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Second order forcing, algebraically closed structures, and large cardinals

by

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Abstract. We investigate four generalizations of the notion "algebraically closed" to a model theoretic context involving weak second order logic. Whereas in first order logic the existence of algebraically closed structures of various sorts is proved by a natural transfinite induction, in our context it is necessary to assume the existence of large cardinals in order to prove the corresponding existence theorems for structures. The present article is devoted to a relatively precise description of the relationship between large cardinals and structures of the special types alluded to. In the final section some examples are discussed.

1. Introduction. If Σ is a class of similar structures, several inequivalent formulations of the notion of an "algebraically closed" structure relative to the class Σ and the appropriate first order language have been studied [4, 11, 13]. The concepts introduced for this purpose may be extended to more powerful languages, such as modal, higher order, or infinitary languages; a detailed treatment of some aspects of the last case is given in [3].

We wish to discuss the algebraically closed structures relative to a second order language L which permits quantification over all non-empty subsets S of the domain of a given structure m such that the cardinality of S is less than the cardinality of the domain of m and to find necessary and sufficient conditions for the existence of such structures. For example, concerning the analogue of Robinson's class G_Σ of the *infinitely generic* structures [10] we will prove:

THEOREM. *The following are equivalent*:

1. *Every increasing function from the ordinals to the ordinals which is continuous at limit ordinals has a regular fixed point.*

2. *For any class Σ of similar structures which is inductive* (*i.e. closed under unions of chains*). *G_Σ is model-consistent with Σ* (*i.e. if m is in Σ then m has an extension m' in G_Σ*).

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Condition 1 is a well-known axiom of large cardinals. We will discuss the significance of condition 2 in section 5 below.

In the next section several formulations of the notion of L-algebraically closed structures will be presented, to be studied in sections 3 and 4. We discuss G_Σ in section 5, concluding in section 6 with some examples and open questions.

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2. Algebraically closed structures. A first order alphabet consistes of the symbols (,), &, $\vee$, $\neg$, $\exists$, $\forall$, infinitely many first order variables $x_1, x_2, \ldots$, and a fixed supply of relation symbols, function symbols, and first order constants. If m is a structure, $m = \langle M, \{R_a\}, \{f_\beta\}\rangle$, then M is the *domain* of m. If A is a set then $|A|$ denotes the cardinality of A, but $|m|$ denotes the domain of m; the cardinality of $|m|$ will be denoted $\|m\|$. A subset S of $|m|$ is *small* iff $|S| < \|m\|$. If T is a theory, then Σ_T denotes the class of models of T.

DEFINITION 2.1. *The language L.*

1. A *second order alphabet* is obtained by adjoining a binary predicate symbol ϵ, infinitely many second order variables $X_1, X_2, \ldots$, and second order constants $\{C_a\}$ to a first order alphabet. Given such a second order alphabet we define *well-formed formula* and related notions in the obvious way, allowing as well-formed such necessarily false formulas as $c_1 \epsilon c_2$, $C_1 \epsilon c_2$, and $C_1 \epsilon C_2$, where c_1, c_2 are first order constants and C_1, C_2 are second order constants.

2. A *weak second order language L* is a language determined syntactically by a given second order alphabet and semantically by interpreting second order variables or constants in a given model m as nonempty small subsets of m, and interpreting ϵ as the membership relation. We also place an important restriction on the substructure relation $m_1 \subseteq m_2$ by requiring that the interpretation of any second order constant defined in m_1 must be the same in both m_1 and m_2.

3. A formula u of L in prenex normal form consists of a quantifier-free matrix preceded by a number of blocks of existential or universal quantifiers. u is in E_n (respectively A_n) if there are n such quantifier blocks and the first block is existential (respectively universal). Sentences in E_n or A_n are also considered to be in E_k and A_k for $k \geqslant n$. Sentences of E_1 are called *existential.*

4. We introduce the restricted quantifiers $(\exists x \epsilon T)$, $(\forall x \epsilon T)$ in the usual way where T is a second order variable or constant (cf. [8]).

$S_0 = P_0$ is the class of formulas of L containing only restricted quantifiers. S_{n+1} (respectively P_{n+1}) is the closure of P_n (respectively S_n) under first and second order unrestricted existential (respectively universal) quantification and restricted existential and universal quantification.

Note that $E_n \subseteq S_n$ and $A_n \subseteq P_n$.

5. Within S_1 we distinguish the class S_1' of formulas of the form

$$\exists X_1 \ldots \exists X_q \exists x_1 \ldots \exists x_p u(X_1, \ldots, X_q, x_1, \ldots, x_p)$$

where u is in S_0.

DEFINITION 2.2. *Notions of L algebraic completeness.* Suppose Σ is a class of similar structures, and L is a weak second order language appropriate to the structures in Σ.

1. A sentence u of L is *Σ-persistent* iff whenever u is true in some structure m in Σ then u is true in all extensions of m in Σ (recall the restriction placed on the notion of extension in 2.1.2). A formula u of L is *Σ-persistent* iff every instance of u is Σ-persistent.

2. Let C be a class of Σ-persistent sentences. A structure m in Σ is *C-complete* over Σ iff for any u in C which is defined in m and true in some extension of m in Σ, u is true in m itself.

3. If Σ is an arbitrary class of similar structures, then E_1 and S_1 are Σ-persistent. E_1-complete structures are also called *existentially complete* and the class of all such is denoted E_Σ. Similarly the class of S_1-complete structures is denoted S_Σ.

4. If C is taken to be the class of all Σ-persistent sentences then C-complete structures are also called *Σ-persistently complete* and the class of all such is denoted Σ'. We do not in general have $(\Sigma')' = \Sigma'$, because the class C' of all Σ'-persistent sentences may be larger than C.

5. Let $\Sigma^0 = \Sigma$ and define inductively $\Sigma^{n+1} = (\Sigma^n)'$, $\Sigma^\infty = \bigcap_n \Sigma^n$. Then $\Sigma = \Sigma^0 \supseteq \Sigma' \supseteq \ldots \supseteq \Sigma^\infty$. It can be proved that $(\Sigma^\infty)' = \Sigma^\infty$, so that this process terminates with Σ^∞.

REMARKS. Each of the classes E_Σ, S_Σ, Σ', and Σ^∞ may be taken to be the class of "algebraically closed" structures of Σ in the weak second order sense. When we wish to distinguish our weak second order notions from the corresponding first order notions we will write E_Σ^{II}, Σ_{II}', $\Sigma_{\mathrm{II}}^\infty$ as opposed to E_Σ^{I}, Σ_{I}', $\Sigma_{\mathrm{I}}^\infty$.

If Γ is a subclass of Σ, Γ is said to be *model-consistent* with Σ iff every structure m in Σ has an extension m' in Γ. It is known that a sufficient condition for the model-consistency of $\Sigma_{\mathrm{I}}^\infty$ (and a fortiori of E_Σ^{I} and Σ_{I}') with Σ is that Σ is inductive (i.e. that the union of a chain of structures in Σ is again in Σ). For the classes E_Σ^{II}, S_Σ, Σ_{II}', and $\Sigma_{\mathrm{II}}^\infty$ the assumption that Σ is inductive must be supplemented by assumptions concerning the existence of special kinds of large cardinals.

