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Recursiveness of initial segments of Kleene's O

by

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Abstract. It is shown that for any constructive ordinal $\alpha \geqslant \omega^2$, there are both recursive and nonrecursive initial segments of the partial ordering $<_0$ of Kleene's O which have order type α.

Let O be Kleene's set of notations for constructive ordinals and let $<_0$ be Kleene's partial ordering of O. If $a \in O$, let $O(a)$ denote $\{b: b <_0 a\}$. It is well known that for any $a \in O$, $O(a)$ is a recursively enumerable (r.e.) subset of O which is well-ordered by $<_0$ with order type $|a|$, the ordinal for which a is a notation. Our purpose here is to determine which constructive ordinals have notations a such that $O(a)$ is recursive (nonrecursive). We prove that every constructive ordinal has a notation a such that $O(a)$ is recursive and in fact that there is a Π^1_1 path P through O such that $O(a)$ is recursive for all a in P. In the other direction we show that the constructive ordinals which have notations a with $O(a)$ nonrecursive are exactly those which are $\geqslant \omega^2$. Our constructions in fact show that if $\alpha \geqslant \omega^2$ is a constructive ordinal and A is an infinite r.e. set (other than ω) then α has a notation a such that $O(a)$ is m-equivalent to A.

Most of our notation is standard. In particular, we use φ_e for the eth partial recursive function and call e an *index* of φ_e. We use the recursion theorem in the following informal style: in the definition of a partial recursive function φ_e, its index may be assumed known in advance. Of course such arguments are easily formalized. An index of a recursive set is any index of its characteristic function. A path through O is a subset of O which is linearly ordered by $<_0$ and contains a notation for each constructive ordinal.

Information on Kleene's O can be found in [1], [2], or [5]. In particular we shall need the binary recursive function $+_0$ which represents ordinal addition in the sense that $|a +_0 b| = |a| + |b|$ for $a, b \in O$. Also $+_0$

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satisfies the following for $a, b, c \in O$:

(P1) $$a <_0 a +_0 b \quad \text{provided} \quad b \neq 1,$$

(P2) $$a \leqslant_0 c <_0 a +_0 b \leftrightarrow (\exists d)[d <_0 b \ \& \ c = a +_0 d].$$

These properties follow at once from [2; XVII, XIX, and XX]. We need one additional property for $+_0$ which will enable us to bound (by c) the existential quantifier which occurs in (P2).

LEMMA 1. *In addition to the properties mentioned above, $+_0$ may be defined so as to satisfy*

(P3) $$a +_0 b \geqslant \max\{a, b\}$$

(*where* max, $\geqslant$ *refer to the standard ordering of* ω) *for all* $a, b \in \omega$, $a \neq 0$.

Proof. Let $+_0$ be a recursive function such that $a +_0 1 = a$; $a +_0 2^b = 2^{a +_0 b}$ $(b \neq 0)$, $a +_0 3.5^e = 3.5^w$ where w is chosen to be $\geqslant \max\{a, 3.5^e\}$ and $\varphi_w(n) = a +_0 \varphi_e(n)$ for $n \in \omega$, and $a +_0 s = 7^{\max\{a,s\}}$ if s is not of the form 2^b or 3.5^e. Since any partial recursive function has arbitrarily large indices which can be effectively found from any given index, an appropriate w to define $a +_0 3.5^e$ can be computed from e and an index of $+_0$ [1]. Hence such a partial recursive function $+_0$ exists by the recursion theorem. The arguments of [2] apply at once to this $+_0$ so (P1) and (P2) hold. (P3) clearly holds except possibly when b is a power of 2. Using this it is immediate to prove (P3) by (strong) finite induction on b. (As usual, such an induction also shows that $+_0$ is a total function.)

The following result shows that there are arbitrarily large constructive ordinals with notations a such that $O(a)$ is recursive.

THEOREM 1. *There is a recursive function h such that whenever $a \in O$, then $h(a) \in O$, $|a| \leqslant |h(a)|$, and $O(h(a))$ is recursive.*

Proof. We want h to satisfy the following: $h(1) = 1$, $h(2^b) = 2^{h(b)}$ for $b \neq 0$, $h(3.5^e) = 3.5^w$ where φ_w is such that

$$\varphi_w(0) = h(\varphi_e(0)),$$

$$\varphi_w(n+1) = \varphi_w(n) +_0 h(\varphi_e(n+1)),$$

and $h(x) = 7$ if x not of the form 2^b or 3.5^e. The existence of such a partial recursive h follows from the recursion theorem because $h(n)$ can be effectively computed from n and an index of h. Furthermore h will be total by the usual inductive argument. It is easy to show by transfinite induction on $|a|$ (using property (P1) of $<_0$) that if $a \in O$ then

[1] Pour-El has pointed out that, because Kleene's function S_1^3 is strictly increasing in each of its arguments, his definition of $+_0$ [2, 22.2] already yields a sufficiently large w and thus his function $+_0$ satisfies Lemma 1 for $a, b \in O$.

