Recursiveness of initial segments of Kleene’s $O$

by

Carl G. Jockusch, Jr.* (Urbana, Ill.)

Abstract. It is shown that for any constructive ordinal $\alpha \geq \omega^2$, there are both recursive and nonrecursive initial segments of the partial ordering $\leq_+ \subset O$ of Kleene’s $O$ which have order type $\alpha$.

Let $O$ be Kleene’s set of notations for constructive ordinals and let $\leq_+$ be Kleene’s partial ordering of $O$. If $a \in O$, let $O(a)$ denote $\{ b \mid b <_+ a \}$. It is well known that for any $a \in O$, $O(a)$ is a recursively enumerable (r.e.) subset of $O$ which is well-ordered by $<_+$ with order type $|a|$, the ordinal for which $a$ is a notation. Our purpose here is to determine which constructive ordinals have notations $a$ such that $O(a)$ is recursive (nonrecursive). We prove that every constructive ordinal has a notation $a$ such that $O(a)$ is recursive and in fact that there is a $\Pi^0_1$ path $P$ through $O$ such that $O(a)$ is recursive for all $a$ in $P$. In the other direction we show that the constructive ordinals which have notations $a$ with $O(a)$ nonrecursive are exactly those which are $\geq \omega^2$. Our constructions in fact show that if $a \geq \omega^2$ is a constructive ordinal and $A$ is an infinite r.e. set (other than $a$) then $a$ has a notation $a$ such that $O(a)$ is $\mathbb{S}$-equivalent to $A$.

Most of our notation is standard. In particular, we use $\varphi_e$ for the $e$th partial recursive function and call $e$ an index of $\varphi_e$. We use the recursion theorem in the following informal style: in the definition of a partial recursive function $\varphi_e$, its index may be assumed known in advance. Of course such arguments are easily formalized. An index of a recursive set is any index of its characteristic function. A path through $O$ is a subset of $O$ which is linearly ordered by $<_+$ and contains a notation for each constructive ordinal.

Information on Kleene’s $O$ can be found in [1], [2], or [5]. In particular we shall need the binary recursive function $+_\omega$, which represents ordinal addition in the sense that $|a +_\omega b| = |a| + |b|$ for $a, b \in O$. Also $+_\omega$

* This research was supported by NSF Grant GP 29229. The author is grateful to G. Kreisel for introducing him to this subject and for much helpful correspondence.
satisfies the following for \( a, b, c \in O \):

\[
(P_1) \quad a +_c b \quad \text{provided} \quad b \neq 1,
\]

\[
(P_2) \quad a +_c c +_c a +_c b = (a +_c b)(a +_c c) \quad \text{where} \quad a +_c 3.5^b = 3.5^a \quad \text{where} \quad a, b \neq 0.
\]

These properties follow at once from \([2; XVII, XIX, and XX]\).

We need one additional property for \(+_c\), which will enable us to bound (by \(c\)) the existential quantifier which occurs in \((P_2)\).

**Lemma 1.** In addition to the properties mentioned above, \(+_c\) may be defined so as to satisfy

\[
(P_3) \quad a +_c b \geq \max\{a, b\}
\]

(where \(\geq\) refers to the standard ordering of \(\omega\)) for all \(a, b \in \omega, a \neq 0\).

**Proof.** Let \(+_c\) be a recursive function such that \(a +_c 1 = a, a +_c a +_c b = 2^{a +_c b} (b \neq 0), a +_c 3.5^b = 3.5^a\) where \(a\) is chosen to be \(\geq \max\{a, 3.5^b\}\) and \(\varphi_n(n) = a +_c \varphi_n(n)\) for \(n \in \omega, a +_c a +_c a = \max\{a, a +_c a +_c a\}\) if \(a\) is not of the form \(2^{a +_c a} +_c a\). Since any partial recursive function has arbitrarily large indices which can be effectively found from any given index, an appropriate \(a\) can be chosen for \(a +_c a +_c a\) can be computed from \(a\) and an index of \(+_c\).

Hence such a partial recursive function \(+_c\) exists by the recursion theorem. The arguments of \([2]\) apply at once to this \(+_c\) so \((P_1)\) and \((P_2)\) hold, \((P_3)\) clearly holds except possibly when \(b = 1\) is a power of 2. Using this it is immediate to prove \((P_3)\) by (strong) finite induction on \(b\). (As usual, such an induction also shows that \(+_c\) is a total function.)