We will make use of the following familiar notions of cardinal arithmetic. A cardinal λ is called *singular* if λ is the sum of fewer than λ cardinals, each of which is less than λ; otherwise λ is *regular*. For cardinals λ, μ, $\lambda^{\underline{\mu}}$ (read "λ to the weak power μ") equals $\sum_{\nu<\mu}\lambda^{\nu}$. λ is a strong limit cardinal iff for $\mu<\lambda$, $2^{\underline{\mu}}<\lambda$.

3. Model-consistency of E_Σ, S_Σ, Σ' and Σ^∞. We wish to discuss the following question in this section: if Σ is an arbitrary inductive class, does it follow that E_Σ, S_Σ, Σ', or Σ^∞ is model-consistent with Σ? In the next section we will restrict our attention to elementary inductive classes.

THEOREM 3.1. *If Σ is an arbitrary inductive class then E_Σ is model-consistent with Σ.*

We need a lemma:

DEFINITION 3.2. A set A in a structure m *meets* the formula $u(x_1, \ldots, x_n)$ iff for $a_1, \ldots, a_n$ in m, if $m \vDash u(a_1, \ldots, a_n)$ then $A \cap \{a_1, \ldots, a_n\} \neq \emptyset$.

LEMMA 3.3. *A necessary and sufficient condition for a structure m to be in E^{II}_Σ is that for every first order formula u in E_1, if A is a set in m which meets u, then $|A| = \|m\|$ or A meets u in every extension of m in Σ.*

Proof. The necessity of the condition on m is obvious. We will show that it is sufficient.

Let m be a structure satisfying the condition of the lemma and let u be a sentence in E_1 satisfied in an extension m' of m. Without loss of generality u may be assumed to have the form

$$\exists X_1 \ldots \exists X_q\, \exists x_1 \ldots \exists x_p \bigvee_{i=1}^{I} \bigwedge_{j=1}^{J_i} v_{ij}(\bar{x}, \bar{X}),$$

where each v_{ij} is a basic formula (i.e., atomic or negated atomic). Let $b_1, \ldots, b_p, B_1, \ldots, B_q$ be respectively elements and small subsets of $|m'|$ such that $m' \vDash \bigvee_{i=1}^{I}\bigwedge_{j=1}^{J_i} v_{ij}(\bar{b}, \bar{B})$, and let i_0 be chosen so that $m' \vDash \bigwedge_{j=1}^{J_{i_0}} v_{i_0 j}(\bar{b}, \bar{B})$. We may assume that $b_1, \ldots, b_n$ are not in m, since otherwise if b_k is in m then the variable x_k may be replaced by a constant naming b_k.

Let w_0 be the conjunction of all first order basic formulas $v_{i_0 j}$, i.e. of all those formulas $v_{i_0 j}$ which do not contain the symbol ϵ. Let w_1 be the conjunction of all formulas of the form $x_k \neq x_r$ such that $b_k \neq b_r$, as well as all formulas of the form $x_k \neq a$, where a is a constant occurring in u. Let $A_1, \ldots, A_r$ be the second order constants occurring in the formulas $v_{i_0 j}$, and let $A = \bigcup_{i=1}^{\gamma} A_i$. By assumption A does not meet the formula $w_0 \,\&\, w_1$ in m' (consider $b_1, \ldots, b_q$), and $|A| < \|m\|$, so by the assumption on m, A does not meet $w_0 \,\&\, w_1$ in m. Let $a_1, \ldots, a_p$ be chosen

in m satisfying $w_0 \,\&\, w_1$, such that $a_1, \ldots, a_p$ all lie outside A. For $1 \leqslant i \leqslant q$, let

$$S_i = \{a\colon \text{for some } j,\ v_{i_0 j} = \text{“}a \in X_i\text{”, or } v_{i_0 j} = \text{“}x_k \in X_i\text{” and } a = a_k\}.$$

It is then fairly easy to check that $m \vDash \bigwedge_{j=1}^{J_{i_0}} v_{i_0 j}(\bar{a}, \bar{S})$, so that $m \vDash u$. ∎

COROLLARY 3.4. *A countable structure is in E^{II}_Σ if and only if it is in E^{I}_Σ.*

Theorem 3.1 will be proved by iterating the construction of the next lemma.

LEMMA 3.5. *Let m be a structure in the inductive class Σ, and suppose that u is a first order existential sentence defined in m. Then m has an extension m' in Σ with the following property:*

if A is any set which meets u in m' then either A meets u in all extensions of m' or $|A| \geqslant \|m\|$.

Furthermore, any extension of m' has the same property.

Proof. We define a chain $\{m_\alpha\colon \alpha < \|m\|\}$ as follows:

1. $m_0 = m$.
2. For γ a limit ordinal, $m_\gamma = \bigcup_{\alpha<\gamma} m_\alpha$.
3. If m_α has been obtained, let $\{A_\beta\colon \beta<\lambda\}$ be a well-ordering of the set of all sets which meet u in m_α. We define another chain $\{m_\alpha^\beta\colon \beta<\lambda\}$ as follows:

a. $m_\alpha^0 = m_\alpha$.

b. For γ a limit ordinal, $m_\alpha^\gamma = \bigcup_{\beta<\gamma} m_\alpha^\beta$.

c. If m_α^β has been obtained, let $m_\alpha^{\beta+1}$ be an extension of m_α^β in which A_β does not meet u, if such an extension exists; otherwise $m_\alpha^{\beta+1} = m_\alpha^\beta$.

Take $m_{\alpha+1} = \bigcup_{\beta<\alpha} m_\alpha^\beta$. Note that $m_{\alpha+1}$ has the following property: if A is a subset of $|m_\alpha|$ which meets u in $m_{\alpha+1}$, then A meets u in all extensions of $m_{\alpha+1}$.

Now set $m' = \bigcup_{\alpha<\|m\|} m_\alpha$, and suppose that A meets u in m'. Let $A_\alpha = A \cap |m_\alpha|$. Then for each α A_α meets u in m_α. If for some α A_α meets u in $m_{\alpha+1}$ then A_α meets u in all extensions of $m_{\alpha+1}$ and a fortiori A meets u in all extensions of m'. On the other hand, if for each α A_α does not meet u in $m_{\alpha+1}$ then for each α $A_{\alpha+1} \supsetneqq A_\alpha$, and it is clear that $|A| \geqslant \|m\|$.

This completes the proof of the first part of the lemma; the additional remark is obvious. ∎

Proof of Theorem 3.1. It is clear from Lemma 3.5 (including the final remark) and the inductivity of Σ that for any structure m in Σ a structure $E(m)$ may be found in Σ which extends m and has the following property:

if u is a first order existential sentence defined in m and A is a subset of $E(m)$ which meets u in $E(m)$, then either A meets u in all extensions of $E(m)$ or $|A| \geq \|m\|$.

Form the chain $m \subseteq E(m) \subseteq E(E(m)) = E^2(m) \subseteq \ldots \subseteq E^n(m) \subseteq \ldots$ and let $E^\infty(m) = \bigcup_n E^n(m)$. Using the criterion of Lemma 3.3, it is easy to see that $E^\infty(m) \in E_\Sigma^{\mathrm{II}}$. ■

We turn now to an examination of the classes S_Σ, Σ', and Σ^∞.

Definition 3.6. A strictly increasing function f from ordinals to ordinals is *normal* iff it is continuous at limit ordinals, i.e. iff $f(\gamma) = \sup_{\alpha<\gamma} f(\alpha)$ for α a limit ordinal.

