Upper and lower Lebesgue-Stieltjes integrals

by

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Abstract. Following Perron's method of introducing major and minor functions we have defined upper and lower integrals of a function defined on a closed interval and have shown that their coincidence gives rise to an integral completely equivalent to that of Lebesgue–Stieltjes. Some properties of upper and lower integrals are established and in particular it is shown that these are AC-ω on the interval. Finally it is shown that if the upper and lower integrals are not equal then the function is not ω-mesurable.

§ 1. Introduction. Let $\omega(x)$ be a non-decreasing function defined on the closed interval $[a, b]$. We extend the definition to all x by taking $\omega(x) = \omega(a)$ for $x < a$ and $\omega(x) = \omega(b)$ for $x > b$. Let S denote the set of points of continuity of $\omega(x)$ and $D = [a, b] - S$. Let S_0 denote the union of pairwise disjoint open intervals (a_i, b_i) in $[a, b]$ on each of which $\omega(x)$ is constant,

$$S_1 = \{a_1, b_1, a_2, b_2 \ldots\}, \qquad S_2 = SS_1 \quad \text{and} \quad S_3 = [a, b]S - (S_0 + S_2).$$

R. L. Jeffery [6] has denoted by $\mathfrak{U}$ the class of functions $f(x)$ defined as follows:

$f(x)$ is defined on the set $[a, b]\, S$ such that $f(x)$ is continuous at every point of $[a, b]S$ with respect to S. If $x_0 \in D$, $f(x)$ tends to a limit as x tends to x_0+ or x_0- over the points of S. These limits are denoted by $f(x_0+)$ and $f(x_0-)$ respectively. Also, $f(x) = f(a+)$ for $x < a$ and $f(x) = f(b-)$ for $x > b$. $f(x)$ may or may not be defined at the points of D.

Suppose $\mathfrak{U}_0 \subset \mathfrak{U}$ contains those functions $f(x)$ in $\mathfrak{U}$ such that for $x_0 \in D$ both $f(x_0+)$ and $f(x_0-)$ are finite.

DEFINITION 1.1. Let $f(x)$ be ω-measurable ([3], [6]) on a ω-measurable set $E \subset [a, b]$ and let $A < f(x) < B$ on E. Let

$$A = y_0 < y_1 < y_2 \ldots < y_n = B$$

be a subdivision of $[A, B]$, and

$$e_i = E \quad (y_i \leqslant f < y_{i+1}, \ i = 0, 1, \ldots, n-1).$$

The limit of $\sum_{i=0}^{n-1} y_i|e_i|_\omega$ as $\max|y_i-y_{i-1}|\to 0$ where $|e_i|_\omega$ denotes the ω-measure of e_i, is called the *Lebesgue–Stieltjes integral* (Def. 3, [6]) of $f(x)$ over E and is written as $\int_E f d\omega$.

This definition may be extended to unbounded functions in the usual way.

Some properties of Lebesgue–Stieltjes integrals are given in [6] and also in [3] and [4]. The purpose of the present paper is to introduce a new definition of the Lebesgue–Stieltjes integral by a modification of the procedure introduced by Perron, and also to study some properties of upper and lower Lebesgue–Stieltjes integrals defined in this paper.

We have denoted the upper and lower ω-derivates [6] of a function $f(x) \in \mathfrak{U}_0$ by $\overline{D}f_\omega(x)$ and $\underline{D}f_\omega(x)$ respectively, and the ω-derivative ([3], [6]) of $f(x)$ by $f'_\omega(x)$. The outer ω-measure of a set E is denoted by $\omega^*(E)$.

If a property P is satisfied at all points of a set A except a set of ω-measure zero, then it is said that P is satisfied ω-almost everywhere in A or at ω-almost all points of A.

We require the following known definitions and results:

Definition 1.2, [6]. A function $f(x)$ defined in $[a, b]$ and in class $\mathfrak{U}_0$ is *absolutely continuous relative to* ω, AC-ω, if for $\varepsilon > 0$ there exists $\delta > 0$ such that for any set of non-overlapping intervals (x_i, x'_i) on $[a, b]$ with $\sum_i \{\omega(x'_i+)-\omega(x_i-)\} < \delta$ the relation

$$\sum_i |f(x'_i+)-f(x_i-)| < \varepsilon$$

is satisfied.

