M) = E_p \vee a^{n+1} M = E_q \vee a^{n+1} M\,, \tag{3.5}$$

for all integers $q \geqslant p$. By combining (3.4) and (3.5) we obtain

$$D_n = \bigwedge_{i \geqslant n} (E_i \vee a^n M) = E_{p+q} \vee a^n M = (E_{p+q} \vee a^{n+1} M) \vee a^n M$$

$$= \Big(\bigwedge_{i \geqslant n+1} (E_i \vee a^{n+1} M)\Big) \vee a^n M = D_{n+1} \vee a^n M$$

which establishes (3.3). It follows from (3.3) that the sequence $\langle D_i \rangle$, $i=1,2,\ldots$, is decreasing and hence Cauchy by Corollary 3.2 which completes the proof.

The sequences $\langle E_i \rangle$ and $\langle D_i \rangle$ in the above theorem have an important relationship which is now established.

THEOREM 3.4. *Let the sequences $\langle E_i \rangle$, $i=1,2,\ldots$, and $\langle D_i \rangle$, $i=1,2,\ldots$, be as in Theorem 3.3. Then the equivalence class determined by $\langle E_i \rangle$ and the equivalence class determined by $\langle D_i \rangle$ are the same.*

Proof. In order to show that $\langle E_i \rangle \sim \langle D_i \rangle$, it is sufficient to show ([6], Theorem 4.4) that, for each nonnegative integer w,

$$E_i \vee a^w M = D_i \vee a^w M\,, \tag{3.6}$$

for all sufficiently large positive integers i. We proceed as follows. Let w be a nonnegative integer. In a manner similar to that used in the proof of Theorem 3.3, for each nonnegative integer n, there exists a positive integer j_n such that $j_n \geqslant n$ and

$$E_i \vee a^n M = E_{j_n} \vee a^n M \quad \text{for all integers } i \geqslant j_n\,.$$

Now, for each nonnegative integer n, let k_n be the least such positive integer j_n. Hence, for each nonnegative integer n, we have that $k_n \geqslant n$ and

$$E_i \vee a^n M = E_{k_n} \vee a^n M \quad \text{for all integers } i \geqslant k_n\,. \tag{3.7}$$

By Theorem 3.1, the sequence $\langle E_i \rangle$, $i=1,2,\ldots$, is Cauchy, so there exists a natural number N such that

$$d_a(E_x, E_y) \leqslant 2^{-w} \quad \text{for all } x, y \geqslant N\,.$$

Thus ([6], Remark 3.6)

$$E_x \vee a^w M = E_y \vee a^w M \quad \text{for all integers } x, y \geqslant N\,. \tag{3.8}$$

Now, let n be a positive integer such that $n \geqslant N+w$. Thus $k_n \geqslant n \geqslant N, w$. Hence from (3.8) we obtain

$$E_{k_n} \vee a^w M = E_n \vee a^w M\,. \tag{3.9}$$

Since $n \geqslant w$, it follows from (3.7) and (3.9) that

$$D_n \vee a^w M = \Big[\bigwedge_{i \geqslant n} (E_i \vee a^n M)\Big] \vee a^w M = (E_{k_n} \vee a^n M) \vee a^w M = E_n \vee a^w M\,.$$

Thus, for all positive integers $i \geqslant N+w$, we have that

$$E_i \vee a^w M = D_i \vee a^w M$$

which establishes (3.6) and completes the proof.

As an immediate Corollary to Theorem 3.3 and Theorem 3.4 we have the following result.

THEOREM 3.5. *Let the sequences $\langle E_i \rangle$, $i=1,2,\ldots$, and $\langle D_i \rangle$, $i=1,2,\ldots$, be as in Theorem 3.3. Then the sequence $\langle D_i \rangle$ is the unique* *completely regular representative of the equivalence class determined by the sequence* $\langle E_i\rangle$.

§ 4. Noetherian completions. Throughout this section Λ is a Noether lattice and Ω is a Noetherian Λ-module. In addition we assume that a is an element of Λ such that the a-adic pseudometric (see [6], § 3) on Λ and Ω is a metric and that the quotient lattice $[a, I]$ is finite dimensional. Thus, the a-adic completions of Λ and Ω ([6], § 6) may be constructed. Throughout this section Ω^* will denote the a-adic completion of Ω, and Λ^* will denote the a-adic completion of Λ. It has been established previously ([6], § 7) that Ω^* is a Λ^*-module. In this section (Theorem 4.8) we show that Λ^* is a Noether lattice and that Ω^* is a Noetherian Λ^*-module given the above assumptions.