Theorem 3.7. *The following are equivalent*:

1. *Every normal function has a regular fixed point.*
2. *For every inductive class Σ of similar structures, S_Σ is model-consistent with Σ.*
3. *For every inductive class Σ of similar structures, Σ' is model-consistent with Σ.*
4. *For every inductive class Σ of similar structures, Σ^∞ is model-consistent with Σ.*

Proof. Evidently $4 \Rightarrow 3 \Rightarrow 2$.

2 implies 1. Let f be a normal function. For α an arbitrary ordinal let M'_α be a set of elements well-ordered by $<'_\alpha$ with order type $f(\alpha)$ and let P'_α contain a single element x_α of M'. Assume for $\alpha \neq \beta$ that $M'_\alpha \cap M'_\beta = \emptyset$. For $\alpha > 0$ let m_α be $\langle \bigcup_{\beta<\alpha} M'_\beta, \{<_\alpha, P_\alpha\}\rangle$ where $P_\alpha = \{x_\beta\colon \beta < \alpha\}$, and $x <_\alpha y$ iff $x \in M'_{\beta_1}$, $y \in M'_{\beta_2}$ where $\beta_1 < \beta_2$ or $(\beta_1 = \beta_2 \;\&\; x <'_{\beta_1} y)$.

Let $\Sigma = \{m_\alpha\}$. Σ is inductive, and the cardinality of any structure in S_Σ is easily seen to be both regular and a fixed point of f.

1 implies 3. We remark first that if Σ is inductive and m is in Σ, then it is an easy matter to construct an extension m' of m in Σ such that every Σ-persistent sentence defined in m and true in some extension of m' in Σ is already true in m'; compare the iterative argument of Theorem 3.2.

If m is an arbitrary structure in Σ define a chain

$$m_0 \subseteq \ldots \subseteq m_\alpha \subseteq \ldots$$

by the following transfinite induction:

a. $m_0 = m$.

b. For γ a limit ordinal $m_\gamma = \bigcup_{\alpha<\gamma} m_\alpha$.

c. $m_{\alpha+1}$ is an extension of m_α in Σ such that any Σ-persistent sentence defined in m_α and true in some extension of $m_{\alpha+1}$ is true in $m_{\alpha+1}$. Furthermore, if it is possible to take $\|m_{\alpha+1}\| > \|m_\alpha\|$, we do so.

Note that we may in fact assume that for every α, $\|m_{\alpha+1}\| > \|m_\alpha\|$, since otherwise it is very easy to show that m_α has an extension m'_α in Σ such that m'_α has no proper extension in Σ, and then m'_α is in Σ' (and even Σ^∞) for trivial reasons. Thus if we define the function f by $f(\alpha) = \|m_\alpha\|$, then f is strictly increasing and hence, in view of clause b above, f is normal.

Let λ be a regular fixed point of f; then λ is a limit cardinal. Consider any Σ-persistent sentence u defined in m_λ and true in some extension m' of m_λ. By the regularity of λ and the construction of m_λ, u is defined in m_α for some $\alpha < \lambda$ and is therefore true in $m_{\alpha+1}$; since it is Σ-persistent, u must hold in m_λ. Thus m_λ is in Σ', and evidently m_λ extends m.

1 implies 4. Let Σ be an inductive class of structures. Assuming 1, we have just shown that Σ' is model-consistent with Σ. Retaining the assumption that 1 holds, we may prove by a very similar argument that if Σ^n is model consistent with Σ then Σ^{n+1} is model-consistent with Σ^n, from which it follows by induction that for every integer n, Σ^n is model-consistent with Σ. There is however one complication in the proof of the induction step which requires some attention: if Σ is inductive it need not follow that Σ' or any other Σ^n is inductive, so that clause b in the above definition by transfinite induction may carry us out of Σ^n. We modify the clause as follows:

b′. For γ a limit ordinal, $m_\gamma \supseteq \bigcup_{\alpha<\gamma} m_\alpha$ and m_γ is in Σ^n; if $\bigcup_{\alpha<\gamma} m_\alpha \in \Sigma^n$ then $m_\gamma = \bigcup_{\alpha<\gamma} m_\alpha$, and the argument goes much as before (note that if $\|m_\lambda\| = \lambda$ and λ is regular, then $m_\lambda = \bigcup_{\alpha<\lambda} m_\alpha$).

Thus for each n, Σ^n is model-consistent with Σ. Let m be in Σ and define another chain by transfinite induction satisfying:

a. $m_0 = m$.

b. For γ a limit ordinal $m_\gamma = \bigcup_{\alpha<\gamma} m_\alpha$.

c. For γ a limit ordinal and n a nonnegative integer, $m_{\gamma+n+1}$ is an extension of $m_{\gamma+n}$ which lies in Σ^{n+1}.

Let $f(\alpha) = \|m_\alpha\|$ and let λ be the first regular fixed point of f. Then m_λ is in Σ^∞ and m_λ extends m. ■

4. The elementary inductive case.

According to Theorem 3.1, if Σ is an elementary inductive class then E_Σ is model-consistent with Σ. In this section we study the corresponding problem for S_Σ, Σ', and Σ^∞. We begin with S_Σ.

Theorem 4.1. *The following are equivalent*:

1. *If Σ is an elementary inductive class then S_Σ is model-consistent with Σ.*
2. *There are arbitrarily large cardinals λ such that $\lambda^{\underline{\lambda}} = \lambda$.*

Our proof will depend on a downward Löwenheim-Skolem theorem for S_1. It is convenient to introduce the following notation: if u is a sentence with constants $a_1, \ldots, a_p, A_1, \ldots, A_q$ which are defined in a structure m, then m_u is the substructure of m generated by $\{a_1, \ldots, a_p\} \cup A_1 \cup \ldots \cup A_q$. We will occasionally write $\|u\|$ for $\max(\|m_u\|, \aleph_0)$.

LEMMA 4.2. *If m is a structure and u is a sentence of S_1 defined in m then the following are equivalent*:

1. $m \models u$.
2. $\exists m' \subseteq m$, $\|m'\| = \|u\|^+$, $m' \models u$.

Proof. We first sketch a standard reduction of S_1 to S_1' achieved by moving unrestricted quantifiers outward, past restricted quantifiers. Fix a one-to-one function $F\colon |m| \times |m| \to |m|$ and think of it as a ternary relation. We associate to u a sentence $\hat{u}$ in S_1 by iterated applications of the following transformations:

4.2.1. A subformula of the form $\exists x \in A \exists X u'(x, X)$ may be replaced by $\exists X \exists x \in A\, u'(x, X)$.

4.2.2. A subformula of the form $\forall x \in A \exists X u'(x, X)$ may be replaced by $\exists H \forall x \in A\big(u'(x, \{y\colon H(x, y)\})\big)$.

4.2.2 requires some explanation. H is a second order variable, which we may interpret as representing a relation via the function F. (Intuitively, $H(x, y)$ represents "$F(x, y) \in H$".) All that is necessary for a formalization of 4.2.2 is a formalization of the atomic formulas occurring in $u'\big(x, \{y\colon H(x, y)\}\big)$ which involve $\{y\colon H(x, y)\}$; these have the form $t \in \{y\colon H(x, y)\}$, $\{y\colon H(x, y)\} \in t$, or $\{y\colon H(x, y)\} \in \{y\colon H(x, y)\}$. We formalize the first formula as

$$\exists z\big(F(x, t, z) \,\&\, z \in H\big).$$

The other two formulas are tautologically false and may be formalized by $\exists x(x \neq x)$.