Let

$$a \leqslant x_1 < x'_1 \leqslant x_2 < x'_2 \leqslant \ldots \leqslant x_n < x'_n \leqslant b$$

be any subdivision of $[a, b]$. Then the intervals

$$(x_1, x'_1), (x_2, x'_2), \ldots, (x_n, x'_n)$$

are said to form an elementary system [2] in $[a, b]$ which is denoted by

$$I = (x_i, x'_i) \qquad (i = 1, 2, \ldots, n).$$

Let

$$\sigma I = \sum_{i=1}^{n} \{f(x'_i+)-f(x_i-)\}, \qquad |I|_\omega = \sum_{i=1}^{n} \{\omega(x'_i+)-\omega(x_i-)\}.$$

Definition 1.3, [2]. A function $f(x)$ defined on $[a, b]$ and belonging to the class $\mathfrak{U}_0$ is said to be *absolutely continuous above relative to* ω, AC-ω *above*, if for $\varepsilon > 0$ there exists $\delta > 0$ such that for any elementary system I in $[a, b]$ with $|I|_\omega < \delta$ the relation $\sigma I < \varepsilon$ holds. It is said to be *absolutely continuous below relative to* ω, AC-ω *below*, if the relation $\sigma I > -\varepsilon$ holds whenever $|I|_\omega < \delta$.

Theorem 1.1 (cf. Th. 3, [2]). *Let $f(x)$ be defined on $[a, b]$ and be in class $\mathfrak{U}_0$. If $f(x)$ is either* AC-ω *below or* AC-ω *above on $[a, b]$, then $f(x)$ is* BV-ω, [1], *on $[a, b]$.*

Theorem 1.2 (Th. 2.1, [5]). *If $f(x) \in \mathfrak{U}_0$ be* AC-ω *below on $[a, b]$ and if $f'_\omega(x) \geqslant 0$ except on a set of ω-measure zero in $[a, b]$, then $f(x)$ is non-decreasing on $[a, b]S$.*

Theorem 1.3 (cf. Th. 2.5, [5]). *If $f(x) \in \mathfrak{U}_0$ be either* AC-ω *below or* AC-ω *above on $[a, b]$ and if $f(a+) = 0$, then*

$$f(x) = f_1(x)-f_2(x) \qquad \textit{for all } x \in [a, b]S,$$

where $f_1(x)$ and $f_2(x)$ are in class $\mathfrak{U}_0$ and are non-decreasing on $[a, b]$ and $f_1(a+) = f_2(a+) = 0$.

§ 2. Major and minor functions.

Definition 2.1. Let a function $f(x)$ be defined on the closed interval $[a, b]$. A function $M(x) \in \mathfrak{U}_0$ is said to be a *major function* of $f(x)$ on $[a, b]$ if

(a) $M(x)$ is AC-ω below on $[a, b]$,

(b) $M(a-) = 0$,

(c) $\underline{D}M_\omega(x) \geqslant f(x)$ ω-almost everywhere in S_3+D.

Similarly a function $m(x) \in \mathfrak{U}_0$ is said to be a *minor function* of $f(x)$ on $[a, b]$ if

(a′) $m(x)$ is AC-ω above on $[a, b]$,

(b′) $m(a-) = 0$,

(c′) $\overline{D}m_\omega(x) \leqslant f(x)$ ω-almost everywhere in S_3+D.

Lemma 2.1. *If $M(x)$ is a major function and $m(x)$ is a minor function of $f(x)$ on $[a, b]$, then the difference*

$$R(x) = M(x)-m(x)$$

is non-decreasing on $[a, b]S$ and in particular $M(b+) \geqslant m(b+)$.

