Finite dimensional completions in Noether lattices

by

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Abstract. This paper is concerned with the completion of Noetherian lattice modules. It is shown that under relatively general conditions the completion of a Noetherian $A$-module $\mathcal{O}$ is Noetherian as a lattice module over the completion of $A$.

§ 1. Introduction. In [3] B. P. Dilworth began a study of the ideal theory of commutative rings with identity from a lattice theoretic viewpoint and introduced Noether lattices. In a natural manner this theory can be generalized to include the study of module theory as a branch of lattice theory and there are reasons to do so (for example see [7]). Let $R$ be a Noetherian ring and let $M$ be a Noetherian $R$-module. Denote the lattice of ideals of $R$ by $L(R)$ and the lattice of submodules of $M$ by $L(M)$. Then, under the canonical multiplication between elements of $L(R)$ and elements of $L(M)$, it is easily verified that $L(R)$ is a Noether lattice (see [3]) and that $L(M)$ is a Noetherian $L(R)$-module (the definition of a lattice module is given in Section 2). We note here that there exist Noether lattices $A$ and Noetherian $A$-modules $\mathcal{O}$ which are not lattice isomorphic to the lattice of submodules of any Noetherian module over a Noetherian ring (see [1] and [2] for some interesting examples in Noether lattices) and thus the class of Noetherian lattice modules derived from Noetherian rings and modules does not exhaust the class of all Noetherian lattice modules. In this paper we are concerned with the completion (see Section 4) of a Noetherian $A$-module $\mathcal{O}$. This concept is closely related to the $\alpha$-adic completion of Noetherian rings and modules (see Remark 4.10).

In Section 2 we introduce some notation and establish some useful results concerning dimensions of certain sublattices. If $A$ is a Noether lattice, $\mathcal{O}$ is a Noetherian $A$-module, and $b$ is an element of $A$, we show (Theorem 2.2) that the sublattice $[bR, R]$ of $\mathcal{O}$ is finite dimensional, for each $B$ in $\mathcal{O}$, provided that the sublattice $[b, B]$ of $A$ is finite dimensional.

Assume that $\alpha$ is an element of $A$ such that the $\alpha$-adic pseudometric on $\mathcal{O}$ is a metric. Section 3 is devoted to determining the nature of certain Cauchy sequences (Cauchy relative to the $\alpha$-adic metric on $\mathcal{O}$) of ele-
ments of \( \Omega \) which are needed in the subsequent section. Let \( \langle E_i \rangle, i = 1, 2, \ldots, \) be a Cauchy sequence of elements of \( \Omega \) satisfying the condition that: given any natural number \( n \), it follows that \( E_{i+n} \leq E_i \vee \alpha^* M_i \), for all integers \( i \geq n \). It is shown (Theorem 3.5) that the sequence \( \langle E_i \rangle \), where \( D_n = \bigvee (E_i \vee \alpha^* M_i), \), \( n = 1, 2, \ldots, \) is the completely regular representative (see Definition in Section 3) of the equivalence class determined by the Cauchy sequence \( \langle E_i \rangle \). Thus the unique completely regular representative of a Cauchy sequence satisfying the above condition is completely determined. This result is needed in Section 4 to determine the completely regular representatives for meet and residuation of elements in the completion of \( \Omega \).

Let \( c \) be an element of \( A \) such that the \( \alpha \)-adic pseudometric on \( A \) and \( \Omega \) is a metric (see Section 4) and \( \{a, \alpha \} \) is finite dimensional. Let \( A^* \) and \( \Omega^* \) be the \( \alpha \)-adic completions of \( A \) and \( \Omega \), respectively (see Section 4 for definitions). Theorems 4.1 and 4.2 determine the structure of the completely regular representatives for meet and join, respectively, of elements of \( \Omega^* \). It is shown in Theorem 4.3 that \( \Omega^* \) is modular and the completely regular representative for residuation is determined in Theorem 4.4. Theorem 4.6 provides a way to obtain principal elements in \( \Omega^* \) from principal elements in \( \Omega \) and this result is used to show (Theorem 4.7) that the \( A^* \)-module \( \Omega^* \) is principally generated and satisfies the ascending chain condition. Our last result is the following (Theorem 4.8):

Let \( A \) be a Noether lattice, let \( \Omega \) be a Noetherian \( A \)-module, let \( c \) be an element of \( A \) such that the \( \alpha \)-adic pseudometric on \( A \) and \( \Omega \) is a metric, and let \( A^* \) and \( \Omega^* \) be the \( \alpha \)-adic completions of \( A \) and \( \Omega \), respectively. If there exists a natural number \( n \) such that \( \{a, \alpha \} \) is finite dimensional, then

\[ (1.1) \quad A^* \text{ is a Noether lattice, and} \]
\[ (1.2) \quad \Omega^* \text{ is a Noetherian} \ A^* \text{ module.} \]

Finally, an application of Theorem 4.8 to the special case of local and semi-local Noether lattices is given in Remark 4.9.

