Second order forcing, algebraically closed structures, and large cardinals

by

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Abstract. We investigate four generalizations of the notion "algebraically closed" to a model theoretic context involving weak second order logic. Whereas in first order logic the existence of algebraically closed structures of various sorts is proved by a natural transfinite induction, in our context it is necessary to assume the existence of large cardinals in order to prove the corresponding existence theorems for structures. The present article is devoted to a relatively precise description of the relationship between large cardinals and structures of the special types alluded to. In the final section some examples are discussed.

1. Introduction. If $\Sigma$ is a class of similar structures, several inequivalent formulations of the notion of an "algebraically closed" structure relative to the class $\Sigma$ and the appropriate first order language have been studied [4, 11, 13]. The concepts introduced for this purpose may be extended to more powerful languages, such as modal, higher order, or infinitary languages; a detailed treatment of some aspects of the last case is given in [3].

We wish to discuss the algebraically closed structures relative to a second order language $L$ which permits quantification over all non-empty subsets $S$ of the domain of a given structure $m$ such that the cardinality of $S$ is less than the cardinality of the domain of $m$ and to find necessary and sufficient conditions for the existence of such structures. For example, concerning the analogue of Robinson's class $G_2$ of the infinitely generic structures [10] we will prove:

Theorem. The following are equivalent:

1. Every increasing function from the ordinals to the ordinals which is continuous at limit ordinals has a regular fixed point.

2. For any class $\Sigma$ of similar structures which is inductive (i.e. closed under unions of chains), $G_2$ is model-consistent with $\Sigma$ (i.e. if $m$ is in $\Sigma$ then $m$ has an extension $m'$ in $G_2$).

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Condition 1 is a well-known axiom of large cardinals. We will discuss the significance of condition 2 in section 5 below.

In the next section several formulizations of the notion of $L$-algebraically closed structures will be presented, to be studied in sections 3 and 4. We discuss $G_2$ in section 5, concluding in section 6 with some examples and open questions.

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2. Algebraically closed structures. A first order alphabet consists of the symbols $\{\ldots, \), $\&, \vee, \neg, \mathit{A}, \mathit{V}, \mathit{S}_{\ldots}$, infinitely many first order variables $x_1, x_2, \ldots$, and a fixed supply of relation symbols, function symbols, and first order constants. If $m$ is a structure, $m = \langle M, \{R_m\}, \{f_m\}\rangle$, then $M$ is the domain of $m$. If $A$ is a set then $|A|$ denotes the cardinality of $A$, but $|m|$ denotes the domain of $m$; the cardinality of $|m|$ will be denoted $|m|$. A subset $S$ of $|m|$ is small iff $|S| < |m|$. If $T$ is a theory, then $\Sigma_T$ denotes the class of models of $T$.

**Definition 2.1. The language $L$.**

1. A second order alphabet is obtained by adjoining a binary predicate symbol $\epsilon$, infinitely many second order variables $X_1, X_2, \ldots$, and second order constants $\langle C_1 \rangle$ to a first order alphabet. Given such a second order alphabet we define well-formed formula and related notions in the obvious way, allowing as well-formed such necessarily false formulas as $c_1 \epsilon c_2, c_1 \epsilon c_2, c_1 \epsilon c_1$, where $c_1, c_2$ are first order constants and $c_1, c_2$ are second order constants.

2. A second order language is a language determined syntactically by a given second order alphabet and semantically by interpreting second order variables or constants in a given model as nonempty small subsets of $m$, and interpreting $\epsilon$ as the membership relation. We also place an important restriction on the substructure relation $m \subseteq m'$ by requiring that the interpretation of any second order constant defined in $m'$ must be the same in both $m$ and $m'$.

3. A formula $\alpha$ of $L$ in prenex form consists of a quantifier-free matrix preceded by a number of blocks of existential or universal quantifiers. $\alpha$ is in $E_n$ (respectively $A_n$) if there are $n$ such quantifier blocks and the first block is existential (respectively universal). Sentences in $E_n$ or $A_n$ are also considered to be in $E_k$ and $A_k$ for $k \geq n$. Sentences of $E_1$ are called existential.

4. We introduce the restricted quantifiers $\exists x \epsilon T, (\forall x \epsilon T)$ in the usual way where $T$ is a second order variable or constant (cf. [8]).

$S_0$ is the class of formulas of $L$ containing only restricted quantifiers. $S_{n+1}$ (respectively $P_{n+1}$) is the closure of $P_n$ (respectively $S_n$) under first and second order unrestricted existential (respectively universal) quantification and restricted existential and universal quantification.

Note that $E_n \subseteq S_n$ and $A_n \subseteq S_n$.

5. Within $S_1$ we distinguish the class $R_{\alpha}$ of formulas of the form

$$\exists x_1, \ldots, x_n, \exists y \alpha(x_1, \ldots, x_n, y)$$

where $\alpha$ is in $S_1$.

**Definition 2.2. Notions of $L$ algebraic completeness.** Suppose $\Sigma$ is a class of similar structures, and $L$ is a weak second order language appropriate to the structures in $\Sigma$.

1. A sentence $\alpha$ of $L$ is $\Sigma$-persistent iff whenever $\alpha$ is true in some structure $m$ in $\Sigma$ then $\alpha$ is true in all extensions of $m$ in $\Sigma$ (recall the restriction placed on the notion of extension in 2.1.2). A formula $\gamma$ of $L$ is $\Sigma$-persistent iff every instance of $\gamma$ is $\Sigma$-persistent.

2. Let $C$ be a class of $\Sigma$-persistent sentences. A structure $m$ in $\Sigma$ is $C$-complete over $\Sigma$ iff for any $\alpha$ in $C$ which is defined in $m$ and true in some extension of $m$ in $\Sigma$, $\alpha$ is true in $m$ itself.

3. If $\Sigma$ is an arbitrary class of similar structures, then $E_1$ and $A_1$ are $\Sigma$-persistent. $E_1$-complete structures are also called existentially complete and the class of all such is denoted $E_1\subseteq$. Similarly the class $S_1\subseteq$-complete structures is denoted $S_1\subseteq$.

4. If $C$ is taken to be the class of all $\Sigma$-persistent sentences then $C$-complete structures are also called $\Sigma$-persistently complete and the class of all such is denoted $\Sigma_1$. We do not in general have $(\Sigma_1) = \Sigma$, because the class $C'$ of all $\Sigma$-persistent sentences may be larger than $C$.

5. Let $\Sigma = \Sigma_1$ and define inductively $\Sigma^{n+1} = (\Sigma^n) \supseteq \Sigma^n = \bigcap_{i=1}^{\infty} \Sigma_i$.

Then $\Sigma = \Sigma_1 \supseteq \Sigma_2 \supseteq \ldots \supseteq \Sigma_i$. It can be proved that $(\Sigma^n) = \Sigma^n$, so that this process terminates with $\Sigma_\omega$.

**Remarks.** Each of the classes $E_1, S_1, \Sigma$, and $\Sigma_\omega$ may be taken to be the class of "algebraically closed" structures of $\Sigma$ in the weak second order sense. When we wish to distinguish our weak second order notions from the corresponding first order notions we will write $E_1, S_1, \Sigma_1, S_\omega$ as opposed to $E_1, S_1, \Sigma_1, S_\omega$.