The following result shows that there are arbitrarily large constructive ordinals with notations \(a\) such that \(O(a) = O\).

**Theorem 1.** There is a recursive function \(h\) such that whenever \(a \in O\), then \(h(a) \in O\). \(h(a) \in O\).

**Proof.** We want \(h\) to satisfy the following: \(h(1) = 1, h(2^b) = 2^{a +_c b}\) for \(b \neq 0, h(3.5^b) = 3.5^a\) where \(\varphi_n\) is such that \(\varphi_n(0) = h(\varphi_n(0))\), \(\varphi_n(n +_c 0) = h(\varphi_n(n +_c 0))\), and \(h(a) = \gamma\) if \(a\) not of the form \(2^{a +_c a} +_c a\) or \(3.5^a\). The existence of such a partial recursive \(h\) follows from the recursion theorem because \(h(a)\) can be effectively computed from \(a\) and an index of \(h\). Furthermore \(h\) will be total by the usual inductive argument. It is easy to show by transfinite induction on \(\omega\) (using \(P_1\)) of \(< a\) that \(h(a) \in O\) then

\[
h(a) \in O \quad \text{and} \quad [a] \subset [h(a)].\]

To show that \(O(h(a))\) is recursive for \(a \in O\), we define a recursive function \(f\) such that \(f(a)\) is an index of the characteristic function of \(O(h(a))\) for \(a \in O\). Again \(f\) is obtained using the recursion theorem (alternatively the recursion lemma \([5, p. 396]\) could be applied.) Since \(O(h(1)) = O\) and \(O(2^{a +_c b}) = O(h(a)) \cup [h(a)]\) for \(a \in O\), the only interesting case in the definition of \(f(a)\) is where \(x\) is of the form \(3.5^n\).

For this case we need to be able to find an index for \(O(3.5^n)\) effectively from the indices of \(O[h(\varphi_n(n))]\) for \(n \in \omega\). More precisely, it will suffice to prove the following lemma:

**Lemma 2.** There is a recursive function \(t(e, j)\) such that \(3.5^n \in O\) and \(\varphi_n(j)\) is an index of \(O[h(\varphi_n(n))]\) for all \(n\), then \(t(e, j)\) is an index of \(O[h(3.5^n)]\).

For once Lemma 2 is established, \(f(3.5^n)\) is defined to be \(f(e, j)\) where \(j\) is an index of \(f(e)\). Then it is immediate by transfinite induction on \(\omega\) that \(f(a)\) is an index of \(O(h(a))\) for \(a \in O\).

Since the definition of \(O(3.5^n)\) uses the function \(+_c\), the following lemma is crucial to the proof of Lemma 2.

**Lemma 3.** If \(a, b \in O\) and \(O(a)\), \(O(b)\) are recursive then \(O(a +_c b)\) is recursive.

**Proof.** Since \(\leq\) is a linear ordering of \(O(a +_c b)\) and \(+_c\) has properties \((P_2)\) and \((P_3)\),

\[
O(a +_c b) = O(a) \cup \{c : \mathbb{E} d(D)(d \in O(b) \& c = a +_c d)\}.
\]

Thus if \(a \in O\), \(O(b)\) are recursive, \((P_4)\) exhibits \(O(a +_c b)\) as the union of two recursive sets.

To define \(t(e, j)\) as required by Lemma 2, assume that \(3.5^n \in O\) and \(\varphi_n(0)\) is an index of \(O[h(\varphi_n(0))]\). Let \(h(3.5^n) = 3.5^n\). Then \(O[h(\varphi_n(0))]\) is recursive for each \(n\), it follows from Lemma 2 by induction on \(n\) that \(O[h(\varphi_n(0))]\) is recursive for each \(n\). Since the proof of Lemma 2 is uniform we can in fact effectively compute from \(e, j, n\) an index of \(O[h(\varphi_n(0))]\). We claim now that

\[
eq O(3.5^n) \leftrightarrow (E n)(\exists c \in O[h(\varphi_n(n +_c 1))]\).
\]

The claim implies that \(O(3.5^n)\) is recursive in view of the uniform recursiveness of \(O[h(\varphi_n(0))]\) and in fact yields an index of \(O(3.5^n)\) uniformly from \(e\) and \(j\). Then \(t(e, j)\) is defined to be this index.