Evidently for structures of regular cardinality the formulas occurring in 4.2.2 are equivalent. However in the singular case the desired relation H need not be small. Nonetheless, if the transform $\hat{u}$ is defined by successive transformations of the type described in 4.2.1 and 4.2.2 the following properties are clearly satisfied:

4.2.3. $\hat{u}$ is in S_1', $\hat{u} = \exists X_1 \ldots \exists X_q \hat{u}'(X_1, \ldots, X_q)$ where $\hat{u}'$ contains no bound second order variables (and is in S_1).

4.2.4. If $m \models \hat{u}$ then $m \models u$. If $\|m\|$ is regular and $m \models u$ then $m \models \hat{u}$.

4.2.5. If $m \models u$ then there are sets $A_1, \ldots, A_q$ which *may not be small* such that $m \models \hat{u}'(A_1, \ldots, A_q)$.

We will now prove that $1 \Rightarrow 2$, establishing the lemma (the reverse implication is obvious).

Suppose then that $m \models u$, and that we have introduced a suitable function $F\colon |m| \times |m| \to |m|$ and have defined the transform $\hat{u}$. Fix sets $A_1, \ldots, A_q$ according to 4.2.5 such that $m \models \hat{u}'(A_1, \ldots, A_q)$. We may interpret $\hat{u}'(A_1, \ldots, A_q)$ as a first order sentence, construing $A_1, \ldots, A_q$ and all other second order constants in $\hat{u}'$ as unary predicates. Apply the downward Löwenheim-Skolem theorem to m to obtain a substructure m'' of m which is an elementary substructure with respect to all these predicates as well as the ordinary first order predicates occurring in u' and the predicate F; we may assume that $m_u \subseteq m''$, and $\|m''\| = \|u\|$. Let $A_i' = A_i^{m''}$, so that $m'' \models u'(A_1', \ldots, A_q')$. Extend m'' to a structure m' satisfying $m' \subseteq m$, $\|m'\| = \|u\|^+$. Then $m' \models u'(A_1', \ldots, A_q')$ and $A_1', \ldots, A_q'$ are small subsets of $|m'|$, so $m' \models \hat{u}$ (compare 4.2.3). Therefore (4.2.4) $m' \models u$, as desired. ■

Proof of Theorem 4.1.

1 implies 2. Let E be a binary predicate symbol, and take $T = \emptyset$, or any other set of tautologies involving E. Then $\Sigma = \Sigma_T$ is an elementary inductive class.

If m is in S_Σ, $A \subseteq |m|$ is small, $\|m\| = \lambda$, and $B \subseteq A$, then the sentence

4.1.1. $\exists x \forall y \in A[E(y, x) \Leftrightarrow y \in B]$

is in S_1 and is true in an extension of m and hence in m. But as B varies in 4.1.1 this requires the presence of $2^{|A|}$ distinct elements in m. It follows that $\lambda = 2^{\underset{\smile}{\lambda}}$.

Suppose now that λ is singular, and let $\{A_\alpha\colon \alpha < \mu\}$ be a collection of fewer than λ subsets of $|m|$ such that for each α $|A_\alpha| < \lambda$, while $\bigcup_{\alpha<\mu} A_\alpha = |m|$. Since m is in S_Σ, it is easy to see that for each α

4.1.2. $m \models \exists x \forall y \in A_\alpha\big(E(y, x)\big)$.

For each α choose a_α such that $m \models \forall y \in A_\alpha\big(E(y, a_\alpha)\big)$ and let $B = \{a_\alpha\colon \alpha < \mu\}$. Since m is in S_Σ, therefore $m \models \exists x \forall y \in B[\neg E(x, y)]$, a contradiction.

Thus λ is regular and $\lambda = 2^{\underset{\smile}{\lambda}}$, which implies that $\lambda = \lambda^{\underset{\smile}{\lambda}}$ (cf. [1]).

2 implies 1. Let Σ be an elementary inductive class, $\Sigma = \Sigma_T$ and let m be an arbitrary structure in Σ. Choose $\lambda \geqslant |T|^+, \|m\|$, such that $\lambda^{\underset{\smile}{\lambda}} = \lambda$. Extending m if necessary, we may assume that $\|m\| = \lambda$.

If u is a sentence of S_1 which is true in some extension m'' of m in Σ, then u is true in some substructure m' of m'' of cardinality $\leqslant \lambda$ by Lemma 4.2; since Σ is elementary, m and m' have a joint extension of cardinality λ in Σ. Thus we have constructed an extension of m of cardinality λ in which u is satisfied. Since there are $\lambda^{\underset{\smile}{\lambda}} = \lambda$ such sentences defined in m it is clear that we can iterate this construction to obtain an extension m_1 of m in Σ of cardinality λ such that any sentence u of S_1 defined in m and true in some extension of m_1 in Σ is true in m_1. Iterating this least construction λ times and applying the regularity of λ (which is a consequence of $\lambda = \lambda^\lambda$) we see easily that m has an extension in S_Σ of cardinality λ. ∎

The class S_Σ is related to another well-known class of structures, the Σ-homogeneous universal structures, introduced in [6, 7] and more fully investigated in [9, 5].

Definition 4.3. Suppose Σ is a class of structures and m is in Σ.

1. m is *Σ-universal* iff for each $m' \in \Sigma$ such that $\|m'\| \leqslant \|m\|$ there is an embedding $f\colon m' \to m$.

2. m is Σ-homogeneous iff for each $m' \subseteq m$ in Σ such that $\|m'\| < \|m\|$ if $f\colon m' \to m$ is an embedding then f extends to an automorphism of m.

Theorem 4.4. *If Σ is an elementary class of structures, m is Σ-homogeneous universal, and $\|m\| > \aleph_0$, then m is in S_Σ.*

Proof. Suppose $m \subseteq m' \models u \in S_1$, and u is defined in m. Then u is satisfied by some substructure m'' of m' of cardinality at most $\|u\|^+$. As an easy consequence of the homogeneity and universality of m we can find an embedding f so that

$$\begin{tikzcd} & m'' \arrow[dd, dashed, "f"] \\ m_u \arrow[ur] \arrow[dr] & \\ & m \end{tikzcd} \quad \text{commutes}.$$

It then follows that $m \models u$. ∎

There is an analog to Theorem 3.5 which for Σ elementary inductive relates the model-consistency of Σ^∞ to the existence of fixed points for definable normal functions. The rest of this section is devoted to this theorem and related results, some of which apply to Σ' (Corollary 4.11 and Theorem 4.15).

Let Z be the usual language of set theory, whose only predicate symbol is the binary predicate symbol $\in$ denoting the membership relation, augmented by the following unary function symbols: T, P, S, R intended to denote respectively transitive closure, pair set, power set, and the function $R(x) = \{y\colon \mathrm{rk}(y) < x\}$ defined for ordinals x, where rk is the ordinary rank function defined by $\mathrm{rk}(y) = \sup(\mathrm{rk}(z)+1\colon z \in y)$, and containing in addition the constant ω denoting the integers.

We will be interested in the (S_n, P_n)-hierarchy defined relative to Z, in which quantifiers of the form:

$$\forall x \in F(y_1, \ldots, y_n) \quad \text{or} \quad \exists x \in F(y_1, \ldots, y_n),$$

for F a term of Z may be ignored. Let us denote the sets of formulas thus defined by S_n^*, P_n^*. It follows from Lemma 5, p. 12 in [8] that formulas in S_n^* and P_n^* are equivalent with respect to Zermelo–Fraenkel set theory to formulas in S_{n+1} and P_{n+1} respectively, for $n \geqslant 1$.