If $\Gamma$ is a subclass of $\Sigma$, $\Gamma$ is said to be model-consistent with $\Sigma$ iff every structure in $\Gamma$ has an extension in $\Sigma$. It is known that a sufficient condition for the model-consistency of $\Sigma$ is that $\Sigma$ is inductive (i.e. that the union of a chain of structures in $\Sigma$ is again in $\Sigma$). For the classes $E_1, S_1, \Sigma_1, \Sigma_2$, and $\Sigma_\omega$ the assumption that $\Sigma$ is inductive must be supplemented by assumptions concerning the existence of special kinds of large cardinals.

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We will make use of the following familiar notions of cardinal arithmetic. A cardinal $\kappa$ is called singular if $\kappa$ is the sum of fewer than $\kappa$ cardinals, each of which is less than $\kappa$; otherwise $\kappa$ is regular. For cardinals $\kappa, \mu$, $\kappa^\mu$ (read $\kappa$ to the weak power $\mu^\mu$) equals $\sum_{\alpha<\kappa} \kappa^\mu$. $\kappa$ is a strong limit cardinal iff for $\mu<\kappa$, $\kappa^\mu<\kappa$.

3. Model-consistency of $\mathbb{E}_2$, $\mathbb{E}_3$, $\Sigma$ and $\Sigma^\omega$. We wish to discuss the following question in this section: if $\Sigma$ is an arbitrary inductive class, does it follow that $\mathbb{E}_2$, $\mathbb{E}_3$, $\Sigma$, or $\Sigma^\omega$ is model-consistent with $\Sigma^\omega$? In the next section we will restrict our attention to elementary inductive classes.

Theorem 3.1. If $\Sigma$ is an arbitrary inductive class then $\mathbb{E}_2$ is model-consistent with $\Sigma$.

We need a lemma:

**Definition 3.2.** A set $A$ in a structure $m$ meets the formulas $u(a_1, \ldots, a_n)$ iff for $a_1, \ldots, a_n$ in $m$, if $m \models u(a_1, \ldots, a_n)$ then $A \cap \{a_1, \ldots, a_n\} \neq \emptyset$.

**Lemma 3.3.** A necessary and sufficient condition for a structure $m$ to be in $\mathbb{E}_2$ is that for every first order formula $u$ in $E_1$, if $A$ is a set in $m$ which meets $u$, then $|A| = |m|$ or $A$ meets $u$ in every extension of $m$ in $\Sigma$.

Proof. The necessity of the condition on $m$ is obvious. We will show that it is sufficient.

Let $m$ be a structure satisfying the condition of the lemma and let $u$ be a sentence in $E_1$ satisfied in an extension $m'$ of $m$. Without loss of generality $m$ may be assumed to have the form

$$\exists X_1 \ldots \exists X_k \exists a_1 \ldots \exists a_p \bigvee_{i=1}^k \bigwedge_{j=1}^{j_i} v_{i}(\vec{X}_i, \vec{X})$$

where each $v_i$ is a basic formula (i.e., atomic or negated atomic). Let $b_1, \ldots, b_k, B_1, \ldots, B_p$ be respectively elements and small subsets of $m'$ such that $m' \models \bigvee_{i=1}^k \bigwedge_{j=1}^{j_i} v_{i}(\vec{B}_i, \vec{B})$, and let $b_i$ be chosen so that $m' \models \bigvee_{i=1}^k \bigwedge_{j=1}^{j_i} v_{i}(\vec{B}_i, \vec{B})$.

We may assume that $b_1, \ldots, b_k$ are not in $m$, since otherwise if $b_k$ is in $m$, then the variable $a_k$ may be replaced by a constant naming $b_k$.

Let $w_0$ be the conjunction of all first order basic formulas $v_0$ which do not contain the symbol $e$. Let $w_0$ be the conjunction of all formulas of the form $a_0 \neq a$, such that $b_0 \neq b_1$, as well as all of $u$, where $a_0 \neq a$, where $a$ is a constant occurring in $u$. Let $A_1, \ldots, A_p$ be the second order constants occurring in the formulas $v_{i}$, and let $A = \bigcup A_i$. By assumption $A$ does not meet the formula $w_0$ in $m'$, (consider $b_1, \ldots, b_k$, and $|A| < |m'|$, so by the assumption on $m$, $A$ does not meet $w_0$ in $m$. Let $a_1, \ldots, a_p$ be chosen in $m$ satisfying $w_0 \land w_1$, such that $a_1, \ldots, a_p$ all lie outside $A$. For $1 \leq i \leq p$, let

$$S_i = \langle a_i \rangle$$

for some $f_i$, $v_{i} = (\langle a_i \rangle)$, or $v_{i} = (\langle a_i \rangle)$, and $a = a_0$.

It is then fairly easy to check that $m \models \bigwedge_{i=1}^p v_{i}(\vec{B}_i, \vec{B})$, so that $m \models u$.■

**Corollary 3.4.** A countable structure is in $\mathbb{E}_2$ if and only if it is in $\mathbb{E}_2^\omega$.

Theorem 3.1 will be proved by iterating the construction of the next lemma.

**Lemma 3.5.** Let $m$ be a structure in the inductive class $\Sigma$, and suppose that $u$ is a first order existential sentence defined in $m$. Then $m$ has an extension $m'$ in $\Sigma$ with the following property:

- if $A$ is any set which meets $u$ in $m'$ then either $A$ meets $u$ in all extensions of $m'$ or $|A| > |m|$.

Further, any extension of $m'$ has the same property.

Proof. We define a chain $(m_\alpha; \alpha < |m|)$ as follows:

1. $m_0 = m$.
2. For $\alpha$ a limit ordinal, $m_\alpha = \bigcup m_\beta$.
3. If $m_\beta$ has been obtained, let $(A; \beta < \lambda)$ be a well-ordering of the set of all sets which meet $u$ in $m_\beta$. We define another chain $(m'_\beta; \beta < \lambda)$ as follows:
   - $m'_\beta = m_\beta$.
   - For $\beta$ a limit ordinal, $m'_\beta = \bigcup m'_\gamma$.
   - If $m'_\gamma$ has been obtained, let $m'_\gamma + 1$ be an extension of $m'_\gamma$ in which $A_\gamma$ does not meet $u$, if such an extension exists; otherwise $m'_\gamma + 1 = m'_\gamma$.

Take $m_{\lambda+1} = \bigcup m'_\beta$. Note that $m_{\lambda+1}$ has the following property:

- If $A$ is a subset of $|m|$ which meets $u$ in $m_{\lambda+1}$ then $A$ meets $u$ in all extensions of $m_{\lambda+1}$.

Now set $m' = \bigcup m_\alpha$, and suppose that $A$ meets $u$ in $m'$. Let $A_\alpha = A \cap |m_\alpha|$. Then for each $A_\alpha$, if $A_\alpha$ meets $u$ in $m_\alpha$, if for some $A_\alpha$, $A_\alpha$ meets $u$ in $m_{\lambda+1}$, then $A_\alpha$ meets $u$ in all extensions of $m_{\lambda+1}$; otherwise $A_\alpha$ does not meet $u$ in $m_{\lambda+1}$, then for each $A_\alpha$, $A_\alpha$ meets $u$ in $m_{\lambda+1}$, $A_\alpha$, and it is clear that $|A| > |m|$.