To verify \((5)\) observe that the implication from left to right is immediate from the definition of \(\leq\), even without the bound on the quantifier. For the converse, assume \(c \leq h(3.5^n)\). If \(c < h(\varphi_n(0))\) then simply let \(n = 0\) to satisfy the right hand side. Otherwise let \(n\) be the unique

---

\(^{(1)}\) P. D. E. P. has pointed out that, since Kleene's function \(\#\) is strictly increasing in both of its arguments, his definition of \(+_c\) \([2, 226]\) already yields a sufficiently large \(a\) and thus his function \(+_c\) satisfies Lemma 1 for \(a, b \in O\).
number such that
\[ \varphi(n) \leq c \implies \varphi(n+1) = \varphi(n) + a \wedge \varphi(n+1) = \varphi(n) + a \wedge \varphi(n+1) = \varphi(n) + a . \]
By property (P2) of \( \prec_1 \), there is a \( d \) such that \( \varphi(n) + d = c \). By (P3), we have \( \varphi(n) \leq c \). But also by (P3), \( \varphi_0 \) is nondecreasing with respect to the standard ordering of \( \omega \). Finally, \( \varphi_0(n) \neq \varphi_0(n+1) \) by (P1) because \( \varphi(n+1) \neq 1 \). Therefore \( \varphi_0 \) is strictly increasing and so \( n \leq \varphi(n) < c \).
Thus the \( n \) we have defined satisfies the right-hand side of (6).

In defining \( t(e,j) \) we assumed that \( e \) and \( f \) were as in the hypothesis of Lemma 2. It remains only to point out that this definition can be formally carried out for any pair \( e, j \) so that \( t \) is a total recursive function.

The proof of Lemma 2, and thus of Theorem 1, is complete.

It is easy to show that if \( a \prec b \) and \( O(b) \) is recursive, then \( O(a) \) is recursive (cf. Proposition 1). From this and Theorem 1, it follows that each constructive ordinal has a notation \( a \) such that \( O(a) \) is recursive. We shall now apply the methods of Feferman-Specter [1] to obtain a stronger result.

Let \( \prec_1, O^* \) be as in Feferman-Specter [1]. In particular \( \prec_1 \) is an r.e. partial ordering whose restriction to \( O \) is \( \prec_1 \). Let \( \Omega^* \) be a \( \Sigma_1 \) subset of \( O \) such that the \( H_n^1 \) paths through \( O \) are precisely the sets of the form \( O \cap C(a) \) for \( a \in O^* \). Here, \( C(a) = \Omega(a) \) denotes the b; \( \prec_1 \leq a \). Observe that \( \Omega(a) = \Omega(a) \) for \( a \in O \). The following lemma, which will only be needed in the special case where \( A = (a; C(a) \text{ recursive}) \), sums up the information we need from [1]

**Lemma 4.** If \( A \) is a \( \Sigma_1 \) subset of \( O \) such that

(i) \( a \in O \setminus \Omega^* \leq b \implies a \prec b \prec A \), and

(ii) \( a \in O \setminus \Omega^* \leq [b \in A \setminus O \setminus \Omega^* \leq [b \leq b] \),

then there is a \( H_n^1 \) path \( P \) through \( O \) such that \( A \supseteq P \).

**Proof.** If \( A \cap (O^* \setminus O) \) is nonempty since then one can choose \( a \in A \cap (O^* \setminus O) \) and take \( P = P \setminus O \cap \Omega(a) \), Suppose for a contradiction that \( A \cap (O^* \setminus O) \) is empty, i.e., \( A \setminus O \subseteq O \). Then \( A \setminus O \) is a \( \Sigma_1 \) subset of \( O \) and hence the corresponding set of ordinals is bounded above by a constructive ordinal (cf. the second proof of [1, 26] or [6, p. 184]). But since \( A \setminus O \subseteq \Omega(a) \), this contradicts (ii).

We now want to apply Lemma 4 with \( A = (a; C(a) \text{ recursive}) \). Since \( O(a) = \Omega(a) \) for \( a \in O \) it follows that at once from Theorem 1 that hypothesis (ii) of Lemma 4 is satisfied for this choice of \( A \). Also it is a routine exercise to verify that this \( A \) is arithmetical and hence \( \Sigma_1 \). We now show that (i) also holds.

**Proposition 1.** If \( a \in O^* \), \( O(a) \text{ is recursive}, \) and \( b \prec a \), then \( O(b) \text{ is recursive} \).