For the remainder of this section Σ_0 will denote the class of all structures of the form $m = \langle |m|, \{B^m\}\rangle$ where B^m is a binary relation on m. Σ_0 is inductive and elementary.

Lemma 4.5. *Suppose m is in Σ_0' and $u(x_1, x_2, y)$ is the formula $B(x_1, y)$ & & $B(y, x_2)$. Then:*

1. *In m, $\forall X \exists Y[u\colon X \times X \xrightarrow{1\text{-}1} Y]$ in the sense that*

$$m \models \forall X \exists Y \forall x_1, x_2 \in X\, E!\, y \in Y\big(u(x_1, x_2, y)\big) \;\&\; \forall x_1', x_2' \in X\big(u(x_1', x_2', y)\big) \Rightarrow$$
$$\Rightarrow (x_1 = x_1' \;\&\; x_2 = x_2').$$

2. *$\|m\|$ is an uncountable regular strong limit cardinal.*

Proof. Obvious. ∎

Lemma 4.5 serves to indicate the power of the language L when it is interpreted in models in Σ_0'. According to part 1, we may quantify over small relations of any fixed rank ("arity"), as well as small subsets. The regularity of $\|m\|$ eliminates all the pathology which might otherwise accompany such quantification, the uncountability of $\|m\|$ permits us to discuss syntactical and semantical notions such as satisfaction in L, since ω may be embedded as a small subset of any such $|m|$, and the fact that $\|m\|$ is a strong limit cardinal will be useful when we need to replace an ordinal $a < \|m\|$ by a larger cardinal, as well as in dealing with restricted quantification of the form $\exists x \in S(y)$, or $\forall x \in S(y)$.

We will now describe a transformation which associates to each formula u of Z a formula U in the language of Σ_0. It will be convenient to denote the structure $\langle R(a), \{\in|_{R(a)}\}\rangle$ by $r(a)$.

Definition 4.6.

1. Let $\mathrm{Rk}(X, Y, H)$ be a formalization in the language of Σ_0 of:

$$\langle Y, \{H\}\rangle \simeq r(X)$$

(or, somewhat more precisely, of $\forall S \exists T(u\colon S \times S \xrightarrow{1\text{-}1} T)$ & $\langle Y, \{H\}\rangle \simeq r(X)$; the truth of the first conjunct is required to adequately formalize the second).

Let $w(x_1, \ldots, x_q; X, Y, H)$ be the formula:

$$x_1 \in Y \;\&\; \ldots \;\&\; x_q \in Y \;\&\; \mathrm{Rk}(X, Y, H).$$

Let $\mathrm{Ext}(X, Y, H; X', Y', H')$ be:

$$\mathrm{Rk}(X, Y, H) \;\&\; \mathrm{Rk}(X', Y', H') \;\&\; Y \subset Y' \;\&\; H = H' \cap Y \times Y.$$

Thus $w(x_1, \ldots, x_q; X, Y, H)$ says that $x_1, \ldots, x_q$ may be construed as sets of rank $< |X|$ via the isomorphism of $\langle Y, \{H\}\rangle$ with $r(|X|)$, while $\mathrm{Ext}(X, Y, H; X', Y', H')$ means

$$r(|X|) \simeq \langle Y, \{H\}\rangle \subseteq \langle Y', \{H'\}\rangle \simeq r(|X'|).$$

2. Let $u(x_1, \ldots, x_q)$ be a formula of Z containing constants $a_1, \ldots, a_p$ denoting sets of rank $< \lambda$, where λ is some infinite cardinal, and suppose m is a structure in Σ_0 of cardinality at least λ^+. Choose A, S, E satisfying $\mathrm{Rk}(A, S, E)$ in m such that $|A| = \lambda$, and let $a_1', \ldots, a_p'$ correspond to $a_1, \ldots, a_p$ under the isomorphism $\langle S, \{E\}\rangle \simeq r(|A|)$. We define the transform $U(y_1, \ldots, y_q;\ X, Y, H)$ corresponding to u by induction on the complexity of u, in such a way that U formalizes the following notion:

$y_1, \ldots, y_q$ correspond to sets $x_1, \ldots, x_q$ of rank $< |X|$ via the isomorphism $\langle Y, \{H\}\rangle \simeq r(|X|)$ and $u(x_1, \ldots, x_q)$ is true.

We may suppose without loss of generality that the function symbols T, P, S, R and the constant ω do not occur in u, since any formula u' is equivalent to such a formula u. The definition of U is as follows:

2.1. If u is atomic, so that u is $t_1 \in t_2$ where t_1 and t_2 are variables x_j or constants a_j, then U is:

$$H(t_1', t_2') \ \&\ w(t_1', t_2'';\ X, Y, H) \ \&\ \mathrm{Ext}(A, S, E;\ X, Y, H)$$

where for $i = 1, 2$ if $t_i = a_j$ (respectively x_j) we take $t_i' = a_j'$ (respectively y_j).

2.2. If $u = u_1 \& u_2$, $u_1 \vee u_2$, or $\neg u_1$ and U_1, U_2 are defined corresponding to u_1, u_2, then $U = U_1 \& U_2$, $U_1 \vee U_2$, or $\neg U_1$, respectively.

2.3. If $u(x_1, \ldots, x_q) = \exists x u'(x, x_1, \ldots, x_q)$ and $U'(y, y_1, \ldots, y_q;\ X, Y, H)$ is defined then $U(y_1, \ldots, y_q;\ X, Y, H)$ is:

$$\exists X', Y', H', y\big(\mathrm{Ext}(X, Y, H;\ X', Y', H') \ \&\ U'(y, y_1, \ldots, y_q;\ X', Y', H')\big).$$

Lemma 4.7. *Suppose $u(x_1, \ldots, x_q)$ is a formula of Z and all of the constants mentioned in u have rank $< \lambda$ where λ is an infinite cardinal, that m is in Σ_0, $\|m\| > \lambda$, m satisfies 4.3.1 and 4.3.2, and that $A, S, E, a_1', \ldots, a_p'$ have been determined as in Definition 4.6. Let $U(y_1, \ldots, y_q;\ X, Y, H)$ be the formula associated with u. Then for all $s_1', \ldots, s_q'$ in $|m|$ and all small X, Y, H such that $m \models w(s_1', \ldots, s_q';\ X, Y, H) \ \&\ \mathrm{Ext}(A, S, E;\ X, Y, H)$,*

$$m \models U(s_1', \ldots, s_q';\ X, Y, H) \quad \textit{if and only if} \quad r(|m|) \models u(s_1, \ldots, s_q)$$

where s_i' corresponds to s_i via the isomorphism $\langle Y, \{H\}\rangle \simeq r(|X|)$. Furthermore, if u is in S_{n+1}^* then U is Σ_0^n-persistent.

Proof. The first statement involves a trivial induction on the complexity of u; the fact that $\|m\|$ is a limit cardinal enters in the treatment of the existential quantifier.

For the second statement it suffices to prove the following assertions:

1. If u is in S_0^* then U is Σ_0-persistent.

2. If u was obtained from u' by existential or restricted quantification (in the sense of Z) and if U' is Σ_0^n-persistent then U is Σ_0^n-persistent.