$h(a) \in O$ and $|a| \leqslant |h(a)|$. To show that $O(h(a))$ is recursive for $a \in O$, we define a recursive function f such that $f(a)$ is an index of (the characteristic function of) $O(h(a))$ for $a \in O$. Again f is obtained using the recursion theorem (alternatively the recursion lemma [5, p. 398] could be applied.) Since $O(h(1)) = \emptyset$ and $O(2^{h(a)}) = O(h(a)) \cup \{h(a)\}$ for $a \in O$, the only interesting case in the definition of $f(x)$ is where x is of the form 3.5^e. For this case we need to be able to find an index for $O(h(3.5^e))$ effectively from the indices of $O(h(\varphi_e(n)))$, $n \in \omega$. More precisely, it will suffice to prove the following lemma:

LEMMA 2. *There is a recursive function $t(e, j)$ such that if $3.5^e \in O$ and $\varphi_j(n)$ is an index of $O(h(\varphi_e(n)))$ for all n, then $t(e, j)$ is an index of $O(h(3.5^e))$.*

For once Lemma 2 is established, $f(3.5^e)$ is defined to be $t(e, j)$ where j is an index of $f \circ \varphi_e$. Then it is immediate by transfinite induction on $|a|$ that $f(a)$ is an index of $O(h(a))$ for $a \in O$.

Since the definition of $h(3.5^e)$ uses the function $+_0$, the following lemma is crucial to the proof of Lemma 2.

LEMMA 3. *If $a, b \in O$ and $O(a)$, $O(b)$ are recursive then $O(a +_0 b)$ is recursive.*

Proof. Since $<_0$ is a linear ordering of $O(a +_0 b)$ and $+_0$ has properties (P2) and (P3),

(P4) $$O(a +_0 b) = O(a) \cup \{c\colon (\exists d)_{\leqslant c}[d \in O(b) \ \& \ c = a +_0 d]\}.$$

Thus if $O(a)$, $O(b)$ are recursive, (P4) exhibits $O(a +_0 b)$ as the union of two recursive sets.

To define $t(e, j)$ as required by Lemma 2, assume that $3.5^e \in O$ and $\varphi_j(n)$ is an index of $O(h(\varphi_e(n)))$. Let $h(3.5^e) = 3.5^w$. Then since $O(h(\varphi_e(n)))$ is recursive for each n, it follows from Lemma 2 by induction on n that $O(\varphi_w(n))$ is recursive for each n. Since the proof of Lemma 2 is uniform we can in fact effectively compute from e, j, and n an index of $O(\varphi_w(n))$. We claim now that

(5) $$c \in O(3.5^w) \leftrightarrow (\exists n)_{\leqslant c}[c \in O(\varphi_w(n+1))].$$

The claim implies that $O(3.5^w)$ is recursive in view of the uniform recursiveness of $O(\varphi_w(n+1))$ and in fact yields an index of $O(3.5^w)$ uniformly from e and j. Then $t(e, j)$ is defined to be this index.

To verify (5) observe that the implication from right to left is immediate from the definition of $<_0$, even without the bound on the quantifier. For the converse, assume $c <_0 3.5^w$. If $c <_0 \varphi_w(0)$ then we simply let $n = 0$ to satisfy the right hand side. Otherwise let n be the unique number such that

$$\varphi_w(n) \leqslant_0 c <_0 \varphi_w(n+1) = \varphi_w(n) +_0 h(\varphi_e(n+1)) .$$

By property (P2) of $+_0$, there is a d such that $\varphi_w(n) +_0 d = c$. By (P3), $\varphi_w(n) \leqslant c$. But also by (P3), φ_w is nondecreasing with respect to the standard ordering of ω. Finally $\varphi_w(n) \neq \varphi_w(n+1)$ by (P1) because $\varphi_e(n+1) \neq 1$. Therefore φ_w is strictly increasing and so $n \leqslant \varphi_w(n) \leqslant c$. Thus the n we have defined satisfies the right-hand side of (5).

In defining $t(e, j)$ we assumed that e and j were as in the hypothesis of Lemma 2. It remains only to point out that this definition can be formally carried out for any pair e, j so that t is a total recursive function. The proof of Lemma 2, and thus of Theorem 1, is complete.

It is easy to show that if $a <_0 b$ and $O(b)$ is recursive, then $O(a)$ is recursive (cf. Proposition 1). From this and Theorem 1, it follows that each constructive ordinal has a notation a such that $O(a)$ is recursive. We shall now apply the methods of Feferman–Spector [1] to obtain a stronger result.