Proof. By Theorem 1.1 we get that the functions $M(x)$ and $m(x)$ are BV-ω on $[a, b]$. So, it follows that (Th. 6.2, [3]) the ω-derivatives $M'_\omega(x)$ and $m'_\omega(x)$ exist finitely ω-almost everywhere in $[a, b]$. Hence for ω-almost all points in $[a, b]$

$$R'_\omega(x) = M'_\omega(x)-m'_\omega(x) = \underline{D}M_\omega(x)-\overline{D}m_\omega(x) \geqslant 0,$$

(by (c) and (c′)). Since $R(x)$ is AC-ω below on $[a, b]$, we get by Theorem 1.2 that $R(x)$ is non-decreasing on $[a, b]S$. So, if ξ and η are two points of $[a, b]S$ with $\xi < \eta$ we get

$$R(\eta) \geqslant R(\xi) .$$

Proceeding to the limit as $\xi \to a+$ and $\eta \to b-$ over S, we get

$$R(b-) \geqslant R(a+) = 0 .$$

So

$$M(b-) \geqslant m(b-) , \quad \text{or} \quad M(b+) \geqslant m(b+) .$$

§ 3. Upper and lower integral functions.

DEFINITION 3.1. Let the function $f(x)$ possesses major functions $M(x)$ on $[a, b]$. We define the function $U(x)$ by

$$U(x) = \begin{cases} 0 & \text{for} \quad x < a , \\ \inf\{M(x_0)\} & \text{for} \quad x = x_0 \in [a, b]S , \\ U(b-) & \text{for} \quad x > b . \end{cases}$$

Then $U(x)$ is said to be the *upper integral function* of $f(x)$ on $[a, b]$.

Similarly, if the function $f(x)$ possesses minor functions $m(x)$ and if we define the function $L(x)$ by

$$L(x) = \begin{cases} 0 & \text{for} \quad x < a , \\ \sup\{m(x_0)\} & \text{for} \quad x = x_0 \in [a, b]S , \\ L(b-) & \text{for} \quad x > b , \end{cases}$$

then $L(x)$ is said to be the *lower integral function* of $f(x)$ on $[a, b]$.

If $f(x)$ has a major function $M(x)$ and a minor function $m(x)$, it has both the upper and the lower integral functions and

$$m(x) \leqslant U(x) \leqslant M(x) ,$$

and

$$m(x) \leqslant L(x) \leqslant M(x)$$

for all $x \in [a, b]S$. Throughout the paper, we assume that both major and minor functions of $f(x)$ exist.

THEOREM 3.1. *Let the function $f(x)$ be defined on $[a, b]$. If $M(x)$ and $m(x)$ be respectively a major function and a minor function of $f(x)$ on $[a, b]$ and if $U(x)$ and $L(x)$ be the upper and the lower integral functions of $f(x)$, then each of the differences*

$$M(x) - U(x) \quad \textit{and} \quad L(x) - m(x)$$

is non-decreasing on $[a, b]S$.

Proof. Let x_1 and x_2 be any two points of continuity of $\omega(x)$ with $a \leqslant x_1 < x_2 \leqslant b$. Let $M_1(x)$ be a major function of $f(x)$ on $[a, b]$ such that

$$M_1(x_1) - U(x_1) < \varepsilon ,$$

where $\varepsilon > 0$ is chosen arbitrarily. Then the function $M_2(x)$ defined by

$$M_2(x) = \begin{cases} 0 & \text{for} \quad x < a , \\ M_1(x) & \text{for} \quad x \in [a, x_1]S , \\ M_1(x_1) + M(x) - M(x_1) & \text{for} \quad x \in [x_1, b]S , \\ M_2(b-) & \text{for} \quad x > b , \end{cases}$$

is also a major function of $f(x)$ on $[a, b]$. Hence

$$U(x_2) \leqslant M_2(x_2) = M_1(x_1) + M(x_2) - M(x_1)$$
$$< U(x_1) + \varepsilon + M(x_2) - M(x_1) ,$$

or

$$M(x_1) - U(x_1) < \{M(x_2) - U(x_2)\} + \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, it follows that $M(x) - U(x)$ is non-decreasing on $[a, b]S$.

The second difference may be treated analogously. This proves the theorem.

THEOREM 3.2. *Let the function $f(x)$ be defined on $[a, b]$. If $U(x)$, $L(x)$ be the upper and the lower integral functions of $f(x)$ on $[a, b]$, then $U(x) - L(x)$ is non-decreasing on $[a, b]S$.*

Proof. Suppose x_1, x_2 are any two points of $[a, b]S$ with $x_1 < x_2$. Let $M(x)$ be a major function and $m(x)$ be a minor function of $f(x)$ on $[a, b]$ such that

$$M(x_2) - U(x_2) < \tfrac{1}{2}\varepsilon , \quad L(x_2) - m(x_2) < \tfrac{1}{2}\varepsilon ,$$

where $\varepsilon > 0$ is chosen arbitrarily. From the definitions of upper and lower integral functions it follows that

$$U(x_1) - L(x_1) \leqslant M(x_1) - m(x_1)$$
$$\leqslant M(x_2) - m(x_2) \quad \text{(by Lemma 2.1)}$$
$$< U(x_2) - L(x_2) + \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$U(x_1) - L(x_1) \leqslant U(x_2) - L(x_2) ;$$

and so the difference $U(x) - L(x)$ is non-decreasing on $[a, b]S$.