This completes the proof of the first part of the lemma; the additional remark is obvious. ■

Proof of Theorem 3.1. It is clear from Lemma 3.5 (including the final remark) and the inductivity of $\Sigma$ that for any structure $m$ in $\Sigma$ a structure $\mathbb{E}(m)$ may be found in $\Sigma$ which extends $m$ and has the following property:

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if \( u \) is a first order existential sentence defined in \( m \) and \( A \) is a subset of \( E(m) \) which meets \( u \) in \( E(m) \), then either \( A \) meets \( u \) in all extensions of \( E(m) \) or \( |A| \geq |m| \).

Form the chain \( m \subseteq E(m) \subseteq E(E(m)) = E^2(m) \subseteq \ldots \subseteq E^2(m) \subseteq \ldots \) and let \( E^2(m) = \bigcup E^2(m) \). Using the criterion of Lemma 3.5, it is easy to see that \( E^2(m) \in E(m) \).

We turn now to an examination of the classes \( S_x, \Sigma, \) and \( \Sigma^\omega \).

**Definition 3.6.** A strictly increasing function \( f \) from ordinals to ordinals is normal if it is continuous at limit ordinals, i.e., if \( f(\gamma) = \sup_{\alpha < \gamma} f(\alpha) \)

for \( \alpha \) a limit ordinal.

**Theorem 3.7.** The following are equivalent:

1. Every normal function has a regular fixed point.
2. For every inductive class \( \Sigma \) of similar structures, \( S_x \) is model-consistent with \( \Sigma \).
3. For every inductive class \( \Sigma \) of similar structures, \( \Sigma^\omega \) is model-consistent with \( \Sigma \).
4. For every inductive class \( \Sigma \) of similar structures, \( \Sigma^\omega \) is model-consistent with \( \Sigma \).

**Proof.** Evidently 4 \( \Rightarrow \) 3 \( \Rightarrow \) 2.

3 implies 1. Let \( f \) be a normal function. For an arbitrary ordinal \( \alpha \) let \( M_0 \) be a set of elements well-ordered by \( \alpha \) with order type \( f(\alpha) \) and let \( P_0 \) contain a single element \( a \) of \( M_0 \). Assume for \( \alpha > 0 \) that \( M_\alpha = \bigcup M_\alpha \) \( \times \) \( \bigcup M_\alpha \), where \( P_\alpha = (\{a : \alpha \in \alpha \} \cup \bigcup P_\alpha) \), and \( a < b \) if \( \alpha \in \bigcup M_\alpha \), \( y \in M_\beta \) where \( \beta < \alpha \) or \( (\beta = \alpha \) and \( a < \beta \) \).

Let \( \Sigma = \{m_0\} \). \( \Sigma \) is inductive, and the cardinality of any structure in \( S_x \) is easily seen to be both regular and a fixed point of \( f \).

1 implies 3. We remark first that if \( \Sigma \) is inductive and \( m \) is in \( \Sigma \), then it is easy to construct an extension \( m^* \) of \( m \) in \( \Sigma \) such that every \( \Sigma \)-persistent sentence defined in \( m \) and true in some extension of \( m \) in \( \Sigma \) is already true in \( m^* \); compare the iterative argument of Theorem 3.3.

If \( M \) is an arbitrary structure in \( \Sigma \) define a chain

\[ m_0 \subseteq m_1 \subseteq \ldots \subseteq m_n \subseteq \ldots \]

by the following transfinite induction:

a. \( m_0 = m \).

b. For a limit ordinal \( m_n = \bigcup m_\alpha \).

c. \( m_{n+1} \) is an extension of \( m_n \) such that any \( \Sigma \)-persistent sentence defined in \( m_n \) and true in some extension of \( m_{n+1} \) is true in \( m_{n+1} \). Furthermore, if it is possible to take \( |m_{n+1}| > |m_n| \), we do so.

Note that we may in fact assume that for every \( a, |m_{n+1}| \geq |m_n| \).

Since otherwise it is very easy to show that \( m_n \) has an extension \( m_{n+1} \) in \( \Sigma \) such that \( m_{n+1} \) has no proper extension in \( \Sigma \), and then \( m_n \) is in \( \Sigma \) for \( \omega \) reasons. Thus if we define the function \( f \) by \( f(\alpha) = |m_{n+1}| \), then \( f \) is strictly increasing and hence, in view of clause b above, \( f \) is normal.

Let \( \lambda \) be a regular fixed point of \( f \); then \( \lambda \) is a limit cardinal. Consider any \( \Sigma \)-persistent sentence \( u \) defined in \( m_\omega \) and true in some extension \( m_\omega \) of \( m_\omega \). By the regularity of \( \lambda \) and the construction of \( m_\omega \), \( u \) is defined in \( m_\omega \) for some \( \alpha < \lambda \) and is therefore true in \( m_\alpha \); since it is \( \Sigma \)-persistent, \( u \) must hold in \( m_\omega \). Thus \( m_\omega \) is in \( \Sigma \), and evidently \( m_\omega \) extends \( m_\omega \).

1 implies 4. Let \( \Sigma \) be an inductive class of structures. Assuming 1, we have just shown that \( \Sigma \) is model-consistent with \( \Sigma \). Retaining the assumption that \( \lambda \) holds, we may prove by a very similar argument that if \( \Sigma \) is model consistent with \( \Sigma \) then \( \Sigma^{\omega+1} \) is model-consistent with \( \Sigma^{\omega+1} \) from which it follows by induction that for every integer \( n, \Sigma^n \) is model-consistent with \( \Sigma \). There is however one complication in the proof of the induction step which requires some attention: if \( \Sigma \) is inductive it need not follow that \( \Sigma \) or any other \( \Sigma \) is inductive, so that clause b in the above definition by transfinite induction may carry us out of \( \Sigma \). We modify the clause as follows:

b'. For \( \gamma \) a limit ordinal, \( m_\gamma = \bigcup m_\alpha \) and \( m_\gamma \) is in \( \Sigma^\gamma \); if \( \bigcup m_\alpha \times \alpha \), then \( m_\gamma = \bigcup m_\alpha \), and the argument goes much as before (note that if \( |m_\alpha| = \lambda \) and \( \lambda \) is regular, then \( m_\alpha = \bigcup m_\alpha \).

Thus for each \( m, \Sigma^\alpha \) is model-consistent with \( \Sigma \). Let \( m \) be in \( \Sigma \) and define another chain by transfinite induction satisfying:

a. \( m_0 = m \).

b. For \( \gamma \) a limit ordinal \( m_\gamma = \bigcup m_\alpha \).

c. For \( \gamma \) a limit ordinal and \( u \) a nonnegative integer, \( m_{\gamma+1} \) is an extension of \( m_{\gamma+1} \) which lies in \( \Sigma_{\gamma+1} \).