The first two theorems determine the structure of the completely regular representatives of $B\wedge C$ and $B\vee C$, for B and C in Ω^*. Theorem 4.3 shows that Ω^* is modular and Theorem 4.4 establishes the completely regular representative for residuation. Theorem 4.6 provides a useful tool for obtaining principal elements in the Λ^*-module Ω^*, and finally, the last theorem establishes our goal.

THEOREM 4.1. *Let B and C be elements of Ω^*. Let the sequences $\langle B_n\rangle$ and $\langle C_n\rangle$ be the completely regular representatives of B and C, respectively. Then*

(4.1) *The sequence $\langle B_n\wedge C_n\rangle$ is Cauchy and is a representative of $B\wedge C$.*

(4.2) *The sequence $\langle \bigwedge_{i\geqslant n} [(B_i\wedge C_i)\vee a^nM]\rangle$, $n = 1, 2, \ldots$ is the completely regular representative of $B\wedge C$.*

Proof. Since the sequences $\langle B_i\rangle$ and $\langle C_i\rangle$ are completely regular, they are decreasing. Hence the sequence $\langle B_i\wedge C_i\rangle$ is decreasing and thus Cauchy (Corollary 3.2). Let D be the equivalence class determined by $\langle B_i\wedge C_i\rangle$ and let $\langle D_i\rangle$ be the completely regular representative of D. Thus, by Theorem 3.5, we have that

$$D_n = \bigwedge_{i\geqslant n} [(B_i\wedge C_i)\vee a^nM] \quad \text{for} \quad n = 1, 2, \ldots$$

Since $D_n\leqslant B_n$ and $D_n\leqslant C_n$, for all integers $n\geqslant 1$, it follows ([6], Proposition 5.10) that $D\leqslant B$ and $D\leqslant C$, and consequently $D\leqslant B\wedge C$.

Suppose F is an element of Ω^* such that $F\leqslant B$ and $F\leqslant C$, and let the sequence $\langle F_i\rangle$ be the completely regular representative of F. Then $F_n\leqslant B_n$ and $F_n\leqslant C_n$, for all integers $n\geqslant 1$ ([6], Proposition 5.9), and hence $F_n\leqslant B_n\wedge C_n$, for all integers $n\geqslant 1$. Consequently

$$F_n = \bigwedge_{i\geqslant n} (F_i\vee a^nM) \leqslant \bigwedge_{i\geqslant n} [(B_i\wedge C_i)\vee a^nM] = D_n,$$

for all integers $n\geqslant 1$. Thus $F\leqslant D$. We conclude that $D = B\wedge C$ which establishes (4.1) and (4.2) and completes the proof.

The following result is Proposition 4.7 of [6] and is included here for completeness of this section. The reader is referred to [6] for the proof.

THEOREM 4.2. *Let B and C be elements of Ω^*. Let the sequences $\langle B_i\rangle$ and $\langle C_i\rangle$ be the completely regular representatives of B and C, respectively. Then the sequence $\langle B_i\vee C_i\rangle$ is Cauchy and is the completely regular representative of $B\vee C$.*

The above two theorems are used in the next result to show that Ω^* is modular.

THEOREM 4.3. *The lattice Ω^* is modular.*

Proof. Let A, B and C be elements of Ω^* such that $A\geqslant B$. Let the sequences $\langle A_i\rangle$, $\langle B_i\rangle$, and $\langle C_i\rangle$ be the completely regular representatives of A, B, and C, respectively. Since $A\geqslant B$, we have $A_i\geqslant B_i$, for all integers $i\geqslant 1$ ([6], Proposition 5.9). Let $\langle D_i\rangle$ and $\langle E_i\rangle$ be the completely regular representatives of $A\wedge(B\vee C)$ and $B\vee(A\wedge C)$, respectively.