THEOREM 3.3. *The upper integral function $U(x)$ associated to a function $f(x)$ on $[a, b]$ is in class $\mathfrak{U}_0$ and is* AC-ω *below on $[a, b]$, and the lower integral function $L(x)$ associated to $f(x)$ belongs to the class $\mathfrak{U}_0$ and is* AC-ω *above on $[a, b]$.*

Proof. We first show that $U(x)$ belongs to the class $\mathfrak{U}_0$. Let $\{\varepsilon_n\}$ be a sequence of positive constant terms such that $\varepsilon_n \to 0$ as $n \to \infty$, and let for each ε_n, $M_n(x)$ be a major function of $f(x)$ on $[a, b]$ such that for $x \in [a, b]S$

$$0 \leqslant M_n(x) - U(x) < \varepsilon_n .$$

Hence for $x \in [a, b]S$, $U(x)$ is the limit of a uniformly convergent sequence of functions every member of which belongs to class $\mathfrak{U}_0$ and so $U(x)$ is continuous on $[a, b]S$ and $U(x) \in \mathfrak{U}_0$.

Let $M(x)$ be a major function of $f(x)$ on $[a, b]$ such that

$$M(b-) - U(b-) < \tfrac{1}{2}\varepsilon \tag{1}$$

where $\varepsilon > 0$ is chosen arbitrarily. Since $M(x)$ is AC-ω below on $[a, b]$ there exists a $\delta > 0$ such that for any elementary system $I = \{(x_i, x_i')\}$ on $[a, b]$ with $|I|_\omega < \delta$

$$\sum_i \{M(x_i'+) - M(x_i-)\} > -\tfrac{1}{2}\varepsilon . \tag{2}$$

Then

$$\sum_i \{U(x_i'+) - U(x_i-)\}$$
$$= \sum_i \{M(x_i'+) - M(x_i-)\} - \sum_i [\{M(x_i'+) - U(x_i'+)\} - \{M(x_i-) - U(x_i-)\}]$$
$$> \sum_i \{M(x_i'+) - M(x_i-)\} - \tfrac{1}{2}\varepsilon \quad \text{(by (1) and Th. 3.1)}$$
$$> -\varepsilon \quad \text{(by (2))} .$$

Hence $U(x)$ is AC-ω below on $[a, b]$. The second part of the theorem can be proved analogously.

Note 3.1. If $f(x)$ is bounded on $[a, b]$, then its upper and lower integral functions are AC-ω on $[a, b]$. For if

$$A \leqslant f(x) \leqslant B , \quad x \in [a, b] ,$$

then $A\{\omega(x) - \omega(a-)\}$ and $B\{\omega(x) - \omega(a-)\}$ are respectively a minor and a major function of $f(x)$ on $[a, b]$. Let $U(x)$ be the upper integral function of $f(x)$. By Theorem 3.1 we get for $x \in [a, b]S$

$$U(x) = B\{\omega(x) - \omega(a-)\} + \pi(x) , \tag{3}$$

where $\pi(x) \in \mathfrak{U}_0$ and is non-increasing on $[a, b]S$. Then from (3) it follows that $U(x)$ is AC-ω above on $[a, b]$ and so by Theorem 3.3, $U(x)$ is AC-ω on $[a, b]$.

In a similar way it can be shown that the lower integral function is also AC-ω on $[a, b]$.

THEOREM 3.4. *The upper integral function is a major function and the lower integral function is a minor function.*

Proof. We consider the upper integral function in detail, the proof for the lower integral function being similar.