Let \( f(\alpha) = |m_\alpha| \) and let \( \lambda \) be the first regular fixed point of \( f \). Then \( m_\lambda \) is in \( \Sigma^\lambda \) and \( m_\lambda \) extends \( m \).

4. The elementary inductive case. According to Theorem 3.1, if \( \Sigma \) is an elementary inductive class then \( \Sigma_x \) is model-consistent with \( \Sigma \). In this section we study the corresponding problem for \( \Sigma_x, \Sigma, \) and \( \Sigma^{\omega} \).

We begin with \( \Sigma_x \).

**Theorem 4.1.** The following are equivalent:

1. If \( \Sigma \) is an elementary inductive class then \( \Sigma_x \) is model-consistent with \( \Sigma \).

2. There are arbitrarily large cardinals \( \lambda \) such that \( \Sigma_x = \lambda \).
Our proof will depend on a downward Löwenheim-Skolem theorem for $S_t$. It is convenient to introduce the following notation: if $u$ is a sentence with constants $a_1, ..., a_p, A_1, ..., A_q$ which are defined in a structure $m$, then $m^u$ is the substructure of $m$ generated by $(a_1, ..., a_p) \cup A_1 \cup ... \cup A_q$. We will occasionally write $[u]$ for max(Im$u$, $\alpha$).

**Lemma 4.2.** If $m \models u$, then $u$ is a sentence of $S_t$, defined in $m$ then the following are equivalent:

1. $m \models u$.
2. $\exists m^u \subseteq m$, cardinality $|m^u| = |u|^{\aleph_0}$, $m^u \models u$.

**Proof.** We first sketch a standard reduction of $S_t$ to $S_t'$ by moving unrestricted quantifiers outward, past restricted quantifiers. Fix a one-to-one function $F: |m| \times |m| \to |m|$ and think of it as a ternary relation. We associate to $u$ a sentence $\tilde{u}$ in $S_t'$ by iterating applications of the following transformations:

1. **A**. Subformula of the form $\exists x \in A \exists X u'[x, X]$ may be replaced by $\exists x \in A \exists X u'[x, X]$. 

2. **B**. Subformula of the form $\forall x \in A \exists X u'[x, X]$ may be replaced by $\exists X \forall x \in A \exists X u'[x, X]$. 

3. **C**. Subformula of the form $\exists X \forall x \in A \exists X u'[x, X]$ may be replaced by $\exists X \forall x \in A \exists X u'[x, X]$.

4. **D**. Subformula of the form $\exists X \forall x \in A \exists X u'[x, X]$ may be replaced by $\exists X \forall x \in A \exists X u'[x, X]$. 

**Proof of Theorem 4.1.**

1 implies 2. Let $E$ be a binary predicate symbol, and take $\Sigma = \emptyset$, or any other set of tautologies involving $E$. Then $\Sigma = \Sigma_E$ is an elementary inductive class.

If $m$ is in $S_{2,1}$, $\Sigma \subseteq |m|$ is small, $\|m\| = \lambda$, and $B \subseteq A$, then the sentence $\forall y \exists x \in A (E(y, x) \iff y \in B)$ is in $S_t$ and is true in an extension of $m$ and hence in $m$. But as $B$ varies in $S_{2,1}$ this requires the presence of $2^{\aleph_0}$ distinct elements in $m$. It follows that $\lambda = 2^\lambda$.

Suppose now that $\lambda$ is singular, and let $\{A_\alpha : \alpha < \mu\}$ be a collection of fewer than $\lambda$ subsets of $|m|$ such that for each $\alpha$, $|A_\alpha| < \lambda$, while $\bigcup A_\alpha = |m|$. Since $m$ is in $S_{2,1}$, it is easy to see that for each $\alpha$.

4.1.1. $m \models \forall x \in A \exists y \in B (E(x, y))$.

For each $\alpha$ choose $a_\alpha$ such that $m \models \forall x \in A (E(y, a_\alpha))$ and let $B = \{a_\alpha : \alpha < \mu\}$. Since $m$ is in $S_{2,1}$, therefore $m \models \forall x \exists y \in B (E(x, y))$, a contradiction.

Thus $\lambda$ is regular and $\lambda = 2^\lambda$, which implies that $\lambda = 2^\lambda$ (cf. [1]).

2 implies 1. Let $\Sigma$ be an elementary inductive class, $\Sigma = \Sigma_E$ and let $m$ be an arbitrary structure in $\Sigma$. Choose $\lambda \geq \|\Sigma\|$, $|m|$, such that $\lambda \geq \lambda$. Extending $m$ if necessary, we may assume that $|m| = \lambda$.

If $u$ is a sentence of $S_t$ which is true in some extension $m''$ of $m$ in $\Sigma$, then $m$ is true in some substructure $m'$ of $m''$ of cardinality $< \lambda$ by Lemma 4.2; since $\Sigma$ is elementary, $m$ and $m'$ have a joint extension of cardinality $\lambda$ in $\Sigma$. Thus we have constructed an extension of $m$ of cardinality $\lambda$ in which $u$ is satisfied. Since there are $\lambda^\alpha$ such sentences defined in $m$ it is clear that we can iterate this construction to obtain an extension $m_1$ of $m$ in $\Sigma$ of cardinality $\lambda$ such that any sentence $u$ of $S_t$ defined in $m_1$ and true in some extension of $m_1$ in $\Sigma$ is true in $m_1$. Iterating
For the remainder of this section \( S_\infty \) will denote the class of all structures of the form \( m = \langle m, \{ B^m \} \rangle \) where \( B^m \) is a binary relation on \( m \).

**Lemma 4.5.** Suppose \( m \in S_\infty \) and \( u(x_1, x_2, y) \) is the formula \( B(x_1, y) \land B(y, x_2) \).

1. In \( m \), \( \forall X \exists Y \exists z \forall u \exists x \forall y \exists x' \exists x'' \exists x''' \exists x'''' \exists x''''

2. \( |m| \) is an uncountable regular strong limit cardinal.

**Proof.** Obvious.

**Lemma 4.5** serves to indicate the power of the language \( L \) when it is interpreted in models of \( S_\infty \). According to part 1, we may quantify over small relations of any fixed rank ("arity") as well as small subsets. The regularity of \( |m| \) eliminates all the pathology which might otherwise accompany such quantification, the uncountability of \( |m| \) permits us to discuss syntactical and semantical notions such as satisfaction in \( L \), since \( \omega \) may be embedded as a small subset of any such \( |m| \) and the fact that \( |m| \) is a strong limit cardinal will be useful when we need to relate a cardinal \( \alpha \) to an \( \omega \) by a larger cardinal, as well as in dealing with restricted quantification of the form \( \forall u \in \alpha \forall x \in \alpha (y) \), or \( \forall u \in \alpha \forall x \in \alpha (y) \).

We will now describe a transformation which associates to each formula \( u \) of \( L \) a formula \( \bar{u} \) in the language of \( S_\infty \). It will be convenient to denote the structure \( \langle B(a), \{ e_{\infty} \} \rangle \) by \( r(a) \).