Let n be a positive integer. Since Ω is modular, $\langle B_i\rangle$ is completely regular, and $[a^nM, M]$ is finite dimensional (Corollary 2.3), it follows that

$$D_n = \bigwedge_{i\geqslant n} [(A_i\wedge(B_i\vee C_i))\vee a^nM] = \bigwedge_{i\geqslant n} [(B_i\vee(A_i\wedge C_i))\vee a^nM]$$
$$= \bigwedge_{i\geqslant n} [B_n\vee((A_i\wedge C_i)\vee a^nM)] = B_n\vee \bigwedge_{i\geqslant n} ((A_i\wedge C_i)\vee a^nM) = E_n$$

by Theorems 4.1 and 4.2, and Theorem 4.10 of [6]. Thus, for each integer $n\geqslant 1$, we have $D_n = E_n$. It follows that

$$A\wedge(B\vee C) = B\vee(A\wedge C)$$

and hence that Ω^* is modular as claimed.

We need the next result in order to work with residuation in Ω^*.

THEOREM 4.4. *Let A and B be elements of Ω^*. Let the sequences $\langle A_n\rangle$ and $\langle B_n\rangle$ be the completely regular representatives of A and B, respectively. Then*

(4.3) *The sequence $\langle A_n:B_n\rangle$ is Cauchy and is a representative of $A:B$.*

(4.4) *The sequence $\langle \bigwedge_{i\geqslant n} [(A_i:B_i)\vee a^n]\rangle$, $n = 1, 2, \ldots$, is the completely regular representative of $A:B$ in Λ^*.*

Proof. Since the sequences $\langle A_n\rangle$ and $\langle B_n\rangle$ are completely regular, it follows that

$$A_{n+1}:B_{n+1} \leqslant (A_{n+1}\vee a^nM):B_{n+1} = (A_{n+1}\vee a^nM):(B_{n+1}\vee a^nM) = A_n:B_n,$$

for each integer $n \geqslant 1$. Thus the sequence $\langle A_n{:}B_n\rangle$ is decreasing and hence Cauchy by Corollary 3.2. Let b in Λ^* be the equivalence class determined by the sequence $\langle A_n{:}B_n\rangle$. Let the sequences $\langle b_n\rangle$ and $\langle c_n\rangle$ be the completely regular representatives of b and $A{:}B$, respectively. Note that by Theorem 3.5 we have

$$b_n = \bigwedge_{i\geqslant n} [(A_i{:}B_i)\vee a^n] \quad \text{for} \quad n = 1, 2, \ldots$$

In order to complete the proof it is sufficient to show that $b = A{:}B$.

Since, for each integer $n \geqslant 1$, we have that

$$b_n B_n = \Big(\bigwedge_{i\geqslant n} [(A_i{:}B_i)\vee a^n]\Big) B_n \leqslant [(A_n{:}B_n)\vee a^n] B_n \leqslant A_n\,,$$

it follows from Propositions 5.10 and 5.13 of [6] that $bB \leqslant A$, and consequently $b \leqslant A{:}B$. Since $(A{:}B)B \leqslant A$, we have that

$$c_i B_i \leqslant c_i B_i \vee a^i M \leqslant A_i \quad \text{for all integers } i \geqslant 1 \tag{4.5}$$

([6], Proposition 5.9 and Corollary 5.15). Hence, for each integer $n \geqslant 1$, it now follows from (4.5) that

$$c_n = \bigwedge_{i\geqslant n} (c_i \vee a^n) \leqslant \bigwedge_{i> n} [(A_i{:}B_i)\vee a^n] = b_n\,,$$

and therefore that $A{:}B \leqslant b$ by Proposition 5.10 of [6]. Thus $A{:}B = b$ and the proof is complete.

We will make use of the following characterization of principal elements in the process of proving Theorem 4.6. The reader is referred to Lemma 2.1 of [4] for a proof.

LEMMA 4.5. *Let Ω be an Λ-module and let A be an element of Ω. If Ω and Λ are modular, then A is a principal element of Ω if and only if:*

$$C \wedge A = (C{:}A)A \tag{4.6}$$

and

$$bA{:}A = b \vee (0{:}A)\,, \tag{4.7}$$

for all b in Λ and for all C in Ω.