Let $U(x)$ be the upper integral function of $f(x)$ on $[a, b]$. We show that

$$\underline{D}U_\omega(x) \geqslant f(x) \quad \omega\text{-almost everywhere in } [a, b] . \tag{4}$$

Assume on the contrary that

$$f(x) - \underline{D}U_\omega(x) > 0$$

on a set of positive outer ω-measure. Since $U'_\omega(x)$ exists finitely ω-almost everywhere in $[a, b]$, there exists a positive number ε such that

$$f(x) - U'_\omega(x) > \varepsilon$$

at the points of a set $E \subset S_3 + D$ where

$$\omega^*(E) > 0 .$$

We first assume that

$$\omega^*(ES_3) = p > 0 .$$

Let $M(x)$ be a major function of $f(x)$ such that

$$M(b-) - U(b-) < \tfrac{1}{4}p\varepsilon . \tag{5}$$

Put

$$R(x) = M(x) - U(x) .$$

Then $R(x)$ is in class $\mathfrak{U}_0$ and non-decreasing on $[a, b]S$ and

$$R'_\omega(x) = M'_\omega(x) - U'_\omega(x)$$

ω-almost everywhere on $[a, b]$. The set $E_1 \subset S_3$ where

$$M'_\omega(x) - U'_\omega(x) > \varepsilon$$

is ω-measurable and E_1 contains ES_3 with the possible exception of a set of ω-measure zero. Hence

$$|E_1|_\omega \geqslant p .$$

Let $P \subset E_1$ be a closed set with

$$|P|_\omega > \tfrac{1}{2}p .$$

Let $[\alpha, \beta]$ be the smallest closed interval containing P. If $x \in P$, there exists a null sequence $\{h_i\}$ $(h_i > 0, x+h_i \in S)$ such that

$$R(x+h_i) - R(x) > \varepsilon\{\omega(x+h_i) - \omega(x)\} . \tag{6}$$

Let F denote the family of closed intervals $[x, x+h_i]$ thus associated with the set P. Then by Theorem 1.1 [3] there exists a finite family of pairwise disjoint closed intervals $\Delta_1, \Delta_2, \ldots, \Delta_n$ of F for which

$$\sum_{i=1}^{n} |\Delta_i P|_\omega > |P|_\omega - \tfrac{1}{4}p . \tag{7}$$

Write

$$\Delta_i = [x_i, x_i + k_i] \quad (i = 1, 2, \ldots, n) .$$

We may suppose that

$$x_1 < x_2 < \ldots < x_n \quad \text{and} \quad x_1 = \alpha , \quad x_n + k_n = \beta .$$

Then

$$x_i + k_i < x_{i+1} \quad (i = 1, 2, \ldots, n-1) .$$

We have

$$\begin{aligned} R(\beta) - R(\alpha) &\geqslant \sum_{i=1}^{n} \{R(x_i + k_i) - R(x_i)\} \\ &> \varepsilon(|P|_\omega - \tfrac{1}{4}p) \quad (\text{by (6) and (7)}) , \\ &> \tfrac{1}{4}p\varepsilon , \end{aligned}$$

which contradicts (5) because $R(x)$ is non-negative and non-decreasing on $[a, b]S$.

We arrive at a similar contradiction if we assume that ED is different from a null set. The proof of the relation (4) is now complete. Obviously, $U(a-) = 0$. Hence by Theorem 3.3, it follows that $U(x)$ is a major function of $f(x)$ on $[a, b]$.

§ 4. Modified Perron-Stieltjes integral and Lebesgue-Stieltjes integral. If $f(x)$ has major and minor functions and if $L(x)$ and $U(x)$ are its lower and upper integral functions on $[a, b]$, then $L(b+)$ and $U(b+)$ are finite and $L(b+) \leqslant U(b+)$. If

$$L(b+) = U(b+) ,$$

we say that $f(x)$ has *modified Perron-Stieltjes integral* on $[a, b]$ which is

equal to this common value. The integral is denoted by

$$(\text{MPS}) \int_a^b f(x)\, d\omega .$$

More generally, we say that $L(b+)$ and $U(b+)$ are respectively the *lower* and *upper* MPS-integrals of $f(x)$ on the closed interval $[a, b]$. We observe that a necessary and a sufficient condition that $f(x)$ should be MPS-integrable on $[a, b]$ is that to every $\varepsilon > 0$ there correspond a major function $M(x)$ and a minor function $m(x)$ such that