§ 2. Notation and preliminary results. A lattice is said to be multiplicative in case it is a complete lattice on which there is defined a commutative, associative, join distributive multiplication such that the unit element of the lattice is an identity for the multiplication. Let \( \Omega \) be a complete lattice and let \( A \) be a multiplicative lattice. Elements of \( \Omega \) will be denoted by \( A, B, C, \ldots \). The null element and unit element of \( \Omega \) will be denoted by 0 and \( M \), respectively. Elements of \( A \) will be denoted by \( a, b, c, \ldots \). The null element and the unit element of \( A \) will be denoted by 0 and \( I \), respectively. \( \Omega \) is said to be an \( A \)-module in case there is a multiplication between elements of \( A \) and elements of \( \Omega \), denoted \( aA \), for \( a \in A \) and \( A \in \Omega \), such that

1. \( (ab)A = a(bA) \);
2. \( (\bigvee b_i)(\bigvee B_j) = \bigvee b_iB_j \);
3. \( IA = A \);
4. \( 0A = 0 \).

\[ \text{for all } a, b, \] \( A \in \Omega \).

Let \( \Omega \) be an \( A \)-module. For \( A \) and \( B \) in \( \Omega \), \( A \leq B \) will denote the largest \( B \) in \( \Omega \) such that \( bB \leq A \). An element \( A \) in \( \Omega \) is said to be meet principal in case \( \langle (b \land (B \leq A)) \rangle = A \).

\[ \text{for all } b \leq A \text{ and for all } B \in \Omega; \ A \text{ is said to be join principal in case} \]
\[ b \lor (B \leq A) = (bA \lor B) : A \]

\[ \text{for all } b \leq A \text{ and for all } B \in \Omega; \ A \text{ is said to be principal in case} A \text{ is both meet and join principal. \( \Omega \) is called principally generated if each} \]
\[ \text{element of \( \Omega \) is the join (finite or infinite) of principal elements of \( \Omega \). If \( \Omega \) is principally generated, modular, and satisfies the ascending chain condition, \( \Omega \) is said to be a Noetherian} \ A \text{-module. If \( A \) is a Noetherian} \ A \text{-module, \( \Omega \) is called a Noether lattice. For other properties and general definitions related to Noetherian lattices and Noetherian lattice modules the reader should consult the references.} \]

The following lemma is useful in the proof of Theorem 2.2. The reader is referred to Lemma 2.5 of [4] for a proof.

**Lemma 2.1.** Let \( \Omega \) be an \( A \)-module. Let \( A \) be a principal ideal of \( \Omega \), let \( a \) and \( b \) be elements of \( A \) such that \( a \leq b \), let \( A \) be modular, and let \( \langle b, A \rangle \), \( b \leq a \). Then the map \( \varphi: [a, b] \rightarrow [a, A], b \leq a \) defined by \( \varphi([x]) \) is a lattice isomorphism of \( [a, b] \) onto \( [a, A] \).

We now establish the following results on dimensions which will be required later.

**Theorem 2.2.** Let \( A \) be a Noether lattice, let \( \Omega \) be a Noetherian \( A \)-module, and let \( B \) be an element of \( A \). If \( \langle a, B \rangle \) is finite dimensional, then \( \langle bB, B \rangle \) is finite dimensional for all elements \( B \in \Omega \).

**Proof.** Assume \( \langle b, B \rangle \) is finite dimensional and let \( B \) be an arbitrary element of \( \Omega \). Since \( \Omega \) is Noetherian there are principal elements \( b_i, B_2, \ldots, B_k \) in \( \Omega \) such that \( B = B_1 \vee \ldots \vee B_k \). Define \( T_0 = b \cup B \) and define \( T_j = B \vee B_{j+1} \), for each \( j, 0 \leq j \leq k-1 \). Thus,

\[ T_j = b \vee B_j \vee B_{j+1} \] \( \quad \text{for } \quad 1 \leq j \leq k. \]

Since each \( B_m \), \( 1 \leq m \leq k \), is principal we obtain

\[ (2.1) \quad [T_j, T_{j+1}] = [T_j, B_j \vee B_{j+1}] \cong [T_j, B_{j+1}, T_{j+1}] \]

\[ = [(T_j, B_{j+1}) B_{j+1}, B_{j+1}] \]
for each $j$, $0 \leq j \leq k-1$, by the isomorphism theorems. Also, for each $j$, $0 \leq j \leq k-1$, we have

$$0:B_{j+1}/\sim I \cong (0:B_{j+1}) \leq T_j:B_{j+1},$$

so that

$$(2.2) \quad [T_j:B_{j+1}, I] \cong ([T_j:B_{j+1}]B_{j+1}, I)$$

for each $j$, $0 \leq j \leq k-1$, by Lemma 2.1. From (2.1) and (2.2) it now follows that

$$(2.3) \quad [T_j, T_{j+1}] \cong [T_j:B_{j+1}, I]$$

for each $j$, $0 \leq j \leq k-1$. Now, observe also that

$$bB_j \leq bB \leq \ldots \leq B = T_1,$$

so that

$$b \leq (T_1:B_{j+1})$$

for each $j$, $0 \leq j \leq k-1$. Consequently, $[T_j:B_{j+1}, I]$ is finite dimensional, and hence $[T_1, T_{j+1}]$ is finite dimensional from (2.3), for each $j$, $0 \leq j \leq k-1$. Hence, since

$$bB = T_k \leq T_{k-1} \leq \ldots \leq T_1 = bB \leq B,$$

it follows that $[bB, B]$ is finite dimensional which completes the proof.