3. If $u = \neg u'$ and U' is Σ_0^n-persistent then $U = \neg U'$ is Σ_0^{n+1}-persistent.

1 and 3 are thoroughly trivial and 2 is straightforward. ∎

Lemma 4.8. *Suppose that Σ_0^{n-1} is model-consistent with Σ_0, or, for $n = 0$, that there are arbitrarily large regular strong limit cardinals. If u is a sentence of S_{n+1}^*, m is in Σ_0^n, m is a regular uncountable strong limit cardinal, u is defined in $r(\|m\|)$, and $r(\|m\|) \models u$, then $V \models u$, i.e. u is true.*

Proof. Since $\|m\|$ is regular we may assume that u is a subformula of a formula of the form

4.8.1. $\exists x_1 \forall x_2 \ldots Q x_{n+1} v(x_1, \ldots, x_{n+1})$

where Q is $\forall$ or $\exists$ and v is in S_0^* (cf. [8]).

The proof then proceeds by induction on n and the complexity of u, and is immediate if u is in S_0^* or if u was obtained by existential or restricted quantification (in particular we may assume $n > 0$).

Assume therefore that

$$u(x_1, \ldots, x_k) = (\forall x) u'(x, x_1, \ldots, x_k)$$

is in S_{n+1}^* (hence in P_n^*, in view of 4.8.1), and that $r(\|m\|) \models u$ while $V \models \neg u$. Then for some s, $V \models \neg u'(s)$. u' is in S_{n-1}^*.

Take an extension m' of m in Σ_0^{n-1} such that $\|m'\| > \mathrm{rk}(s)$, and $\|m'\|$ is a regular strong limit cardinal. By the induction hypothesis, if $r(\|m'\|) \models u'(s)$ then $V \models u'(s)$; hence $r(\|m'\|) \models \neg u'(s)$. Then $r(\|m'\|) \models \neg u$ which is in S_n^*.

Let U be defined in m corresponding to u as described in 4.6. Note that Lemma 4.7 applies to U in m' as well as in m. By that lemma, $m' \models \neg U$ and $\neg U$ is Σ_0^{n-1}-persistent. By definition of Σ_0^n it follows that $m \models \neg U$; hence $r(\|m\|) \models \neg u$. This contradiction completes the proof of the lemma. ∎

Definition 4.9. A relation $\mathfrak{R}(x_1, \ldots, x_n)$ is *definable* iff there is a formula $u(x_1, \ldots, x_n)$ of Z (which may mention sets $b_1, \ldots, b_p$ in V) such that for all $a_1, \ldots, a_n$

$$\mathfrak{R}(a_1, \ldots, a_n) \quad \text{if and only if} \quad V \models u(a_1, \ldots, a_n).$$

$\mathfrak{R}$ is said to be S_n^*- or P_n^*-*definable* iff the formula u may be taken in S_n^* or P_n^* respectively. In particular we may speak of definable classes and functions.

Lemma 4.10. *If $n > 0$, m is in Σ_0^n and Σ_0^n is model-consistent with Σ_0 then $\|m\|$ is a regular fixed point of any S_n^*-definable normal function f whose defining formula $u(x, y)$ is defined in $r(\|m\|)$.*

Proof. Let f be defined by $u(x, y)$ in S_n^*, and let $U(x, y;\ X, Y, H)$ be the corresponding formula defined in m relative to some fixed choice of A, S, E. For $\lambda < \|m\|$, $\mu = f(\lambda)$, choose m' in Σ_0^n such that $m \subseteq m'$

and $\|m'\| > \mu$. If $r(\|m'\|) \models \neg u(\lambda, \mu)$ then $V \models \neg u(\lambda, \mu)$; therefore $r(\|m'\|) \models u(\lambda, \mu)$.

Fix A', S', E' and λ' in $|m|$ such that

$$m \models w(\lambda'; A', S', E') \,\&\, \mathrm{Ext}(A, S, E; A', S', E'),$$

and such that λ' corresponds to λ. Then:

$$m' \models \exists y\, U(\lambda', y; A', S', E').$$

which is Σ_0^{n-1}-persistent, so $m \models \exists y\, U(\lambda', y; A', S', E')$. Let s' in $|m|$ be such that $m \models U(\lambda', s'; A', S', E')$ and let s be the set corresponding to s'. Then $V \models u(\lambda, s)$; it follows that $s = \mu$. Thus μ' is in $|m|$, and therefore $\|m\| \geqslant \mu$. Thus $\|m\| \geqslant \sup_{\lambda<\|m\|} f(\lambda)$. The lemma follows immediately. ■

COROLLARY 4.11. *If $n > 0$ and Σ_0^n is model-consistent with Σ_0 then there are arbitrarily large cardinals λ such that λ is a regular fixed point of all S^*-definable normal functions defined over $r(\lambda)$.*

We will obtain a partial converse to this corollary by studying certain definable normal functions associated with definable classes of structures.

DEFINITION 4.12. If Γ is a class of similar structures, the *Löwenheim function* f_Γ for Γ is defined as follows:

$f_\Gamma(\lambda) = \mu$ iff μ is the least cardinal $\geqslant \lambda$ such that for every structure m in Γ such that $|m|$ is in $R(\lambda)$, if u is defined in m and true in some extension of m in Γ, then u is true in some extension m' of m in Γ such that $|m'|$ is in $R(\mu)$.

LEMMA 4.13. *Suppose $n > 0$, Γ is an S_n^*-definable class of similar structures, and f is the Löwenheim function for Γ. Then:*

1. *f is P_{n+1}^*-definable.*
2. *Γ' is S_{n+2}^*-definable.*

Proof. Both assertions may be verified directly. We remark that such quantifications as: ($\forall u$ defined in m) may be formalized by restricted quantifications since all such formulas u may be construed as finite functions from ω to $\omega \cup |m|$, and hence lie in $S\Big(S\Big(S\big(T\big(P(\omega, |m|)\big)\big)\Big)\Big)$.

COROLLARY 4.14. *If Σ is a elementary then for each n Σ^n is S_{2n}^*-definable and the corresponding Löwenheim function f_n is P_{2n+1}^*-definable.*

We can now augment Corollary 4.11 by

THEOREM 4.15. *Suppose that there are arbitrarily large cardinals λ such that λ is a regular fixed point of every S_{2n}^*-definable normal function defined over $r(\lambda)$, and that Σ is an elementary inductive class. Then Σ^n is model-consistent with Σ.*

Proof Let f_k be the Löwenheim function for Σ^k, and let $f(\lambda) = \sup\big(f_0(\lambda), \ldots, f_{n-1}(\lambda)\big)$. Then f is P_{2n-1}^*-definable. f may not be normal

because it need not be strictly increasing. However f is dominated by the S_{2n}^*-definable normal function g determined by

$$g(\lambda) = \sup\big(g(\mu)+1\colon\ \mu<\lambda\big)+f(\lambda) \quad \text{for all } \lambda.$$

If m is a structure in Σ and $m \in R(\lambda)$, let $\mu_1, \ldots, \mu_n$ be regular fixed points of g, and hence also of f, such that $\lambda < \mu_1 < \mu_2 < \ldots < \mu_n$. It is easy to show inductively that there is a chain

$$m = m_0 \subseteq m_1 \subseteq m_2 \subseteq \ldots \subseteq m_n$$

such that for $0 \leqslant i \leqslant n$, $m_i \in \Sigma^i$. Thus the theorem is proved. ■

COROLLARY 4.16. *The following are equivalent:*

1. *Every definable normal function has a regular fixed point.*
2. *For every elementary inductive class Σ and every integer n, Σ^n is model-consistent with Σ.*

THEOREM 4.17. *The following are equivalent:*

1. *There are arbitrarily large cardinals λ such that λ is a regular fixed point of every normal function definable over λ (i.e., by a formula u defined in $r(\lambda)$).*
2. *For every elementary inductive class Σ, Σ^∞ is model-consistent with Σ.*

Proof. 2 implies 1. It follows from Lemma 4.8 that for any structure m in Σ^∞, $\|m\|$ is a regular fixed point of every normal function defined over $\|m\|$.