**Proof.** By [1, Definitions 3.1 and 3.3] if \( a \in O^* \), then \( O(a) \) is linearly ordered by \( \prec_1 \). If \( b \prec \Delta \), we have

\[ c \prec b \leftrightarrow (c \prec a \wedge c \neq b \wedge b \prec \Delta) \]

Thus if \( O(a) \) is recursive, then \( O(b) \) is co-r.e., i.e., the complement of an r.e. set. Since \( C(b) \) is always r.e., it follows that \( O(b) \) is recursive.

Now we have from Theorem 1, Lemma 4, and Proposition 1:

**Corollary 1.** There is a \( H_n^1 \) path \( P \) through \( O \) such that \( O(a) \text{ is recursive for all } a \in O^* \).

If \( a \in O \), let \( O_a \) be the set of all numbers in \( O(a) \) of the form 1 or \( 3.5^a \). It is easy to show that \( O(a) \equiv O_a \) for any \( a \in O \), \( a \neq 1 \). If \( a \in O \) and \( |a| < \omega^3 \), it follows that \( O(a) \) is recursive because \( O_a \) is finite. The following result was obtained independently by Parikh [4].

**Theorem 2.** There exists \( a \in O \) such that \( |a| = \omega^3 \) and \( O(a) \) is nonrecursive.

**Proof.** Let \( A \) be an r.e. nonrecursive set, and let \( f \) be a 1-1 recursive function whose range is \( A \). For \( y \in \omega \), let \( y^* \) be the unique element of \( O \) such that \( y^* \equiv y \). We want to define a recursive function \( g \) such that for all \( n, s, \) and \( y \)

\[ g(y) = \begin{cases} y^* & \text{if } x = f(0) \\ 3.5^{g(y)} + y^* & \text{if } x = f(n+1) \\ \text{undefined} & \text{if } x \notin A \end{cases} \]

The existence of such a recursive function \( g \) is a consequence of the recursion theorem because if we take the index of \( g \) as given, the right-hand side of (6) becomes a partial recursive function \( v(x, y) \) of \( x \) and \( y \) (which is single-valued because \( f \) is one-one). Then \( g(x) \) is defined to be an index of the partial recursive function \( v(x, y) \).

It is easy to see by induction on \( n \) that \( 3.5^n \in O, 3.5^n \in O, 3.5^{g(n)} \in O \), and \( 3.5^{g(n)} = (n+1) \). Let \( l \) be such that \( \varphi(l) = 3.5^{g(l)} \), and let \( a = 3.5^l \). Clearly \( a \in O \) and \( |a| = \omega^3 \). Also

\[ a \in A \iff 3.5^{g(l)} \leq a \]

For if \( x = f(n) \), \( 3.5^{g(l)} = \varphi(n) \leq 3.5^l \) and if \( x \notin A \) then \( \varphi(n) \) is everywhere divergent so \( 3.5^{g(l)} \notin O \). From (7) and the nonrecursiveness of \( A \) it follows that \( O(a) \) is nonrecursive.

The following Corollary combines and sharpens some of our results.

**Corollary 3.** If \( A \) is an infinite r.e. set other than \( \omega \), and \( a \) is a constructive ordinal \( \geq \omega^3 \), then \( a \) has a notation \( c \) such that \( O(c) \equiv \Delta \).
Proof. Assume first that \( \alpha = \alpha^2 \), and let \( a \) be as in Theorem 2. Then \( A \subseteq O(\alpha) \) by (7). Since \( g(\alpha) \) in the proof of Theorem 2 need only be the index of a certain partial recursive function, there is no difficulty in modifying the definition of \( g \) so that it is strictly increasing. The range of \( g \) is recursive. Since \( [5,5^0(\alpha)] = \omega \cdot (\alpha+1) \) we have for \( \alpha \neq 0 \):

\[
\alpha \in O_1(\alpha) \iff \exists \gamma \in \text{range } g \cdot g^{-1}(\gamma) \in A.
\]

Therefore \( O(\alpha) \subseteq O_1(\alpha) \subseteq A \), since \( A \neq \emptyset \). Now if \( \alpha \gg \alpha^2 \Rightarrow \beta \) be such that \( \alpha^2 \vdash \beta = \alpha \). Let \( b \) be a notation for \( \beta \). By the previous case it suffices to show that \( O(\alpha^2+b) \subseteq O(\alpha) \). But (P4) in the proof of Lemma 3 shows that \( O(\alpha+b) = O(\alpha) \). By the disjoint unions of \( O(\alpha) \) with a recursive set, from which \( O(\alpha+b) = O(\alpha) \) follows immediately. Thus \( \alpha \vdash a \vdash b \).