As a useful corollary we obtain the following result.

**Corollary 2.3.** Let $A$ be a Noether lattice and let $\Omega$ be a Noetherian $A$-module. Let $b$ be an element of $A$ and let $\theta$ be an element of $\Omega$. If $[b, I]$ is finite dimensional, then for each number $n$, $[b^n, B]$ is finite dimensional.

**Proof.** Assume that $[b, I]$ is finite dimensional and let $n$ be a natural number. We have that

$$b^nB \leq b^{n-1}B \leq \ldots \leq bB \leq B.$$

By Theorem 2.2, each quotient $[b^nB, b^{n-1}B]$ is finite dimensional, for $1 \leq k \leq n$. Since $\Omega$ is modular it follows that $[b^nB, B]$ is finite dimensional as claimed.

**Corollary 2.4.** Let $A$ be a Noether lattice and let $b$ be an element of $A$. If $[b, I]$ is finite dimensional, then $[b^n, I]$ is finite dimensional, for each natural number $n$.

**Proof.** Apply Corollary 2.3 to the Noetherian $A$-module $A$.

**Corollary 2.5.** Let $A$ be a Noether lattice and let $b$ be an element of $A$. Then the following three conditions are equivalent:

1. $[b, I]$ is finite dimensional.
2. $[b^n, I]$ is finite dimensional for each natural number $n$.
3. $[b^n, I]$ is finite dimensional for some natural number $n$.

**Proof.** This follows from the modularity of $A$ and Corollary 2.4.

**§ 3. Representatives.** Throughout this section $A$ is a Noether lattice and $\Omega$ is a Noetherian $A$-module. We assume that $a$ is an element of $A$ such that the $a$-adic pseudometric (see [8], § 3) on $\Omega$ is a metric and that the quotient lattice $[a, I]$ is finite dimensional (see [8], § 2).

In this section we will establish a few results concerning certain Cauchy sequences of elements of $\Omega$ which will be needed in later sections of this paper. We begin with the following.

**Theorem 3.1.** Let $\{E_i\}, i = 1, 2, \ldots$, be a sequence of elements of $\Omega$ such that, given a natural number $n$, it follows that

$$E_{i+1} \leq E_i \vee a^nM$$

for all integers $i \geq n$.

Then the sequence $\{E_i\}, i = 1, 2, \ldots$, is Cauchy.

**Proof.** Let $\varepsilon > 0$. Choose $n$ to be the least natural number $k$ such that $2^{-k} < \varepsilon$. Consider the sequence

$$\langle E_i \vee a^nM \rangle$$

for $i = 1, 2, \ldots$.

From (3.1) it follows that this sequence is decreasing in $i$, for $i \geq n$. Also note that

$$a^nM \leq E_i \vee a^nM$$

for $i = 1, 2, \ldots$.

Consequently, since $[a^nM, M]$ is finite dimensional by Corollary 2.3, there exists a natural number $s$ such that

$$E_i \vee a^nM = E_i \vee a^sM$$

for all integers $i, j \geq s$.

It follows that

$$d(E_i, E_j) \leq 2^{-s} < \varepsilon$$

for all integers $i, j \geq s$.

Hence the sequence $\{E_i\}, i = 1, 2, \ldots$, is Cauchy and the proof is complete.

Since any decreasing sequence of elements of $\Omega$ satisfies the conditions of the above theorem, we obtain the following.

**Corollary 3.2.** Let $\{E_i\}, i = 1, 2, \ldots$, be a decreasing sequence of elements of $\Omega$. Then $\langle E_i \rangle$ is a Cauchy sequence.

Recall ([8], Definition 4.7) that a Cauchy sequence $\{E_i\}$ of elements of $\Omega$ is said to be regular in case

$$E_i \vee a^nM = E_{i+n} \vee a^nM,$$

for all positive integers $i$, and completely regular in case

$$E_i = E_{i+n} \vee a^nM,$$

for all positive integers $i$. 

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THEOREM 3.3. Let \( \langle E_i \rangle, i = 1, 2, \ldots \), be a sequence of elements of \( \Omega \) such that given a natural number \( n \), it follows that
\[
E_{i+1} \preceq E_i \oplus a^n M
\]
for all integers \( i \geq n \).