1 implies 2. Suppose $\Sigma = \Sigma_T$ and $\lambda > \mathrm{rk}(T)$ is a regular fixed point of all normal functions definable over λ, and in particular of all the Löwenheim functions f_n associated with the classes Σ^n. Then it follows that

$$\{\{m \in \Sigma^n\colon\ |m| \in R(\lambda)\}\colon\ n \in \omega\}$$

is a definable set of sets. Thus starting with any m in Σ such that $|m| \in R(\lambda)$, we may obtain a chain

$$[m = m_0 \subseteq m_1 \subseteq m_2 \subseteq \ldots \subseteq m_\gamma \subseteq m_{\gamma+1} \subseteq \ldots \quad (\gamma < \lambda);$$

such that for each limit ordinal γ and integer n, $m_{\gamma+n} \in \Sigma^n$, and $|m_{\gamma+n}| \in R(\lambda)$; under the assumptions on λ, this may be carried out in ZFC. Let $m' = \bigcup_{\alpha<\lambda} m_\alpha$. Then $m' \in \Sigma^\infty$. This completes the proof. ■

5. Forcing; G_Σ. There are two notions of algebraically closed structure with respect to first order logic which have been developed recently by Abraham Robinson, namely the notions of "generic structure" with respect to either finite or infinite forcing. The latter generalizes to the weak second order language L in an obvious way, and is intimately related to our class Σ^∞; see Theorem 5.5 below. The first order case is described in detail in [10] and [2].

DEFINITION 5.1. If Σ is a class of structures and m is a structure in Σ, we define the forcing relation $m \Vdash u$ (m *forces* u) with respect to Σ for sentences u of L by induction on the complexity of u:

1. For u atomic, $m \Vdash u$ iff $m \models u$ (this applies to sentences of the form $a \in A$ as well as first order atomic sentences).
2. $m \Vdash u_1 \,\&\, u_2$ iff $m \Vdash u_1$ and $m \Vdash u_2$.
3. $m \Vdash u_1 \vee u_2$ iff $m \Vdash u_1$ or $m \Vdash u_2$.
4. $m \Vdash \neg u$ iff there is no extension m' of m in Σ such that $m' \Vdash u$.
5. $m \Vdash (\exists x) u(x)$ iff for some a in $|m|$, $m \Vdash u(a)$.
6. $m \Vdash (\exists X) u(X)$ iff for some small set A contained in $|m|$, $m \Vdash u(A)$.

DEFINITION 5.2. A structure m of Σ is *generic* iff for each sentence u of L defined in m, $m \Vdash u$ if and only if $m \models u$. The class of generic structures in Σ is denoted G_Σ.

The following standard lemma of forcing theory may be proved without difficulty:

LEMMA 5.3. *Suppose m is a structure in Σ, u is defined in m, $m \Vdash u$, and m' is an extension of m in Σ. Then $m' \Vdash u$.*

COROLLARY 5.4. *If m_1, m_2 are generic and $m_1 \subseteq m_2$ then $m_1 \prec_L m_2$.*

THEOREM 5.5. *If G_Σ or Σ^∞ is model-consistent with Σ, then $G_\Sigma = \Sigma^\infty$.*

Proof. We will simply sketch the argument; details for the first order case may be found in [2].

First one proves that G_Σ and Σ^∞ are both model-complete, i.e. that for $m_1 \subseteq m_2$ in Γ where $\Gamma = G_\Sigma$ or Σ^∞, $m_1 \prec_L m_2$; for G_Σ this is just Corollary 5.4. Next it may be proved that if Γ is an arbitrary model-complete model-consistent subclass of Σ, then $\Gamma \subseteq G_\Sigma \cap \Sigma^\infty$. The proof of this fact simply consists of a direct verification that structures in Γ satisfy the defining conditions for G_Σ as well as for Σ^∞.

From these results the theorem follows at once. ■

COROLLARY 5.6. *The following are equivalent:*

1. *Every normal function has a regular fixed point.*
2. *For every inductive class Σ, G_Σ is model-consistent with Σ.*

Although this corollary is an utterly trivial consequence of Theorems 3.5 and 5.5, it should be noted that in pathological cases G_Σ and Σ^∞ may actually exhibit radically different behavior.

The model-consistency of G_Σ or Σ^∞ is the basis for any real theory of these classes, and it can be used to characterize Σ^∞ abstractly as the L-model-companion of Σ (cf. [1], sections 1 and 6). We give the following application of model-consistency as an example.

THEOREM 5.7. *If Σ is an elementary class of structures and Σ^∞ is model-consistent with Σ then Σ^∞ is closed under L-elementary substructure, i.e. if $m_1 \prec_L m_2$ and m_2 is in Σ^∞, then m_1 is in Σ^∞.*

Proof. It is evidently sufficient to apply the following to $\Gamma = \Sigma^n$ for $n = 0, 1, 2, \ldots$:

5.7.1. If Σ is elementary, $\Gamma \subseteq \Sigma$ is closed under L-elementary substructure, and Γ is model-consistent with Σ, then Γ' is closed under L-elementary substructure.

To prove 5.7.1, suppose $m_1 \prec_L m_2$ and m_2 is in Γ'. Since Σ is elementary, m_1 is in Σ. Let m_3 be any extension of m_1, and let u be a Γ-persistent sentence true in m_3. Then by a standard compactness argument a structure m_4 can be found in Σ such that

$$\begin{tikzcd}[row sep=small] & m_2 \arrow[dr, dashed] & \\ m_1 \arrow[ur] \arrow[dr] & & m_4 \\ & m_3 \arrow[ur, dashed] & \end{tikzcd} \quad \text{commutes}.$$

By the model-consistency of Γ, m_4 has an extension m_5 in Γ. The $m_3 \models u \Rightarrow m_5 \models u \Rightarrow m_2 \models u \Rightarrow m_1 \models u$. It follows that m_1 is in Γ'. ■

6. Examples and open questions. Motivated by various formulations of the notion of algebraic completeness in first order logic, we have described four classes of special structures: E_Σ, S_Σ, Σ', and Σ^∞. It is clear that for interesting classes Σ of algebraic structures, Σ' and Σ^∞ are extremely difficult to "compute". On the other hand E_Σ and S_Σ are fairly well-behaved, according to Lemma 3.3 and Theorem 4.4. We might also consider the class S'_Σ of S'_1-complete structures, but as yet we know of no example in which S_Σ and S'_Σ differ.

It should be possible to describe E_Σ and S_Σ explicitly for a number of interesting classes Σ. We give two simple examples:

EXAMPLE 6.1. If Σ is the class of all fields, then E_Σ is the class of algebraically closed fields (as in the first order case) while S_Σ is the class of uncountable algebraically closed fields.