THEOREM 4.6. *Let $\langle B_i\rangle$ be a Cauchy sequence of elements of Ω. If each B_i is a principal element of Ω, then the equivalence class determined by $\langle B_i\rangle$ is a principal element of the Λ^*-module Ω^*.*

Proof. Assume each B_i, $i = 1, 2, \ldots$, is a principal element of Ω. By selecting a subsequence of $\langle B_i\rangle$, we may assume without loss of generality that $\langle B_i\rangle$ is a regular Cauchy sequence. Let D in Ω^* denote the class determined by $\langle B_i\rangle$ and let $\langle D_i\rangle$ be the completely regular

representative of D. Since Ω^* and Λ^* are modular (Theorem 4.3) we shall show that D is principal by establishing (4.6) and (4.7) of Lemma 4.5. Note that $D_n = B_n \vee a^n M$, for each $n \geqslant 1$ ([6], Corollary 4.13 and Theorem 4.14).

Let C be an element of Ω^* and let $\langle C_i\rangle$ be the completely regular representative of C. For each integer $n \geqslant 1$, a routine computation shows that

$$\begin{aligned} C_n \wedge D_n &= C_n \wedge (B_n \vee a^n M) = a^n M \vee \big(C_n{:}(B_n \vee a^n M)\big) B_n \\ &= a^n M \vee (C_n{:}D_n) B_n \end{aligned} \tag{4.8}$$

since M is modular and B_n is a principal element of M. Using Theorem 4.1 and Theorem 4.4 above, it now follows from (4.8) that $C \wedge D = (C{:}D)D$ by ([6], Corollary 4.6 and (5.16)). Thus (4.6) of Lemma 4.5 is established.

Now, let b be an element of Λ^* and let $\langle b_i\rangle$ be the completely regular representative of b. A routine computation shows that, for each integer $n \geqslant 1$,

$$\begin{aligned} (b_n D_n \vee a^n M){:}D_n &= \big(b_n (B_n \vee a^n M) \vee a^n M\big){:}D_n = b_n \vee \big(a^n M{:}(B_n \vee a^n M)\big) \\ &= b_n \vee (a^n M{:}B_n) \end{aligned} \tag{4.9}$$

because each B_n is principal. By ([6], Corollary 5.15 and Corollary 4.13) and Theorem 4.4, it follows from (4.9) that $bD{:}D = b \vee (0{:}D)$ which establishes (4.7) of Lemma 4.5 and completes the proof.

In order to establish our main theorem we require the following result.

THEOREM 4.7. *The Λ^*-module Ω^* is principally generated and satisfies the ascending chain condition.*

Proof. It was shown in Theorem 6.3 of [6] that Ω^* satisfies the ascending chain condition. Thus, we need only establish here that Ω^* is principally generated as an Λ^*-module. For this, since Ω^* satisfies the ascending chain condition, it is sufficient to show that: if given any two elements A and B of Ω^* with $A < B$, there exists a principal element D in Ω^* such that $D \nleqslant A$ and $D \leqslant B$.

Thus, let A and B be elements of Ω^* with $A < B$. Let $\langle A_i\rangle$ and $\langle B_i\rangle$ be the completely regular representatives of A and B, respectively. Since $A \leqslant B$ and $A \neq B$ it follows that $A_i \leqslant B_i$, $i = 1, 2, \ldots$, and that there exists a natural number n such that $A_i < B_i$ for all integers $i \geqslant n$. In particular, $A_n < B_n$, and hence, since Ω is principally generated, there exists a principal element D_n in Ω such that $D_n \leqslant B_n$ and $D_n \nleqslant A_n$. We will now construct inductively a sequence D_{n+j}, $j = 1, 2, \ldots$, of principal elements of Ω such that

$$D_{n+j} \leqslant (D_{n+j-1} \vee a^{n+j-1} M) \wedge B_{n+j} \tag{4.10}$$

and

$$D_{n+j} \nleqslant A_n . \tag{4.11}$$

Suppose D_{n+j}, $j = 1, 2, \ldots, k$, have been chosen such that (4.10) and (4.11) are satisfied. If

$$(D_{n+k} \vee a^{n+k}M) \wedge B_{n+k+1} \leqslant A_n ,$$

then

$$D_{n+k} \leqslant (D_{n+k} \vee a^{n+k}M) \wedge B_{n+k} = (D_{n+k} \vee a^{n+k}M) \wedge (B_{n+k+1} \vee a^{n+k}M)$$
$$= \big((D_{n+k} \vee a^{n+k}M) \wedge B_{n+k+1}\big) \vee a^{n+k}M \leqslant A_n ,$$

since the sequence $\langle B_i \rangle$ is completely regular, which contradicts our assumption that (4.11) holds for D_{n+k}. Consequently we have that

$$(D_{n+k} \vee a^{n+k}M) \wedge B_{n+k+1} \nleqslant A_n . \tag{4.12}$$

From (4.12) and the fact that Ω is principally generated we can choose a principal element D_{n+k+1} in Ω such that

$$D_{n+k+1} \leqslant (D_{n+k} \vee a^{n+k}M) \wedge B_{n+k+1}$$

and

$$D_{n+k+1} \nleqslant A_n$$

which completes the inductive construction.