For each natural number \( n \), define \( D_n \) by
\[
D_n = \bigwedge_{i=n}^\omega (E_i \oplus a^n M).
\]
Then the sequence \( \langle D_i \rangle, i = 1, 2, \ldots \), is a completely regular Cauchy sequence.

Proof. In order to establish that the sequence \( \langle D_i \rangle, i = 1, 2, \ldots \), is a completely regular Cauchy sequence we must show that it is Cauchy and that
\[
D_k = D_{k+1} \oplus a^n M \quad \text{for all integers } k \geq n.
\]
Thus, let \( n \geq 1 \). Since, from (3.2), the sequence
\[
\langle E_i \oplus a^n M \rangle \quad \text{for } i = 1, 2, \ldots
\]
is decreasing in \( i \), for \( i \geq n \), and since
\[
a^n M \preceq E_i \oplus a^n M \quad \text{for } i = 1, 2, \ldots,
\]
it follows from Corollary 2.3 that there exists a natural number \( j \) such that
\[
\bigwedge_{i=n}^\omega (E_i \oplus a^n M) = E_j \oplus a^n M = E_k \oplus a^n M \quad \text{for all integers } k \geq j.
\]
Similarly, for the sequence
\[
\langle E_i \oplus a^{n+k} M \rangle \quad \text{for } i = 1, 2, \ldots,
\]
there exists a natural number \( p \) such that
\[
\bigwedge_{i=n+1}^\omega (E_i \oplus a^{n+k} M) = E_p \oplus a^{n+k} M = E_q \oplus a^{n+k} M,
\]
for all integers \( q \geq p \). By combining (3.4) and (3.5) we obtain
\[
D_n = \bigwedge_{i=n}^\omega (E_i \oplus a^n M) = E_{p+q} \oplus a^n M = (E_{p+q} \oplus a^{n+k} M) \oplus a^n M
\]
\[
= \bigwedge_{i=n+1}^\omega (E_i \oplus a^{n+k} M) \oplus a^n M = D_{k+1} \oplus a^n M
\]
which establishes (3.3). It follows from (3.3) that the sequence \( \langle D_i \rangle, i = 1, 2, \ldots \), is decreasing and hence Cauchy by Corollary 3.2 which completes the proof.

The sequences \( \langle E_i \rangle \) and \( \langle D_i \rangle \) in the above theorem have an important relationship which is now established.
completely regular representative of the equivalence class determined by the sequence \((E_i)\).

§ 4. Noetherian completions. Throughout this section \(A\) is a Noether lattice and \(\Omega\) is a Noetherian \(A\)-module. In addition we assume that \(a\) is an element of \(A\) such that the \(a\)-adic pseudometric (see [6], § 3) on \(A\) and \(\Omega\) is a metric and that the quotient lattice \([a, I]\) is finite dimensional. Thus, the \(a\)-adic completions of \(A\) and \(\Omega\) ([6], § 6) may be constructed. Throughout this section \(\Omega^*\) will denote the \(a\)-adic completion of \(\Omega\), and \(\Omega^*\) will denote the \(a\)-adic completion of \(\Omega^*\). It has been established previously ([6], § 7) that \(\Omega^*\) is a \(\Omega\)-module. In this section (Theorem 4.8) we show that \(\Omega^*\) is a Noetherian \(\Omega\)-module given the above assumptions.

The first two theorems determine the structure of the completely regular representations of \(B \land C\) and \(B \lor C\), for \(B\) and \(C\) in \(\Omega^*\). Theorem 4.3 shows that \(\Omega^*\) is modular and Theorem 4.4 establishes the completely regular representation for residuation. Theorem 4.6 provides a useful tool for obtaining principal elements in the \(\omega^*\)-module \(\Omega^*\), and finally, the last theorem establishes our goal.

**Theorem 4.1.** Let \(B\) and \(C\) be elements of \(\Omega^*\). Let the sequences \(\langle B_i \rangle\) and \(\langle C_i \rangle\) be the completely regular representatives of \(B\) and \(C\), respectively. Then

\[(4.1)\] The sequence \(\langle B_i \land C_i \rangle\) is Cauchy and is a representative of \(B \land C\).

\[(4.2)\] The sequence \(\langle \bigwedge_{i=1}^n (B_i \lor C_i) \rangle\), \(n = 1, 2, \ldots\), is the completely regular representative of \(B \lor C\).

**Proof.** Since the sequences \(\langle B_i \rangle\) and \(\langle C_i \rangle\) are completely regular, they are decreasing. Hence the sequence \(\langle B_i \land C_i \rangle\) is decreasing and thus Cauchy (Corollary 3.2). Let \(D\) be the equivalence class determined by \(\langle B_i \land C_i \rangle\) and let \(\langle D_i \rangle\) be the completely regular representative of \(D\). Thus, by Theorem 3.3, we have that

\[D_n = \bigwedge_{i=1}^n [(B_i \land C_i) \lor a^* M] \quad \text{for} \quad n = 1, 2, \ldots\]

Since \(D_n \leq B_n\) and \(D_n \leq C_n\), for all integers \(n \geq 1\), it follows ([6], Proposition 5.10) that \(D \leq B\) and \(D \leq C\), and consequently \(D \leq B \land C\).