**Definition 4.6.**

1. Let \( \bar{u}(X, Y, H) \) be a formalization in the language of \( S_\infty \) of:

2. \( [Y, (H)] = r(X) \)

(or, somewhat more precisely, of \( \forall S \exists T(u: S \times S \rightarrow T) \land \langle Y, (H) \rangle = r(X) \); the truth of the first conjunct is required to adequately formalize the second).

Let \( w(a_1, ..., a_n; X, Y, H) \) be the formula:

\[ a_1 \in Y \land ... \land a_n \in Y \land r(k(x, y, H, X, Y), w(a_1, ..., a_n)) \]

Let \( \bar{u}(X, Y, H; X', Y', H') \) be:

\[ \bar{u}(X, Y, H; X', Y', H') \]

Thus \( w(x_1, ..., a_n; X, Y, H) \) says that \( x_1, ..., a_n \) may be construed as sets of rank \( < |X| \) via the isomorphism of \( \langle X, (H) \rangle \) with \( r(|X|) \), while \( \bar{u}(X, Y, H; X', Y', H') \) means

\[ r(|X|) \approx \langle X, (H) \rangle \subseteq \langle X', (H') \rangle \approx r(|X'|) \]
2. Let \( u(x_1, ..., x_n) \) be a formula of \( \mathcal{Z} \) containing constants \( a_1, ..., a_n \) denoting sets of rank \(< \lambda \), where \( \lambda \) is some infinite cardinal, and suppose \( m \) is a structure in \( \mathcal{Z} \) of cardinality at least \( \lambda^+ \). Choose \( A, S, R, T \) satisfying \( \text{Rk}(A, S, T) \) in \( m \) such that \( |A| = \lambda \), and let \( a'_1, ..., a'_n \) correspond to \( a_1, ..., a_n \) under the isomorphism \( \langle S, T \rangle \cong r(|A|) \). We define the transform \( U(y_1, ..., y_{|\mathcal{Z}|}; X, Y, H) \) corresponding to \( u \) by induction on the complexity of \( u \), in such a way that \( U \) formalizes the following notion:

\[
y_1, ..., y_{|\mathcal{Z}|} \text{ correspond to sets } x_1, ..., x_n \text{ of rank } < |X| \text{ via the isomorphism } \langle X, (H) \rangle \cong r(|X|) \text{ and } u(x_1, ..., x_n) \text{ is true.}
\]

We may suppose without loss of generality that the function symbols \( T, P, S, R \) and the constant \( 0 \) do not occur in \( u \), since any formula \( u' \) equivalent to such a formula \( u \). The definition of \( U \) is as follows:

2.1. If \( u \) is atomic, then \( u = t_1 t_2 \) where \( t_1 \) and \( t_2 \) are variables \( x_i \) or constants \( a_i \), then \( U \) is:

\[
H'(t'_1, t'_2) \cup U(t'_1, t'_2; X, Y, H) \cup \text{Ext}(A, S, T; X, Y, H)
\]

where for \( i = 1, 2 \) if \( t_i = a_i \) (respectively \( x_i \)) we take \( t'_i = a'_i \) (respectively \( x_i \)).

2.2. If \( u = u_1 u_2 \), \( u = \bar{u}_1 \), \( u = \neg u_1 \) or \( u = U_1 \), then \( U = U_1 \cup U_2 \cup \bar{U}_1 \), respectively.

2.3. If \( u(x_1, ..., x_n) = \exists x u(x_1, ..., x_n, y_1, ..., y_{|\mathcal{Z}|}; X, Y, H) \) is defined then \( U(y_1, ..., y_{|\mathcal{Z}|}; X, Y, H) :=

\[
\exists X', Y', H', \exists \text{Ext}(X, Y, H; X', Y', H') \cup U'(y_1, ..., y_{|\mathcal{Z}|}; X', Y', H').
\]

**Lemma 4.7.** Suppose \( u(x_1, ..., x_n) \) is a formula of \( \mathcal{Z} \) and all of the constants mentioned in \( u \) have rank \(< \lambda \) where \( \lambda \) is an infinite cardinal, that \( m \) is in \( \mathcal{S}_\lambda \), \(|m| > \lambda \), \( m \) satisfies 4.3.1 and 4.3.2, and that \( A, S, T, a_1, ..., a_n \) have been determined as in Definition 4.6. Let \( U(y_1, ..., y_{|\mathcal{Z}|}; X, Y, H) \) be the formula associated with \( u \). Then for all \( z_1, ..., z_n \) in \( m \) and all small \( X, Y, H \) such that \( m \models u(z_1, ..., z_n; X, Y, H) \), \( m \models U(z_1, ..., z_n; X, Y, H) \) if and only if \( r(|m|) \models u(z_1, ..., z_n) \).

**Proof.** The first statement is trivial by Lemma 4.1. The second is true since \( r(|m|) \) is a limit cardinal in the treatment of the existential quantifier.

For the second statement it suffices to prove the following assertions:

1. If \( u \) is in \( \mathcal{S}_\lambda \) then \( U \) is \( \mathcal{S}_\lambda \)-persistent.
2. If \( u \) was obtained from \( w \) by existential or restricted quantification (in the sense of \( \mathcal{Z} \)) and \( U \) is \( \mathcal{S}_\lambda \)-persistent then \( U \) is \( \mathcal{S}_\lambda \)-persistent.

3. If \( u = \neg \bar{w} \) and \( U \) is \( \mathcal{S}_\lambda \)-persistent then \( U \models \neg \bar{U} \) is \( \mathcal{S}_\lambda \)-persistent.

1 and 3 are thoroughly trivial and 2 is straightforward.

**Lemma 4.8.** Suppose that \( \mathcal{S}_\lambda \) is model-consistent with \( \mathcal{S}_\Lambda \), or, for \( \mu = 0 \), that there are arbitrarily large regular strong limit cardinals. If \( \mu \) is a sentence of \( \mathcal{S}_\lambda \), \( m \) is in \( \mathcal{S}_\lambda \), \( m \) is a regular uncountable strong limit cardinal, \( u \) is defined in \( r(|m|) \), and \( r(|m|) \models u \), then \( V \models u \), i.e. \( \mu \) is true.

**Proof.** Since \( |m| \) is regular we may assume that \( u \) is a subformula of a formula of the form

\[
\text{\( Q \in \mathcal{S}_\Lambda \) and \( \bar{v} \) is in \( \mathcal{S}_\lambda \) (cf. [8]).}
\]

The proof then proceeds by induction on \( u \) and the complexity of \( u \), and is immediate if \( u \) is in \( \mathcal{S}_\lambda \) or if \( u \) was obtained by existential or restricted quantification (in particular we may assume \( u > 0 \)).

Assume therefore that

\[
u(x_1, ..., x_n) = (\forall x) u(x, x_1, ..., x_n)
\]

is in \( \mathcal{S}_\lambda \) (hence in \( \mathcal{P}_0 \), in view of 4.8.1), and that \( r(|m|) \models u \) while \( V \not\models \bar{u} \).

Then for some \( x, V \models \neg \bar{u}(x) \). \( u \) is in \( \mathcal{S}_\lambda \). Take an extension \( m' \) of \( m \) in \( \mathcal{S}_\lambda \) such that \( |m'| > \text{rk}(s) \) and \( |m'| \) is a regular strong limit cardinal. By the induction hypothesis, if \( r(|m'|) \models u \) then \( V \models u(x) \) (hence \( r(|m'|) \models \bar{u}(x) \)). Then \( r(|m'|) \models \bar{u} \) which is \( \mathcal{S}_\lambda \).