Proof. If A is the class of algebraically closed fields and B is the class of uncountable algebraically closed fields, then $A \supseteq E_\Sigma \supseteq S_\Sigma \supseteq B$, the last inclusion following from Theorem 3.10. All fields in $A - B$ are countable; therefore, by Corollary 3.5, $A = E^{\mathrm{II}}_\Sigma$. Finally, to see that $(A - B) \cap S_\Sigma = \emptyset$, consider the formula

$$\exists X\big(\forall x \in X[x^2 \in x] \,\&\, \forall x, y \in X[x^2 = y^2 \Rightarrow x = y] \\ \&\, \exists x \in X \forall y \in X(y^2 \neq x)\big). \quad ■$$

EXAMPLE 6.2. If Σ is the class of ordered sets, then E_Σ is the class of densely ordered sets without first or last element such that no interval is small, while S_Σ is the class of uncountable saturated ordered sets.

Proof. The first statement is a straightforward consequence of Lemma 3.3. For the second part, since each saturated ordered set is homogeneous universal, each uncountable one is in S_Σ. It is obvious that each ordered set in S_Σ is saturated; to see that each is uncountable consider the formula:

$$\exists X \forall x \in X \exists y \in X (x < y) . \blacksquare$$

In Examples 6.1 and 6.2 E^{II}_Σ coincides with the class of structures in E^{I}_Σ in which all infinite definable subsets of $|m|$ have cardinality $\|m\|$. The next example shows that this last property is not a sufficient condition for member-ship in E^{II}_Σ.

EXAMPLE 6.3. Let Q, R be binary predicate symbols and let $E_1, E_2, \ldots$ be unary predicate symbols. Consider the following axiom scheme:

1. $\forall x, y \big(R(x, y) \Leftrightarrow Q(x, y) \,\&\, Q(y, x)\big)$.
2. R is an equivalence relation.
3. $\forall x, y, z, w \big(Q(x, y) \Rightarrow [Q(x, z) \,\&\, Q(w, y) \Leftrightarrow R(x, w) \,\&\, R(y, z)]\big)$.
4. $\forall x, y, z \big(Q(x, y) \,\&\, Q(y, z) \Rightarrow R(x, y) \vee R(y, z)\big)$.
5. For each integer n E_n is the unique equivalence class E with respect to R such that $|E| = n$.
6. $\forall x \exists y \big(Q(x, y) \vee Q(y, x) \,\&\, \neg R(x, y)\big)$.

These axioms can be formalized by a first order theory T. Let $\Sigma = \Sigma_T$. If m is a structure in Σ and m' is the graph obtained by reducing m modulo the equivalence relation R then each connected component of m' contains exactly two vertices and Q is asymmetric on m'.

Let m be the particular model defined as follows:

1. Let $A_1, A_2, \ldots, B_1, B_2, \ldots$ be mutually disjoint sets such that for each integer n $|A_n| = n$ and $|B_n| = \aleph_1$. Take $|m| = \bigcup\limits_n (A_n \cup B_n)$.

2. For each integer n and each x in $|m|$, $E_n(x)$ holds if and only if x is in A_n.

3. For x, y in $|m|$, $Q(x, y)$ holds if and only if for some integer n: $x, y \in A_n$; $x, y \in B_n$; $x \in A_n \,\&\, y \in B_n$ and n is odd; *or* $x \in B_n \,\&\, y \in A_n$ and n is even.

4. For x, y in $|m|$, $R(x, y)$ holds if and only if $Q(x, y) \,\&\, Q(y, x)$ holds.

Then m is in Σ. We claim:

m is in E^{I}_Σ, every infinite definable subset of m has cardinality $\|m\|$, and m is not in E^{II}_Σ.

Proof. Let D be the diagram of m, and let $c_1, c_2, \ldots$ be infinitely many distinct new constants. Define:

1. $U_1(y) =$ "$\exists x[Q(y, x) \,\&\, \neg R(x, y)]$";
$U_2(y) =$ "$\exists x[Q(x, y) \,\&\, \neg R(x, y)]$".

2. For $i = 1, 2$, let
$$S_i = \{U_i(c_1)\} \cup \{\forall x\big(E_n(x) \Rightarrow \neg Q(c_1, x) \,\&\, \neg Q(x, c_1)\big) \colon 1 \leqslant n < \infty\}.$$
3. For $i = 1, 2$, let
$$S'_i = \{U_i(c_j) \colon 1 \leqslant j < \infty\} \cup \cup \{\forall x\big(E_n(x) \Rightarrow \neg Q(c_j, x) \,\&\, \neg Q(x, c_j)\big) \colon 1 \leqslant j, n < \infty\}.$$
4. $T' = T \cup D$; for $i = 1, 2$, $T''_i = T' \cup S_i$ and $T'''_i = T' \cup S'_i$.

We will sketch the argument very briefly.

For $i = 1, 2$, T'''_i may be proved complete by a modification of Vaught's test involving the $\aleph_0$-categoricity of T'''_i relative to any finite set of constants in the vocabulary of T'''_i (T'''_i is not itself $\aleph_0$-categorical). It follows from this that T''_i is also complete, and then that T' is complete. Since T' is complete, therefore m is in E^{I}_Σ. Using the completeness of T'_i for $i = 1, 2$, it is not difficult to prove that $|m|$ has no small infinite definable subsets. Finally, note that the countable set $\bigcup\limits_n A_n$ meets $Q(x, y) \,\&\, R(x, y)$ in m but not in all extensions of m in Σ. $\blacksquare$

We close with two open questions:

PROBLEM 6.4. Is there an inductive elementary class Σ with the joint embedding property and amalgamation such that S_Σ contains a structure which is not homogeneous universal? In particular are there groups in S_Σ ($\Sigma =$ the class of all groups) which are not universal? (All such are homogeneous.)

PROBLEM 6.5. Can Theorem 4.13 be improved to yield a natural condition equivalent to:

"for every elementary inductive Σ, Σ' is model-consistent with Σ"?

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Recursiveness of initial segments of Kleene's O

by

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Abstract. It is shown that for any constructive ordinal $\alpha \geqslant \omega^2$, there are both recursive and nonrecursive initial segments of the partial ordering $<_0$ of Kleene's O which have order type α.

Let O be Kleene's set of notations for constructive ordinals and let $<_0$ be Kleene's partial ordering of O. If $a \in O$, let $O(a)$ denote $\{b: b <_0 a\}$. It is well known that for any $a \in O$, $O(a)$ is a recursively enumerable (r.e.) subset of O which is well-ordered by $<_0$ with order type $|a|$, the ordinal for which a is a notation. Our purpose here is to determine which constructive ordinals have notations a such that $O(a)$ is recursive (nonrecursive). We prove that every constructive ordinal has a notation a such that $O(a)$ is recursive and in fact that there is a Π_1^1 path P through O such that $O(a)$ is recursive for all a in P. In the other direction we show that the constructive ordinals which have notations a with $O(a)$ nonrecursive are exactly those which are $\geqslant \omega^2$. Our constructions in fact show that if $\alpha \geqslant \omega^2$ is a constructive ordinal and A is an infinite r.e. set (other than ω) then α has a notation a such that $O(a)$ is m-equivalent to A.

Most of our notation is standard. In particular, we use φ_e for the eth partial recursive function and call e an *index* of φ_e. We use the recursion theorem in the following informal style: in the definition of a partial recursive function φ_e, its index may be assumed known in advance. Of course such arguments are easily formalized. An index of a recursive set is any index of its characteristic function. A path through O is a subset of O which is linearly ordered by $<_0$ and contains a notation for each constructive ordinal.

Information on Kleene's O can be found in [1], [2], or [5]. In particular we shall need the binary recursive function $+_0$ which represents ordinal addition in the sense that $|a+_0 b| = |a|+|b|$ for $a, b \in O$. Also $+_0$

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