Suppose \(F\) is an element of \(\Omega^*\) such that \(F \leq B\) and \(F \leq C\); and let the sequence \(\langle F_i \rangle\) be the completely regular representative of \(F\). Then \(F_i \leq B_i\) and \(F_i \leq C_i\), for all integers \(n \geq 1\) ([6], Proposition 5.8), and hence \(F_n \leq B_n \land C_n\), for all integers \(n \geq 1\). Consequently

\[F_n = \bigwedge_{i=1}^n (F_i \lor a^* M) \leq \bigwedge_{i=1}^n [(B_i \land C_i) \lor a^* M] = D_n,\]

for all integers \(n \geq 1\). Thus \(F \leq D\). We conclude that \(D = B \land C\) which establishes (4.1) and (4.2) and completes the proof.

The following result is Proposition 4.7 of [6] and is included here for completeness of this section. The reader is referred to [6] for the proof.

**Theorem 4.2.** Let \(B\) and \(C\) be elements of \(\Omega^*\). Let the sequences \(\langle B_i \rangle\) and \(\langle C_i \rangle\) be the completely regular representatives of \(B\) and \(C\), respectively. Then the sequence \(\langle B_i \lor C_i \rangle\) is Cauchy and is the completely regular representative of \(B \lor C\).

The above two theorems are used in the next result to show that \(\Omega^*\) is modular.

**Theorem 4.3.** The lattice \(\Omega^*\) is modular.

**Proof.** Let \(A\), \(B\) and \(C\) be elements of \(\Omega^*\) such that \(A \geq B\). Let the sequences \(\langle A_i \rangle\), \(\langle B_i \rangle\), and \(\langle C_i \rangle\) be the completely regular representatives of \(A\), \(B\), and \(C\), respectively. Since \(A \geq B\), we have \(A_i \geq B_i\) for all integers \(i \geq 1\) ([6], Proposition 5.9). Let \(\langle D_i \rangle\) and \(\langle E_i \rangle\) be the completely regular representatives of \(A \land (B \lor C)\) and \(B \lor (A \land C)\), respectively.

Let \(n\) be a positive integer. Since \(\Omega\) is modular, \(\langle D_i \rangle\) is completely regular, and \([a^* M, M]\) is finite dimensional (Corollary 2.3), it follows that

\[D_n = \bigwedge_{i=1}^n [(A_i \land (B_i \lor C_i)) \lor a^* M] = \bigwedge_{i=1}^n [(B_i \lor (A_i \land C_i)) \lor a^* M] = B_n \bigwedge_{i=1}^n [(A_i \land C_i) \lor a^* M] = E_n\]

by Theorems 4.1 and 4.2, and Theorem 4.10 of [6]. Thus, for each integer \(n \geq 1\), we have \(D_n = E_n\). It follows that

\[A \land (B \lor C) = B \lor (A \land C)\]

and hence that \(\Omega^*\) is modular, as claimed.

We need the next result in order to work with residuation in \(\Omega^*\).

**Theorem 4.4.** Let \(A\) and \(B\) be elements of \(\Omega^*\). Let the sequences \(\langle A_i \rangle\) and \(\langle B_i \rangle\) be the completely regular representatives of \(A\) and \(B\), respectively.

Then

\[(4.3)\] The sequence \(\langle A_i \land B_i \rangle\) is Cauchy and is a representative of \(A \land B\).

\[(4.4)\] The sequence \(\langle \bigwedge_{i=1}^n [(A_i \land B_i) \lor a^* M] \rangle\), \(n = 1, 2, \ldots\), is the completely regular representative of \(A \land B\) in \(\Omega^*\).

**Proof.** Since the sequences \(\langle A_i \rangle\) and \(\langle B_i \rangle\) are completely regular, it follows that

\[A_{0,1} = B_{0,1} \leq (A_{0,1} \land B_{0,1}) ; B_{0,1} = (A_{0,1} \lor B_{0,1}) ; (A_{0,1} \land B_{0,1}) = A_{0,1},\]
for each integer \( n \geq 1 \). Thus the sequence \( \langle A_n: B_n \rangle \) is decreasing and hence Cauchy by Corollary 3.2. Let \( b \in A^* \) be the equivalence class determined by the sequence \( \langle A_n: B_n \rangle \). Let the sequences \( \langle b_n \rangle \) and \( \langle a_n \rangle \) be the completely regular representatives of \( b \) and \( A: B \), respectively. Note that by Theorem 3.5 we have

\[
b_n = \bigvee_{n=1}^{\infty} ([A_n: B_n] \vee a^n) \quad \text{for} \quad n = 1, 2, ...
\]

In order to complete the proof it is sufficient to show that \( b = A: B \).