Let \( U \) be defined in \( m \) corresponding to \( u \) as described in 4.6. Note that Lemma 4.7 applies to \( U \) in \( m' \) as well as in \( m \). By that lemma, \( m' \models \neg U \) and \( \neg U \) is \( \mathcal{S}_\lambda \)-persistent. By definition of \( \mathcal{S}_\lambda \) it follows that \( m \models \neg U \); hence \( r(|m|) \models \bar{u} \). This contradiction completes the proof of the lemma.

**Definition 4.9.** A relation \( \mathcal{S}(x_1, ..., x_n) \) is definable iff there is a formula \( u(x_1, ..., x_n) \) of \( \mathcal{Z} \) (which may mention sets \( b_1, ..., b_p \) in \( V \)) such that for all \( a_1, ..., a_n \)

\[
\mathcal{S}(a_1, ..., a_n) \text{ if and only if } V \models u(a_1, ..., a_n).
\]

\( \mathcal{S} \) is said to be \( \mathcal{S}_\lambda \)- or \( \mathcal{P}_0 \)-definable iff the formula \( u \) may be taken in \( \mathcal{S}_\lambda \) or \( \mathcal{P}_0 \) respectively. In particular we may speak of definable classes and functions.

**Lemma 4.10.** If \( \mu = 0 \), \( m \) is in \( \mathcal{S}_\lambda \) and \( \mathcal{S}_\lambda \) is model-consistent with \( \mathcal{S}_\Lambda \) then \( |m| \) is a regular fixed point of any \( \mathcal{S}_\lambda \)-definable normal function \( f \) defining formula \( u(x, y) \) is definable in \( r(|m|) \).

**Proof.** Let \( f \) be defined by \( u(x, y) \) in \( \mathcal{S}_\lambda \) and let \( U(x, y; X, Y, H) \) be the corresponding formula defined in \( \mathcal{Z} \) relative to some fixed choice of \( A, S, T \). For \( \lambda < |m| \), \( \mu = f(\lambda) \), choose \( m' \) in \( \mathcal{S}_\lambda \) such that \( m \subseteq m' \).
and \(|m| > \mu\). If \(r([m]) \models \exists u(\lambda, \mu)\) then \(V \models \exists u(\lambda, \mu)\) and \(r([m]) \models w(\lambda, \mu)\).

Fix \(A', S', \mathcal{E}'\) and \(\lambda'\) in \([m]\) such that:
\[m \models w(\lambda'; A', S', \mathcal{E}') \land \text{Ext}(A, S, \mathcal{E}; A', S', \mathcal{E}'),\]
and such that \(\lambda'\) corresponds to \(\lambda\). Then:
\[m' \models \exists y U(\lambda'; y; A', S', \mathcal{E}')\]
which is \(\Sigma^1_m\)-persistent, so \(m \models \exists y U(\lambda'; y; A', S', \mathcal{E}')\). Let \(s'\) in \([m]\) be such that \(m \models U(\lambda'; s'; A', S', \mathcal{E}')\) and let \(s\) be the set corresponding to \(s'\). Then \(V \models u(\lambda, s)\); it follows that \(s = \mu\). Thus \(\mu'\) is in \([m]\), and therefore \([m] \models \exists u(\lambda, \mu)\). The lemma follows immediately.

**Corollary 4.1.1.** If \(n > 0\) and \(\Sigma^m\) is model-consistent with \(\Sigma^m\) then there are arbitrarily large cardinals \(\lambda\) such that \(\lambda\) is a regular fixed point of all \(\Delta^m\)-definable normal functions defined over \(\mathcal{I}(\lambda)\).

We will obtain a partial converse to this corollary by studying certain definable normal functions associated with definable classes of structures.

**Definition 4.1.2.** If \(\Gamma\) is a class of similar structures, the Löwenheim function \(f_\Gamma\) for \(\Gamma\) is defined as follows:
\[f_\Gamma(\lambda) = \mu \iff \mu\text{ is the least cardinal \(> \lambda\) such that for every structure } m \text{ in } \Gamma \text{ such that } |m| \in B(\lambda), \text{ if } u \text{ is in } m \text{ and true in some extension of } m \text{ in } \Gamma, \text{ then } u \text{ is true in some extension } m' \text{ of } m \text{ in } \Gamma \text{ such that } |m'| \in B(\lambda).\]

**Lemma 4.1.3.** Suppose \(n > 0\), \(\Gamma\) is an \(\Sigma^m\)-definable class of similar structures, and \(f\) is the Löwenheim function for \(\Gamma\). Then:
1. \(f\) is \(\Delta^m\)-definable.
2. \(\Gamma\) is \(\Delta^m\)\(_{\omega\cdot n}\)-definable.

**Proof.** Both assertions may be verified directly. We remark that such quantifications as: (\(\forall u\text{ defined in } m\)) may be formalized by restricted quantifications since all such formulas \(u\) may be construed as finite functions from \(\omega\) to \(\omega \cup |m|\), and hence lie in \(\mathcal{S}(\mathcal{S}(\mathcal{T}(P(\omega), |m|)))\).

**Corollary 4.1.4.** If \(\Sigma\) is a elementary class then for each \(\Sigma^m\) is \(\Sigma^m\)-definable and the corresponding Löwenheim function \(f_{\Sigma^m}\) is \(\Delta^{m+1}\)-definable.

We can now augment Corollary 4.1.1 by

**Theorem 4.1.5.** Suppose that there are arbitrarily large cardinals \(\lambda\) such that \(\lambda\) is a regular fixed point of every \(\Delta^m\)-definable normal function defined over \(\mathcal{I}(\lambda)\), and that \(\Sigma\) is an elementary inductive class. Then \(\Sigma^m\) is model-consistent with \(\Sigma\).

**Proof.** Let \(f_{\Sigma^m}\) be the Löwenheim function for \(\Sigma^m\), and let \(f(\lambda) = \sup\{f_{\Sigma^m}(\lambda), \ldots, f_{\omega}(\lambda)\}\). Then \(f\) is \(\Delta^{m+1}\)-definable. \(f\) may not be normal because it need not be strictly increasing. However \(f\) is dominated by the \(\Delta^m\)-definable normal function \(g\) determined by
\[g(\lambda) = \sup\{g(\mu) + 1 : \mu < \lambda\}
\]
for all \(\lambda\).

If \(m\) is a structure in \(\mathcal{S}\) and \(m \models R(\lambda)\), let \(\mu_1, \ldots, \mu_n\) be regular fixed points of \(g_\lambda\), and hence also of \(f_\lambda\), such that \(\mu_1 < \mu_2 < \ldots < \mu_n\). It is easy to show inductively that there is a chain
\[m = m_0 \subseteq m_1 \subseteq m_2 \subseteq \ldots \subseteq m_n\]
such that for \(0 \leq i < n\), \(m_i \models \Sigma^m\). Thus the theorem is proved.