Let $\lesssim$, O^* be as in Feferman–Spector [1]. In particular $\lesssim$ is an r.e. partial ordering whose restriction to O is $<_0$, and O^* is a Σ_1^1 superset of O such that the Π_1^1 paths through O are precisely the sets of the form $O \cap C(a)$ for $a \in O^* - O$. Here, as in [1], $C(a)$ denotes $\{b: b \lesssim a\}$. Observe that $C(a) = O(a)$ for $a \in O$. The following lemma, which will only be needed in the special case where $A = \{a: C(a) \text{ recursive}\}$, sums up the information we need from [1]

Lemma 4. *If A is a Σ_1^1 subset of ω such that*

(i) $a \in A \cap O^* \ \&\ b \lesssim a \rightarrow b \in A$ *and*

(ii) $a \in O \rightarrow (\exists b)[b \in A \cap O \ \&\ |a| < |b|]$,

then there is a Π_1^1 path P through O such that $A \supseteq P$.

Proof. It suffices to show that $A \cap (O^* - O)$ is nonempty since then one can choose $a \in A \cap (O^* - O)$ and take $P = O \cap C(a)$. Suppose for a contradiction that $A \cap (O^* - O)$ is empty, i.e. $A \cap O^* \subseteq O$. Then $A \cap O^*$ is a Σ_1^1 subset of O and hence the corresponding set of ordinals is bounded above by a constructive ordinal (cf. the second proof of [1, 2.6] or [6, p. 184]). But since $A \cap O^* \subseteq A \cap O$, this contradicts (ii).

We now want to apply Lemma 4 with $A = \{a: C(a) \text{ is recursive}\}$. Since $C(a) = O(a)$ for $a \in O$ it follows at once from Theorem 1 that hypothesis (ii) of Lemma 4 is satisfied for this choice of A. Also it is a routine exercise to verify that this A is arithmetical and hence Σ_1^1. We now show that (i) also holds.

Proposition 1. *If $a \in O^*$, $C(a)$ is recursive, and $b \lesssim a$, then $C(b)$ is recursive.*

Proof. By [1, Definitions 3.1 and 3.3] if $a \in O^*$ then $C(a)$ is linearly ordered by $\lesssim$. If $b \lesssim a$, we have

$$c \lesssim b \leftrightarrow (c \lesssim a \ \&\ c \neq b \ \&\ \neg\, b \lesssim c) .$$

Thus if $C(a)$ is recursive, then $C(b)$ is co-r.e., i.e. the complement of an r.e. set. Since $C(b)$ is always r.e., it follows that $C(b)$ is recursive.

Now we have from Theorem 1, Lemma 4, and Proposition 1:

Corollary 1. *There is a Π_1^1 path P through O such that $O(a)$ is recursive for all $a \in P$.*

If $a \in O$, let $O_l(a)$ be the set of all numbers in $O(a)$ of the form 1 or 3.5^e. It is easy to see that $O(a) \equiv_m O_l(a)$ for any $a \in O$, $a \neq 1$. If $a \in O$ and $|a| < \omega^2$, it follows that $O(a)$ is recursive because $O_l(a)$ is finite. The following result was obtained independently by Parikh [4].

Theorem 2. *There exists $a \in O$ such that $|a| = \omega^2$ and $O(a)$ is non-recursive.*

Proof. Let A be an r.e. nonrecursive set, and let f be a 1-1 recursive function whose range is A. For $y \in \omega$, let y^* be the unique element of O such that $|y^*| = y$. We want to define a recursive function g such that for all n, x, and y

$$\varphi_{g(x)}(y) = \begin{cases} y^* & \text{if} \quad x = f(0) , \\ 3.5^{gf(n)} +_0 y^* & \text{if} \quad x = f(n+1) , \\ \text{undefined} & \text{if} \quad x \notin A . \end{cases} \tag{6}$$

The existence of such a recursive function g is a consequence of the recursion theorem because if we take the index of g as given, the right hand side of (6) becomes a partial recursive function $\tau(x, y)$ of x and y (which is single-valued because f is one-one). Then $g(x)$ is defined to be an index of the partial recursive function $\lambda y \tau(x, y)$.

It is easy to see by induction on n that $3.5^{gf(n)} \in O$, $3.5^{gf(n)} <_0 3.5^{gf(n+1)}$, and $|3.5^{gf(n)}| = \omega \cdot (n+1)$. Let i be such that $\varphi_i(n) = 3.5^{gf(n)}$, and let $a = 3.5^i$. Clearly $a \in O$ and $|a| = \omega^2$. Also

$$x \in A \Leftrightarrow 3.5^{g(x)} <_0 a . \tag{7}$$

For if $x = f(n)$, $3.5^{gf(n)} = \varphi_i(n) <_0 3.5^i$ and if $x \notin A$ then $\varphi_{g(x)}$ is everywhere divergent so $3.5^{g(x)} \notin O$. From (7) and the nonrecursiveness of A it follows that $O(a)$ is nonrecursive.