Now, for $1 \leqslant i \leqslant n-1$, set $D_i = D_n$. It follows from (4.10) and Theorem 3.1 that the sequence $\langle D_i \rangle$, $i = 1, 2, \ldots$, is Cauchy. Let D be the equivalence class determined by the sequence $\langle D_i \rangle$. Since each D_i, $i = 1, 2, \ldots$, is principal by construction, D is a principal element of the Λ^*-module Ω^* by Theorem 4.6.

Suppose for a moment that $D \leqslant A$. Let $\langle E_i \rangle$, $i = 1, 2, \ldots$, be a subsequence of $\langle D_i \rangle$ which is regular ([6], Lemma 4.11). Then

$$E_i \leqslant E_i \vee a^i M \leqslant A_i \leqslant A_n \quad \text{for all integers } i \geqslant n , \tag{4.13}$$

by ([6], Corollary 4.13 and Proposition 5.9). From (4.13) it would follow that $D_j \leqslant A_n$, for all large integers j, which is a contradiction to (4.11). Thus $D \nleqslant A$. It follows from (4.10) that $D \leqslant B$ which completes the proof.

We are now in a position to establish our main theorem.

THEOREM 4.8. *Let Λ be a Noether lattice, let Ω be a Noetherian Λ-module, let b be an element of Λ such that the b-adic pseudometric on Λ and Ω is a metric, and let Λ^* and Ω^* be the b-adic completions of Λ and Ω, respectively. If there exists a natural number n such that $[b^n, I]$ is finite dimensional, then*

(4.14) *Λ^* is a Noether lattice*

and

(4.15) *Ω^* is a Noetherian Λ^*-module.*

Proof. Assume $[b^n, I]$ is finite dimensional. Then (4.15) follows from Corollary 2.5 and Theorems 4.3 and 4.7. Since Λ^* is a Noetherian Λ^*-module, (4.14) follows from (4.15) which completes the proof.

The following note provides an interesting application of Theorem 4.8 to the particular case of local and semi-local Noether lattices which are of special interest (see [5]).

Remark 4.9. Let Λ be a Noether lattice and let Ω be a Noetherian Λ-module. If b is an element of Λ such that the b-adic pseudometric on Λ and Ω is a metric, denote the b-adic completion of Λ by Λ_b^* and the b-adic completion of Ω by Ω_b^*. Define m in Λ by

$$m = \bigwedge \{p \text{ in } \Lambda \mid p \text{ is a maximal element of } \Lambda\} .$$

From Corollary 3.4 of [4], if b is an element of Λ such that $b \leqslant m$, then the b-adic pseudometric on Λ and Ω is a metric and, thus, for $b \leqslant m$, Λ_b^* and Ω_b^* are defined. In conjunction with the above we have the following:

Let b be an element of Λ such that $b \leqslant m$. If

(4.16) *$[b, I]$ is finite dimensional,*

or

(4.17) *Λ is semi-local and $[b, m]$ is finite dimensional,*

then

(4.18) *Λ_b^* is a Noether lattice*

and

(4.19) *Ω_b^* is a Noetherian Λ_b^*-module.*

Proof. Assume that (4.16) or (4.17) holds. If Λ is semi-local and $[b, m]$ is finite dimensional, then, since $[m, I]$ is finite dimensional by Corollary 4.5 of [4], it follows that $[b, I]$ is finite dimensional by the modularity of Ω. Hence, in either case, $[b, I]$ is finite dimensional. It now follows from Theorem 4.8 and the above comments that Λ_b^* is a Noether lattice and Ω_b^* is a Noetherian Λ_b^*-module which completes the proof.