**Corollary 4.1.6.** The following are equivalent:
1. Every definable normal function has a regular fixed point.
2. For every elementary inductive class \(\Sigma\) and every integer \(n\), \(\Sigma^m\) is model-consistent with \(\Sigma\).

**Theorem 4.1.7.** The following are equivalent:
1. There are arbitrarily large cardinals \(\lambda\) such that \(\lambda\) is a regular fixed point of every normal function definable over \(\Sigma\) (i.e., by a formula \(u\) defined in \(\mathcal{I}(\lambda)\)).
2. For every elementary inductive class \(\Sigma\), \(\Sigma^m\) is model-consistent with \(\Sigma\).

**Proof.** 2 implies 1. It follows from Lemma 4.8 that for any structure \(m\) in \(\Sigma^m\), \(|m|\) is a regular fixed point of every normal function defined over \(\mathcal{I}(\lambda)\).

1 implies 2. Suppose \(\Sigma = \Sigma^m\) and \(\lambda > \text{rk}(\mathcal{I})\) is a regular fixed point of all normal functions definable over \(\lambda\), and in particular of all the Löwenheim functions \(f_{\lambda}\) associated with the classes \(\Sigma^m\). Then it follows that
\[\{m \in \Sigma^m : |m| \in B(\lambda)\} = \emptyset\]
is a definable set of sets. Thus starting with any \(m\) in \(\Sigma\) such that \(|m| \in B(\lambda)\), we may obtain a chain
\[m_0 = m \subseteq m_1 \subseteq m_2 \subseteq \ldots \subseteq m_\gamma \subseteq m_{\gamma+1} \subseteq \ldots \subseteq m_\lambda\]
such that for each limit ordinal \(\gamma\) and integer \(n\), \(m_\gamma \in \Sigma^m\), and \(m_\gamma \models R(\lambda)\) under the assumptions on \(\lambda\) this may be carried out in ZFC.

Let \(m' = \bigcup m_\gamma\). Then \(m' \models \Sigma^m\). This completes the proof.

**5. Forcing:** \(G_\Sigma\). There are two notions of algebraically closed structure with respect to first order logic which have been developed recently by Abraham Robinson, namely the notions of "generic structure" with respect to either finite or infinite forcing. The latter generalizes to the weak second order language \(L\) in an obvious way, and is intimately related to our class \(\Sigma^m\); see Theorem 5.5 below. The first order case is described in detail in [10] and [2].
Definition 5.1. If $\Sigma$ is a class of structures and $m$ is a structure in $\Sigma$, we define the forcing relation $m \Vdash u$ (m forces u) with respect to $\Sigma$ for sentences $u$ of $\Sigma$ by induction on the complexity of $u$:  
1. For $u$ atomic, $m \vDash u$ if $m \vDash u$ (this applies to sentences of the form $a \in A$ as well as first order atomic sentences).  
2. $m \vDash u \land v$ if $m \vDash u$ and $m \vDash v$.  
3. $m \vDash u \lor v$ if $m \vDash u$ or $m \vDash v$.  
4. $m \vDash \neg u$ if there is no extension $m'$ of $m$ in $\Sigma$ such that $m' \vDash u$.  
5. $m \vDash (\exists a)u(a)$ if for some $a$ in $|m|$, $m \vDash u(a)$.  
6. $m \vDash (\forall X)u(X)$ if for some small set $A$ contained in $|m|$, $m \vDash u(A)$.  

Definition 5.2. A structure $m$ of $\Sigma$ is generic iff for each sentence $u$ of $L$ defined in $m$, $m \vDash u$ if and only if $m \vDash u$. The class of generic structures in $\Sigma$ is denoted $G_\Sigma$.  

The following standard lemma of forcing theory may be proved without difficulty:  

Lemma 5.3. Suppose $m$ is a structure in $\Sigma$, $u$ is defined in $m$, $m \vDash u$, and $m'$ is an extension of $m$ in $\Sigma$. Then $m' \vDash u$.  

Corollary 5.4. If $m_1$, $m_2$ are generic and $m_1 \subseteq m_2$, then $m_1 \preceq m_2$.  

Theorem 5.5. If $G_\Sigma$ or $\Sigma^m$ is model-consistent with $\Sigma$, then $G_\Sigma = \Sigma^m$.  

Proof. We will simply sketch the argument; details for the first order case may be found in [2].  

First one proves that $G_\Sigma$ and $\Sigma^m$ are both model-complete, i.e. that for $m_1 \subseteq m_2$, $m_1 \preceq m_2$, $G_\Sigma = G_{m_2}$ for $G_\Sigma$ this is just Corollary 5.4. Next it may be proved that if $\Sigma$ is arbitrary model-complete model-consistent subalgebras of $\Sigma$, then $G_{m_2} \subseteq G_{m_1}$. The proof of this fact simply consists of a direct verification that structures in $G_{m_2}$ satisfy the defining conditions for $G_\Sigma$ as well as for $\Sigma^m$.  

From these results the theorem follows at once.  

Corollary 5.6. The following are equivalent:  
1. Every normal function has a regular fixed point.  
2. Every inductive class, $G_\Sigma$ is model-consistent with $\Sigma$.  

Although this corollary is an utterly trivial consequence of Theorems 3.5 and 5.5, it should be noted that in pathological cases $G_\Sigma$ and $\Sigma^m$ may actually exhibit radically different behavior.  

The model-consistency of $G_\Sigma$ or $\Sigma^m$ is the basis for any real theory of these classes, and it can be used to characterize $\Sigma^m$ abstractly as the $L$-model companion of $\Sigma$ (cf. [1], sections 1 and 6). We give the following application of model-consistency as an example.  

Theorem 5.7. If $\Sigma$ is an elementary class of structures and $\Sigma^m$ is model-consistent with $\Sigma$ then $\Sigma^m$ is closed under $L$-elementary substructure, i.e. if $m_1 \preceq m_2$ and $m_2$ is in $\Sigma^m$, then $m_1$ is in $\Sigma^m$.  

Proof. It is evidently sufficient to apply the following to $\Gamma = \Sigma^m$ for $n \geq 0$, 1, 2, ...  

5.7.1. If $\Sigma$ is elementary, $\Gamma \subseteq \Sigma$ is closed under $L$-elementary  
substructure, and $\Gamma$ is model-consistent with $\Sigma$, then $\Gamma$ is closed under  
$L$-elementary substructure.  

To prove 5.7.1, suppose $m_1 \subseteq m_2$, and $m_2$ is in $\Gamma$. Since $\Sigma$ is  
elementary, $m_1$ is in $\Sigma$. Let $m_1$ be any extension of $m_1$, and $u$ be a $\Gamma$-  
consistent sentence true in $m_1$. Then by a standard compactness argument  
a structure $m_2$ can be found in $\Sigma$ such that  

\[ m_1 \xrightarrow{m_2} \text{commutes} \]  

By the model-consistency of $\Gamma$, $m_2$ has an extension $m_2$ in $\Gamma$. The  
$m_1 \vDash u \Rightarrow m_2 \vDash u \Rightarrow m_2 \vDash u \Rightarrow m_2 \vDash u$. It follows that $m_2$ is in $\Gamma$.