In closing we apply our results to partially answer some questions posed by Kreisel [private correspondence] concerning the existence of reducibility relations between \( \Delta^1_0 \) sets and \( \Pi^1_0 \) paths through \( O \). Our first observation comes immediately from the proof of Theorem 2 and was obtained independently by Parikh [4].

Corollary 4. There is a \( \Pi^1_0 \) path \( P \) through \( O \) such that the complete r.e. set \( K \) is many-one reducible to \( P \).

Proof. Let \( A \subseteq K \) be as in the proof of Theorem 2, and let \( P \) be any \( \Pi^1_0 \) path through \( O \) such that \( \alpha \in P \), where \( \alpha \) and \( (a, g) \) are as in the proof of Theorem 2. Then for any \( \delta \), if \( \alpha \in P \) then \( 3.5^0(\delta) \in O(\alpha) \subseteq P \) and if \( \alpha \in K \) then \( 3.5^0(\delta) \in O \). Therefore \( K \prec P \).

In fact it holds in general that if \( \beta \in P \), where \( \beta \) is any path through \( O \), then \( O(\beta) \) is Turing reducible to \( P \). (The proof is the same as the Proposition 1 except that \( O(\alpha) \) is now replaced by \( P \)). We do not know whether this result holds for reducibilities which are stronger than Turing reducibility, but interestingly enough \( \alpha \) itself is a sort of modified converse does.

We call a reducibility \( R \) invertible if whenever a \( \Delta^1_0 \) set \( A \) is \( R \)-reducible to a \( \Pi^1_0 \) path \( P \) through \( O \), then \( A \) is \( R \)-reducible to \( O(\delta) \) for some \( \alpha \in P \). Then, as Kreisel pointed out to me, \( m \)-reducibility is easily seen to be invertible. (If \( A, P \) are as above and \( A = \Delta^1_0(\delta) \) where \( \delta \) is a recursive function, then \( f(\delta) = \Sigma^1_1 \) and hence bounded subset of \( P \). If \( \alpha \in P \) is chosen so that \( \beta \in A \) for \( \beta = f(\delta) \), then clearly \( A = \Delta^1_0(\delta)(\alpha) \) as required.)

Kreisel [5] has shown that enumeration reducibility (as defined in [5]) is invertible and Parikh [4] has shown that truth-table reducibility is invertible. (Although these proofs are more complicated than the one above for many-one reducibility they share the same idea of taking \( A \) and the reduction procedure from \( A \) to \( P \) as given and then inverting the reduction procedure to obtain a (necessarily bounded) subset of \( P \) from this information which yields \( A \) by the same reduction.) From the remarks after Theorem 1 and the cited results of Parikh and Kreisel we obtain immediately the following Corollary which shows that Corollary 4 certainly does not hold for all \( \Pi^1_0 \) paths \( P \) through \( O \).

Corollary 5. There is a \( \Pi^1_0 \) path \( P \) through \( O \) such that any \( \Delta^1_0 \) set which is either enumeration reducible to \( P \) or truth-table reducible to \( P \) is recursive.

It is not known whether Turing reducibility is invertible nor whether the analogue of Corollary 5 holds for Turing reducibility. In fact it is conceivable that Turing reducibility fails very badly to be invertible in the sense that every \( \Delta^1_0 \) set is recursive in every \( \Pi^1_0 \) path \( P \) through \( O \), or even that \( O \) is recursive in every \( \Pi^1_0 \) path through \( O \).

Added in proof (March, 1970), Recently Friedman [7] has shown that there is a \( \Pi^1_0 \) path \( P \) through \( O \) such that \( O \) is recursive in \( P \) (and also \( O(\alpha) \) is recursive for all \( \alpha \in P \)). It follows that Turing reducibility is not invertible.

References

[3] G. Kreisel, Which number theoretic problems can be solved in recursive progressions on \( \Pi^1_0 \) paths through \( O \), J. Symbolic Logic 37 (1972), pp. 311–334.
[7] H. Friedman, Recursion in \( \Pi^1_0 \) paths through \( O \), to appear in Proc. Amer. Math. Soc.

UNIVERSITY OF ILLINOIS
Urbana, Illinois

Accepted by the Editor on 29, 10, 1972.