Note in particular that Theorem 5.9 of [4] and Theorem 8.7 of [6] are just special cases of the preceeding result which in turn is a special case of Theorem 4.8.

Remark 4.10. We point out that the lattice of ideals of the completion $\bar{R}$ of a local ring R can be obtained lattice theoretically from the lattice of ideals of R by using the lattice completion concept discussed above. This is achieved in the following manner. Assume (R, p) is a local ring (commutative and Noetherian with identity) with p-adic (ring) completion $\bar{R}$. Let Λ and $\bar{\Lambda}$ denote the lattice of ideals of R and $\bar{R}$, respectively. It can be shown [8] that $\bar{\Lambda}$ and Λ_p^* (the p-adic lattice completion of Λ — see Remark 4.9) are isomorphic as multiplicative lattices. Thus, the ideal structure of $\bar{R}$ can be determined by purely lattice theoretical means from the lattice of ideals of R.

References

[1] K. P. Bogart, *Structure theorems for regular local Noether lattices*, Michigan Math. J. 15 (1968), pp. 167–176.

[2] — *Nonimbeddable Noether lattices*, Proc. Amer. Math. Soc. 22 (1969), pp. 129–133.

[3] R. P. Dilworth, *Abstract commutative ideal theory*, Pacific J. Math. 12 (1962), pp. 481–498.

[4] E. W. Johnson and J. A. Johnson, *Lattice modules over semi-local Noether lattices*, Fund. Math. 68 (1970), pp. 187–201.

[5] — — and J. P. Lediaev, *A structural approach to Noether lattices*, Canad. J. Math. 22 (1970), pp. 657–665.

[6] J. A. Johnson, *A-adic completions of Noetherian lattice modules*, Fund. Math. 66 (1970), pp. 347–373.

[7] — *Quotients in Noetherian lattice modules*, Proc. Amer. Math. Soc. 28 (1971), pp. 71–74.

[8] — *A note on the lattices of ideals of the completion of a Noetherian ring* (to appear).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON
Houston, Texas

Accepté par la Rédaction le 29. 10. 1973

Locally flat embeddings of Hilbert cubes are flat

by

T. A. Chapman * (Lexington, Ken.)

Abstract. In this paper it is shown that any locally flat embedding of the Hilbert cube Q into a Q-manifold is flat. The techniques employed in the proof of this result also imply that the group of homeomorphisms of $Q\times R^n$ onto itself which are fixed on $Q\times\{0\}$ has exactly two components.

1. Introduction. For topological spaces X and Y, an embedding $i\colon X\to Y$ is said to be *locally flat* (with codimension n) provided that each point of X has a neighborhood U and an open embedding $h\colon U\times R^n\to Y$ such that $h(x,0)=i(x)$, for all $x\in U$. We say that the embedding is *flat* if we can take $U=X$. We use R^n to denote euclidean n-space, Q to denote the Hilbert cube (i.e. the countable infinite product of closed intervals), and by a *Q-manifold* we mean a separable metric manifold modeled on Q. The following is the main result of this paper.

THEOREM 1. *If X is a Q-manifold and $i\colon Q\to X$ is a locally flat embedding, then i is flat.*

Of course this result is false if Q is replaced by a more complicated Q-manifold. For example let $X=M\times Q$, where M is the open Möbius band, let $i_1\colon S^1\to M$ be a homeomorphism of the 1-sphere onto the center circle, and let $i=i_1\times\mathrm{id}\colon S^1\times Q\to M\times Q$. Then i is a codimension 1 locally flat embedding, but i is not flat. (If i were flat, then arbitrarily small neighborhoods of $i_1(S^1)$ in M would be separated by $i_1(S^1)$.) A more general question would be to investigate when locally flat embeddings of Q-manifolds into Q-manifolds have normal bundles (see [2] and [4] for finite-dimensional results).

Let $\mathcal{H}_0(Q\times R^n)$ denote the space of all homeomorphisms of $Q\times R^n$ onto itself (with the CO-topology) which are the identity on $Q\times\{0\}$. The following result is a by-product of the proof of Theorem 1.

THEOREM 2. *$\pi_0(\mathcal{H}_0(Q\times R^n))=2$, for all $n\geqslant 1$. That is, $\mathcal{H}_0(Q\times R^n)$ has exactly two components.*

* Supported in part by NSF Grant GP-28374.