6. Examples and open questions. Motivated by various formulations of the notion of algebraic completeness in first order logic, we have described four classes of special structures: $E_2$, $S_2$, $\Sigma^m$, and $\Sigma^m$. It is clear that for interesting classes $\Sigma$, algebraic structures, $\Sigma^m$ and $\Sigma^m$ are extremely difficult to "compute". On the other hand $E_2$ and $S_2$ are fairly well-behaved, according to Lemma 3.3 and Theorem 4.4. We might also consider the class $S_2$ of $S_2$-complete structures, but as yet we know of no example in which $S_2$ and $S_2'$ differ.

It should be possible to describe $E_2$ and $S_2$ explicitly for a number of interesting classes $\Sigma$. We give two simple examples:  

Example 6.1. If $\Sigma$ is the class of all fields, then $E_2$ is the class of  
algebraically closed fields (as in the first order case) while $S_2$ is the class of  
uncountable algebraically closed fields.  

Proof. If $A$ is the class of algebraically closed fields and $B$ is the  
class of uncountable algebraically closed fields, then $A \supseteq E_2 \supseteq S_2 \supseteq B$,  
the last inclusion following from Theorem 3.10. All fields in $A - B$ are countable; therefore,  
by Corollary 3.5, $A = E_2$. Finally, to see that $A - B$ is present, consider the formula  
\[ \exists X [\forall x \in X (x^2 = x)] \land \forall x, y \in X (x^2 = y^2 = a \land a = y) \]  

& $\exists x \in X \forall y \in X (y^2 \neq a)$.  

Example 6.2. If $\Sigma$ is the class of ordered sets, then $E_2$ is the class of  
densely ordered sets without first or last element such that no interval  
is small, while $S_2$ is the class of uncountable saturated ordered sets.
Proof. The first statement is a straightforward consequence of Lemma 3.3. For the second part, since each saturated ordered set is homogeneous universal, each uncountable one is in $S_2$. It is obvious that each ordered set in $S_2$ is saturated; to see that each is uncountable consider the formula:

$$\exists x \forall y \exists z \exists y \in x (x < y) .$$

In Examples 6.1 and 6.2, $\mathbb{D}^2$ coincides with the class of structures in $\mathbb{D}_2^2$ in which all infinite definable subsets of $[m]$ have cardinality $\|[m]||$. The next example shows that this last property is not a sufficient condition for membership in $\mathbb{D}^2$.

**Example 6.3.** Let $Q$, $R$ be binary predicate symbols and let $E_1, E_2, ...$ be unary predicate symbols. Consider the following axiom scheme:

1. $\forall x, y \in \mathbb{E}(x, y) \iff Q(x, y) \land Q(y, x)$.
2. $R$ is an equivalence relation.
3. $\forall x, y, z \in \mathbb{E}(x, y) \implies Q(x, z) \land Q(w, y) \implies R(x, w) \land R(y, z)$.
4. $\forall x, y, z \in \mathbb{E}(x, y) \land Q(y, z) \implies R(x, y) \lor R(y, z)$.
5. For each integer $n \in \mathbb{E}_n$ is the unique equivalence class $\mathbb{E}$ with respect to $R$ such that $|\mathbb{E}| = n$.

These axioms can be formalized by a first-order theory $T$. Let $\Sigma = \Sigma_p$.

- If $m$ is a structure in $\Sigma$ and $m'$ is the graph obtained by reducing $m$ modulo the equivalence relation $R$ then each connected component of $m'$ contains exactly two vertices and $Q$ is asymmetric on $m'$.

Let $m$ be the particular model defined as follows:

1. Let $A_1, A_2, ...$ be mutually disjoint sets such that for each integer $n \in [A_1] = n$ and $|B_2| = n$. Take $|m| = \bigcup A_n$.
2. For each integer $n$ and each $x \in [m]$, $E[x] \subseteq [m]$ if and only if $x$ is in $A_n$.
3. For $x, y \in [m]$, $Q(x, y)$ holds if and only if for some integer $n$:
   - $x, y \in A_n$;
   - $x \in A_n$ and $y \in A_n$ and $n$ is odd; or
   - $x \in B_n$ and $y \in A_n$ and $n$ is even.
4. For $x, y \in [m]$, $E(x, y)$ holds if and only if $Q(x, y) \land Q(y, x)$ holds.

Then $m$ is in $\Sigma$. We claim that in $\Sigma$, every infinite definable subset of $m$ has cardinality $\|[m]||$ and $m$ is not in $\mathbb{D}^2$.

**Proof.** Let $D$ be the diagram of $m$, and let $c_1, c_2, ...$ be infinitely many distinct new constants. Define:

- $U_1(y) = "\exists z \exists Q(x, y) \land \neg R(x, y)"$;
- $U_2(y) = "\exists z \exists Q(x, y) \land \neg R(x, y)"$.

In Problems 6.4, 6.5. Can Theorems 4.13 be improved to yield a natural condition equivalent to:

"For every elementary inductive $\Sigma$ and $\Sigma'$ is model-consistent with $\Sigma''$?"

References


F. Fundamenta Mathematicae LXXVII
Recursiveness of initial segments of Kleene's $O$

by

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Abstract. It is shown that for any constructive ordinal $a > \omega^a$, there are both recursive and nonrecursive initial segments of the partial ordering $<_a$ of Kleene's $O$ which have order type $a$.

Let $O$ be Kleene's set of notations for constructive ordinals and let $<_a$ be Kleene's partial ordering of $O$. If $a \in O$, let $O(a)$ denote $\{b : b <_a a \}$. It is well known that for any $a \in O$, $O(a)$ is a recursively enumerable (r.e.) subset of $O$ which is well-ordered by $<_a$ with order type $|a|$, the ordinal for which $a$ is a notation. Our purpose here is to determine which constructive ordinals have notations $a$ such that $O(a)$ is recursive (nonrecursive). We prove that every constructive ordinal has a notation $a$ such that $O(a)$ is recursive and in fact that there is a $\Pi_1$ path $P$ through $O$ such that $O(a)$ is recursive for all $a$ in $P$. In the other direction we show that the constructive ordinals which have notations $a$ with $O(a)$ nonrecursive are exactly those which are $> \omega^a$. Our constructions in fact show that if $a > \omega^a$ is a constructive ordinal and $A$ is an infinite r.e. set (other than $a$) then $a$ has a notation $a$ such that $O(a)$ is $\Sigma_1$-equivalent to $A$.

Most of our notation is standard. In particular, we use $\varphi_e$ for the $e$th partial recursive function and call $e$ an index of $\varphi_e$. We use the recursion theorem in the following informal style: in the definition of a partial recursive function $\varphi_e$, its index may be assumed known in advance. Of course such arguments are easily formalized. An index of a recursive set is any index of its characteristic function. A path through $O$ is a subset of $O$ which is linearly ordered by $<_a$ and contains a notation for each constructive ordinal.

Information on Kleene's $O$ can be found in [1], [2], or [5]. In particular we shall need the binary recursive function $+_a$ which represents ordinal addition in the sense that $|a +_b b| = |a| + |b|$ for $a, b \in O$. Also $+_a$

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