Since, for each integer \( n \geq 1 \), we have that

\[
b_n B_n = (\bigvee_{n=1}^{\infty} ([A_n: B_n] \vee a^n)) B_n \leq ([A_n: B_n] \vee a^n) B_n \leq a_n,
\]

it follows from Propositions 5.10 and 5.13 of [6] that \( b B \leq A \), and consequently \( b \leq A: B \). Since \( (A:B) B \leq A \), we have that

\[
(5.5) \quad c_i B_i \leq c_i B_i \vee a^i M < A_i \quad \text{for all integers} \quad i \geq 1
\]

([6], Proposition 5.9 and Corollary 5.15). Hence, for each integer \( n \geq 1 \), it now follows from (5.5) that

\[
a_n = \bigvee_{n=1}^{\infty} (a_n \vee a^n) \leq \bigvee_{n=1}^{\infty} ([A_n: B_n] \vee a^n) = b_n,
\]

and therefore that \( A: B \leq b \) by Proposition 5.10 of [6]. Thus \( A: B = b \) and the proof is complete.

We will make use of the following characterization of principal elements in the process of proving Theorem 4.6. The reader is referred to Lemma 2.1 of [4] for a proof.

**Lemma 4.5.** Let \( \mathcal{Q} \) be an \( A \)-module and let \( A \) be an element of \( \mathcal{Q} \). If \( \mathcal{Q} \) and \( A \) are modular, then \( A \) is a principal element of \( \mathcal{Q} \) if and only if:

\[
(4.6) \quad C \wedge A = (C: A) A
\]

and

\[
(4.7) \quad b A: A = b \vee (0: A),
\]

for all \( b \in A \) and for all \( C \in \mathcal{Q} \).

**Theorem 4.6.** Let \( \langle B_i \rangle \) be a Cauchy sequence of elements of \( \mathcal{Q} \). If each \( B_i \) is a principal element of \( \mathcal{Q} \), then the equivalence class determined by \( \langle B_i \rangle \) is a principal element of the \( A^* \)-module \( \mathcal{Q} \).

**Proof.** Assume each \( B_i, i = 1, 2, \ldots \), is a principal element of \( \mathcal{Q} \). By selecting a subsequence of \( \langle B_i \rangle \), we may assume without loss of generality that \( \langle B_i \rangle \) is a regular Cauchy sequence. Let \( D \) in \( \mathcal{Q} \) denote the class determined by \( \langle B_i \rangle \) and let \( \langle D_i \rangle \) be the completely regular representative of \( D \). Since \( \mathcal{Q} \) and \( A \) are modular (Theorem 4.3) we shall show that \( D \) is principal by establishing (4.6) and (4.7) of Lemma 4.5. Note that \( D_n = B_n \vee a^n M \), for each \( n \geq 1 \) ([6], Corollary 4.13 and Theorem 4.14).

Let \( C \) be an element of \( \mathcal{Q} \) and let \( \langle C_i \rangle \) be the completely regular representative of \( C \). For each integer \( n \geq 1 \), a routine computation shows that

\[
(4.8) \quad C_n \wedge D_n = C_n \wedge (B_n \vee a^n M) = a^n M \vee [C_n: (B_n \vee a^n M)] B_n
\]

\[
= a^n M \vee (C_n: D_n) B_n
\]

since \( M \) is modular and \( B_n \) is a principal element of \( M \). Using Theorem 4.1 and Theorem 4.4 above, it now follows from (4.8) that \( C \wedge D = (C: D) D \) by ([6], Corollary 4.6 and (5.16)). Thus (4.6) of Lemma 4.5 is established.

Now, let \( b \) be an element of \( A^* \) and let \( \langle b_i \rangle \) be the completely regular representative of \( b \). A routine computation shows that, for each integer \( n \geq 1 \),

\[
(4.9) \quad (b_n D_n \vee a^n M): D_n = (b_n D_n \vee a^n M): D_n = b_n \vee (a^n M: (B_n \vee a^n M))
\]

\[
= b_n \vee (a^n M: B_n)
\]

because each \( B_n \) is principal. By ([6], Corollary 5.13 and Corollary 4.13) and Theorem 4.4, it follows from (4.9) that \( b D: D = b \vee (0: D) \) which establishes (4.7) of Lemma 4.5 and completes the proof.

In order to establish our main theorem we require the following result.

**Theorem 4.7.** The \( A^* \)-module \( \mathcal{Q} \) is principally generated and satisfies the ascending chain condition.

**Proof.** It was shown in Theorem 6.3 of [6] that \( \mathcal{Q} \) satisfies the ascending chain condition. Thus, we need only establish here that \( \mathcal{Q} \) is principally generated as an \( A^* \)-module. For this, since \( \mathcal{Q} \) satisfies the ascending chain condition, it is sufficient to show that: if given any two elements \( A \) and \( B \) of \( \mathcal{Q} \) with \( A < B \), there exists a principal element \( D \) in \( \mathcal{Q} \) such that \( D \leq A \) and \( D < B \).