$$M(b+) - m(b+) < \varepsilon .$$

If the MPS-integral exists on $[a, b]$, it exists on every closed subinterval of $[a, b]$. If $f(x)$ is MPS-integrable on $[a, b]$, then the function $G(x)$ defined by

$$G(x) = \begin{cases} 0 & \text{for} \quad x < a , \\ (\text{MPS}) \int_a^x f(t)\, d\omega & \text{for} \quad x \in [a, b] , \\ G(b-) & \text{for} \quad x > b \end{cases}$$

is called the *indefinite* MPS-*integral* of $f(x)$. In the next two theorems we show that the MPS-integral is identical with the Lebesgue–Stieltjes integral.

THEOREM 4.1. *If $f(x)$ is summable* (LS) *on $[a, b]$, then $f(x)$ is* MPS-*integrable on $[a, b]$ and*

$$(\text{MPS}) \int_a^b f(x)\, d\omega = (\text{LS}) \int_a^b f(x)\, d\omega .$$

Proof. Write

$$F(x) = \begin{cases} 0 & \text{for } x < a , \\ (\text{LS}) \int_a^x f(t)\, d\omega & \text{for} \quad x \in [a, b] , \\ F(b-) & \text{for} \quad x > b . \end{cases}$$

Then $F(x)$ is in class $\mathfrak{U}_0$ and is AC-ω on $[a, b]$, and

$$F'_\omega(x) = f(x)$$

ω-almost everywhere in $[a, b]$ (cf. Section 2, [6]). Thus $F(x)$ is both a major and a minor function of $f(x)$ on $[a, b]$ and so it is also the upper and lower integral function of $f(x)$. Hence $f(x)$ is MPS-integrable on

$[a, b]$ and

$$F(b+) = (\mathrm{MPS}) \int_a^b f(x)\,d\omega\,.$$

So

$$(\mathrm{MPS}) \int_a^b f(x)\,d\omega = F(b+) = (\mathrm{LS}) \int_a^b f(x)\,d\omega \qquad \text{(cf. Th. 3.1, [4])}\,.$$

THEOREM 4.2. *If $f(x)$ is* MPS-*integrable on $[a, b]$, then $f(x)$ is summable* (LS) *on $[a, b]$ and the integrals in two senses are equal.*

Proof. Since $f(x)$ is MPS-integrable on $[a, b]$, the upper and lower integral functions associated to $f(x)$ coincide at points of continuity of $\omega(x)$. Hence if we write

$$G(x) = \begin{cases} 0 & \text{for} \quad x < a\,, \\ (\mathrm{MPS}) \int_a^x f(t)\,d\omega & \text{for} \quad x \in [a, b]\,, \\ G(b-) & \text{for} \quad x > b\,, \end{cases}$$

then

$$G(x) = L(x) = U(x) \qquad \text{for} \qquad x \in [a, b]S\,.$$

So, $G(x)$ is in class $\mathfrak{U}_0$ and is AC-ω on $[a, b]$. Further, by Theorem 3.4 we get

$$G'_\omega(x) = f(x)$$

ω-almost everywhere on $[a, b]$. By Theorem 1.1, $G(x)$ is BV-ω on $[a, b]$ and so $G'_\omega(x)$ is summable (LS) (Th. 6.3, [3]) on $[a, b]$. Hence $f(x)$ is also summable (LS) on $[a, b]$. Write

$$F(x) = \begin{cases} 0 & \text{for} \quad x < a\,, \\ (\mathrm{LS}) \int_a^x f(t)\,d\omega & \text{for} \quad x \in [a, b]\,, \\ F(b-) & \text{for} \quad x > b\,. \end{cases}$$

Then

$$F'_\omega(x) = f(x)$$

ω-almost everywhere on $[a, b]$. Hence by Theorem 1 [6],

$$F(x) - G(x) = k \qquad \text{a constant on } S\,.$$

Letting $x \to a+$ over the points of S we get $K = 0$. So

$$F(x) = G(x)\,, \qquad x \in S\,.$$

Therefore

$$(\mathrm{LS}) \int_a^b f(x)\,d\omega = F(b+) = G(b+) = (\mathrm{MPS}) \int_a^b f(x)\,d\omega\,.$$

This completes the proof.