The following Corollary combines and sharpens some of our results.

Corollary 3. *If A is an infinite r.e. set other than ω, and a is a constructive ordinal $\geqslant \omega^2$, then a has a notation c such that $O(c) \equiv_m A$.*

Proof. Assume first that $a=\omega^2$, and let a be as in Theorem 2. Then $A\leqslant_m O(a)$ by (7). Since $g(x)$ in the proof of Theorem 2 need only be the index of a certain partial recursive function, there is no difficulty in modifying the definition of g so that g is strictly increasing. Thus the range of g is recursive. Since $|3.5^{g(x)}|=\omega\cdot(n+1)$ we have for $x\neq 1$

$$x\in O_1(a)\leftrightarrow x \text{ has form } 3.5^e \quad \text{where} \quad e\in \operatorname{range} g \,\&\, g^{-1}(e)\in A\,.$$

Therefore $O(a)\leqslant_m O_1(a)\leqslant_m A$, since $A\neq\omega$. Now if $a\geqslant\omega^2$ let β be such that $\omega^2+\beta=a$. Let b be a notation for β such that $O(b)$ is recursive. (Such a b exists by Corollary 1). Then $|a+_0b|=a$ and we claim that $O(a+_0b)\equiv_m A$. By the previous case it suffices to show that $O(a+_0b)\equiv_m O(a)$. But (P4) in the proof of Lemma 3 shows that $O(a+_0b)$ is the disjoint union of $O(a)$ with a recursive set, from which $O(a+_0b)\equiv_m O(a)$ follows immediately. Thus let $c=a+_0b$.

In closing we apply our results to partially answer some questions posed by Kreisel [private correspondence] concerning the existence of reducibility relations between Δ_1^1 sets and Π_1^1 paths through O. Our first observation comes immediately from the proof of Theorem 2 and was obtained independently by Parikh [4].

Corollary 4. *There is a Π_1^1 path P through O such that the complete* r.e. *set K is many-one reducible to P.*

Proof. Let A be K in the proof of Theorem 2, and let P be any Π_1^1 path through O such that $a\in P$, where a and g are as in the proof of Theorem 2. Then for any x, if $x\in K$ then $3.5^{g(x)}\in O(a)\subseteq P$ and if $x\notin K$ then $3.5^{g(x)}\notin O$ so $3.5^{g(x)}\notin P$. Therefore $K\leqslant_m P$.

In fact it holds in general that if $b\in P$, where P is any path through O, then $O(b)$ is Turing reducible to P. (The proof is the same as the Proposition 1 except that $O(a)$ is now replaced by P). We do not know whether this result holds for reducibilities which are stronger than Turing reducibility, but interestingly enough a sort of modified converse does. Let us call a reducibility R *invertible* if whenever a Δ_1^1 set A is R-reducible to a Π_1^1 path P through O, then A is R-reducible to $O(a)$ for some $a\in P$. Then, as Kreisel pointed out to me, m-reducibility is easily seen to be invertible. (If A, P are as above and $A=f^{-1}(P)$ where f is a recursive function, then $f(A)$ is a Δ_1^1 and hence bounded subset of P. If $a\in P$ is chosen so that $b<_0 a$ for $b\in f(A)$, then clearly $A=f^{-1}(O(a))$ as required.) Kreisel [3] has shown that enumeration reducibility (as defined in [5]) is invertible and Parikh [4] has shown that truth-table reducibility is invertible. (Although these proofs are more complicated than the one above for many-one reducibility they share the same idea of taking A and the reduction procedure from A to P as given and then inverting the reduction procedure to obtain a (necessarily bounded) subset of P from this information which yields A by the same reduction.) From the remarks after Theorem 1 and the cited results of Parikh and Kreisel we obtain immediately the following Corollary which shows that Corollary 4 certainly does not hold for *all* Π_1^1 paths P through O.

Corollary 5. *There is a Π_1^1 path P through O such that any Δ_1^1 set which is either enumeration reducible to P or truth-table reducible to P is recursive.*

It is not known whether Turing reducibility is invertible nor whether the analogue of Corollary 5 holds for Turing reducibility. In fact it is conceivable that Turing reducibility fails very badly to be invertible in the sense that every Δ_1^1 set is recursive in every Π_1^1 path P through O, or even that O is recursive in every Π_1^1 path through O.

Added in proof (March 1975). Recently Friedman [7] has shown that there is a Π_1^1 path P through O such that O is recursive in P (and also $O(a)$ is recursive for all $a\in P$). It follows that Turing reducibility is not invertible.

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