Thus, let \( A \) and \( B \) be elements of \( \mathcal{Q} \) with \( A < B \). Let \( \langle A_i \rangle \) and \( \langle B_i \rangle \) be the completely regular representatives of \( A \) and \( B \), respectively. Since \( A < B \) and \( A < B \) it follows that \( A_i < B_i \), \( i = 1, 2, \ldots \), and that there exists a natural number \( n \) such that \( A_n < B_n \) for all integers \( i > n \). In particular, \( A_n < B_n \) and, hence, since \( \mathcal{Q} \) is principally generated, there exists a principal element \( D_n \) in \( \mathcal{Q} \) such that \( D_n \leq B_n \) and \( D_n \neq A_n \).

We will now construct inductively a sequence \( D_{n+1} \), \( j = 1, 2, \ldots \), of principal elements of \( \mathcal{Q} \) such that

\[
(4.10) \quad D_{n+1} \leq (D_{n+1} \vee a^{n+1} M) \wedge B_{n+1}
\]
and

\[(4.11) \quad D_{n+1} \not\subseteq A_n.\]

Suppose \(D_{n+1}, j = 1, 2, ..., k\) have been chosen such that (4.10) and (4.11) are satisfied. If

\[(D_{n+1} \cup \alpha^{nk}M) \land R_{n+1} \leq A_n,\]

then

\[(D_{n+1} \cup \alpha^{nk}M) \land R_{n+1} = (D_{n+1} \cup \alpha^{nk}M) \land (R_{n+1} \cup \alpha^{nk}M) = (D_{n+1} \cup \alpha^{nk}M) \land \alpha^{nk}M \leq A_n,\]

since the sequence \(\langle D_i \rangle\) is completely regular, which contradicts our assumption that (4.11) holds for \(D_{n+1}\). Consequently we have that

\[(4.12) \quad (D_{n+1} \cup \alpha^{nk}M) \land R_{n+1} \not\subseteq A_n.\]

From (4.12) and the fact that \(\Omega\) is principally generated we can choose a principal element \(D_{n+1} \in \Omega\) such that

\[D_{n+1} \leq (D_{n+1} \cup \alpha^{nk}M) \land R_{n+1}\]

and

\[D_{n+1} = A_n,\]

which completes the inductive construction.

Now, for \(1 \leq i < n\), set \(D_i = D_i\). It follows from (4.10) and Theorem 3.1 that the sequence \(\langle D_i \rangle, i = 1, 2, ..., \), is Cauchy. Let \(D\) be the equivalence class determined by the sequence \(\langle D_i \rangle\). Since each \(D_i, i = 1, 2, ...,\) is principal by construction, \(D\) is a principal element of the \(\Lambda^*\)-module \(\Omega^*\) by Theorem 4.6.

Suppose for a moment that \(D \leq \Lambda\). Let \(\langle E_i \rangle, i = 1, 2, ...,\) be a subsequence of \(\langle D_i \rangle\) which is regular (\(\Omega^*\), Lemma 4.11). Then

\[(4.13) \quad E_i \leq E_i \cup \alpha^{nk}M \leq A_i \leq A_n \quad \text{for all integers} \ i \geq n,\]

by (\(\Omega^*\), Corollary 4.13 and Proposition 5.9). From (4.13) it would follow that \(D_i \leq A_n\) for all integers \(i\), which is a contradiction to (4.11). Thus \(D \not\subseteq \Lambda\). It follows from (4.10) that \(D \not\subseteq \Omega\) which completes the proof.

We are now in a position to establish our main theorem.

**Theorem 4.8.** Let \(\Lambda\) be a Noether lattice, let \(\Omega\) be a Noetherian \(\Lambda^*\)-module, let \(b\) be an element of \(\Lambda\) such that the \(b\)-adic pseudometric on \(\Lambda\) and \(\Omega\) is a metric, and let \(\Lambda^*\) and \(\Omega^*\) be the \(b\)-adic completions of \(\Lambda\) and \(\Omega\), respectively. If there exists a natural number \(n\) such that \([b^n, \Gamma]\) is finite dimensional, then

\[(4.14) \quad \Lambda^* \text{ is a Noether lattice}\]

and

\[(4.15) \quad \Omega^* \text{ is a Noetherian } \Lambda^*\text{-module.}\]

**Proof.** Assume \([b^n, \Gamma]\) is finite dimensional. Then (4.15) follows from Corollary 2.5 and Theorems 4.3 and 4.7. Since \(\Lambda^*\) is a Noetherian \(\Lambda^*\)-module, (4.14) follows from (4.15) which completes the proof.

The following note provides an interesting application of Theorem 4.8 to the particular case of local and semi-local Noether lattices which are of special interest (see [7]).

**Remark 4.9.** Let \(\Lambda\) be a Noether lattice and let \(\Omega\) be a Noetherian \(\Lambda\)-module. If \(b\) is an element of \(\Lambda\) such that the \(b\)-adic pseudometric on \(\Lambda\) and \(\Omega\) is a metric, denote the \(b\)-adic completion of \(\Lambda\) by \(\Lambda^b\) and the \(b\)-adic completion of \(\Omega\) by \(\Omega^b\). Let \(m\) in \(\Lambda\) by

\[m = \bigwedge \{ p \in \Lambda : p \text{ is a maximal element of } \Lambda\} \]

From Corollary 3.4 of [4], if \(b\) is an element of \(\Lambda\) such that \(b \not\subseteq m\), then the \(b\)-adic pseudometric on \(\Lambda\) and \(\Omega\) is a metric and, thus, for \(b \subseteq m\), \(\Lambda^b\) and \(\Omega^b\) are defined. In conjunction with the above we have the following:

Let \(b\) be an element of \(\Lambda\) such that \(b \not\subseteq m\). If

\[(4.16) \quad [b, \Gamma] \text{ is finite dimensional,}\]

or

\[(4.17) \quad \Lambda\text{ is semi-local and } [b, m] \text{ is finite dimensional,}\]

then

\[(4.18) \quad \Lambda^b\text{ is a Noether lattice}\]

and

\[(4.19) \quad \Omega^b\text{ is a Noetherian } \Lambda^b\text{-module.}\]

**Proof.** Assume that (4.16) or (4.17) holds. If \(\Lambda\) is semi-local and \([b, m]\) is finite dimensional, then, since \([m, \Gamma]\) is finite dimensional by Corollary 4.5 of [4], it follows that \([b, \Gamma]\) is finite dimensional by the modularity of \(\Omega\). Hence, in either case, \([b, \Gamma]\) is finite dimensional. It now follows from Theorem 4.8 and the above comments that \(\Lambda^b\) is a Noether lattice and \(\Omega^b\) is a Noetherian \(\Lambda^b\)-module which completes the proof.

Note in particular that Theorem 5.9 of [4] and Theorem 8.7 of [6] are just special cases of the preceding result which in turn is a special case of Theorem 4.8.

**Remark 4.10.** We point out that the lattice of ideals of the completion \(\overline{\mathcal{B}}\) of a local ring \(\mathcal{B}\) can be obtained lattice theoretically from the lattice of ideals of \(\mathcal{B}\) by using the lattice completion concept discussed above. This is achieved in the following manner. Assume \((\mathcal{R}, p)\) is a local ring (commutative and Noetherian with identity) with \(p\)-adic (ring) completion \(\overline{\mathcal{R}}\). Let \(\mathcal{A}\) and \(\overline{\mathcal{A}}\) denote the lattice of ideals of \(\mathcal{R}\) and \(\overline{\mathcal{R}}\), respectively. It can be shown (8) that \(\mathcal{A}\) and \(\overline{\mathcal{A}}\) (the \(p\)-adic lattice com-


Localy flat embeddings of Hilbert cubes are flat

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Abstract. In this paper it is shown that any locally flat embedding of the Hilbert cube Q into a Q-manifold is flat. The techniques employed in the proof of this result also imply that the group of homeomorphisms of $Q \times R^n$ onto itself which are fixed on $Q \times \{0\}$ has exactly two components.

1. Introduction. For topological spaces $X$ and $Y$, an embedding $i: X \rightarrow Y$ is said to be locally flat (with codimension $a$) provided that each point of $X$ has a neighborhood $U$ and an open embedding $h: U \times R^a \rightarrow Y$ such that $h(x, 0) = i(x)$, for all $x \in U$. We say that the embedding is flat if we can take $U = X$. We use $R^n$ to denote euclidean $n$-space, $Q$ to denote the Hilbert cube (i.e. the countable infinite product of closed intervals), and by a $Q$-manifold we mean a separable metric manifold modeled on $Q$. The following is the main result of this paper.

Theorem 1. If $X$ is a $Q$-manifold and $i: Q \rightarrow X$ is a locally flat embedding, then $i$ is flat.

Of course this result is false if $Q$ is replaced by a more complicated $Q$-manifold. For example let $X = M \times Q$, where $M$ is the open Möbius band, let $i_1: S^1 \rightarrow M$ be a homeomorphism of the $1$-sphere onto the center circle, and let $i = i_1 \times id: S^1 \times Q \rightarrow M \times Q$. Then $i$ is a codimension 1 locally flat embedding, but $i$ is not flat. (If $i$ were flat, then arbitrarily small neighborhoods of $i_1(S^1)$ in $M$ would be separated by $i_1(S^1)$.) A more general question would be to investigate when locally flat embeddings of $Q$-manifolds into $Q$-manifolds have normal bundles (see [2] and [4] for finite-dimensional results).

Let $\mathcal{H}(Q \times R^n)$ denote the space of all homeomorphisms of $Q \times R^n$ onto itself (with the CO-topology) which are the identity on $Q \times \{0\}$. The following result is a by-product of Theorem 1.

Theorem 2. $\pi_n(\mathcal{H}(Q \times R^n)) = 2$, for all $n \geq 1$. That is, $\mathcal{H}(Q \times R^n)$ has exactly two components.

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