The modified Perron definition helps in establishing the well known properties of Lebesgue–Stieltjes integral very simply although we are not carrying out the details here. When the upper and lower integral functions $U(x)$ and $L(x)$ are such that

$$U(b+) \neq L(b+)\,,$$

it is then natural in view of the establishment of the identity of MPS-integral and Lebesgue–Stieltjes integral, that we denote $U(b+)$ and $L(b+)$ as the upper and lower Lebesgue–Stieltjes integrals of $f(x)$ on $[a, b]$ with the notations

$$U(b+) = \int_a^{-b} f(x)\,d\omega\,, \qquad L(b+) = \int_{-a}^{b} f(x)\,d\omega\,.$$

§ 5. Upper and lower Lebesgue-Stieltjes integrals. We now establish certain properties of the upper and lower LS-integrals.

THEOREM 5.1. *If the function $f(x)$ has upper and lower* LS-*integrals on $[a, b]$, then the function $|f(x)|$ has also upper and lower* LS-*integrals on $[a, b]$.*

Proof. It is sufficient to show that the function $|f(x)|$ has major and minor functions on $[a, b]$. Let $U(x)$ and $L(x)$ be the upper and lower integral functions of $f(x)$ on $[a, b]$ so that $U(x)$ is AC-ω below on $[a, b]$ and $L(x)$ is AC-ω above on $[a, b]$.

Then by Theorem 1.3, $U(x)$ and $L(x)$ can be expressed in the forms

$$\begin{aligned} U(x) &= \alpha(x) - \beta(x)\,, \\ L(x) &= \theta(x) - \varphi(x)\,, \end{aligned} \qquad x \in [a, b]S\,,$$

where each of the functions $\alpha(x)$, $\beta(x)$, $\theta(x)$ and $\varphi(x)$ is in class $\mathfrak{U}_0$ and non-decreasing on $[a, b]$, and $\alpha(a+) = \beta(a+) = \theta(a+) = \varphi(a+) = 0$. Then for ω-almost all points in $[a, b]$ we get

(8) $$\alpha'_\omega(x) \geqslant U'_\omega(x) \geqslant f(x)\,,$$

and

(9) $$\varphi'_\omega(x) \geqslant -L'_\omega(x) \geqslant -f(x)\,.$$

Write

$$M(x) = \alpha(x) + \varphi(x)\,.$$

Then from (8) and (9) we get

$$\underline{D}M_\omega(x) \geqslant |f(x)|$$

for ω-almost all points in $[a, b]$. Since $M(x)$ is AC-ω below on $[a, b]$, it now easily follows that $M(x)$ is a major function for $|f(x)|$ on $[a, b]$.

The function $m(x) = 0$ is a minor function for $|f(x)|$ on $[a, b]$. Hence $|f(x)|$ has upper and lower LS-integrals.

THEOREM 5.2. *If $f_1(x)$ and $f_2(x)$ have upper and lower* LS-*integrals on* $[a, b]$ *and*

$$f(x) = f_1(x) + f_2(x),$$

then $f(x)$ has upper and lower LS-*integrals on $[a, b]$ and*

$$\overline{\int_a^b} f(x)\,d\omega \leqslant \overline{\int_a^b} f_1(x)\,d\omega + \overline{\int_a^b} f_2(x)\,d\omega, \tag{10}$$

$$\underline{\int_a^b} f(x)\,d\omega \geqslant \underline{\int_a^b} f_1(x)\,d\omega + \underline{\int_a^b} f_2(x)\,d\omega. \tag{11}$$

If one of the functions $f_1(x)$ and $f_2(x)$ is LS-*integrable on $[a, b]$, then each of the inequalities* (10), (11) *becomes an equality.*

Proof. Let $U_1(x)$ and $U_2(x)$ be the respective upper integral functions of $f_1(x)$ and $f_2(x)$ on $[a, b]$. Write

$$M(x) = U_1(x) + U_2(x).$$

Then the function $M(x)$ is in class $\mathfrak{U}_0$ and is AC-ω below on $[a, b]$ and $M(a-) = 0$. Also

$$\underline{D}M_\omega(x) \geqslant \underline{D}U_{1\omega}(x) + \underline{D}U_{2\omega}(x) \geqslant f_1(x) + f_2(x)$$

at ω-almost all points in $[a, b]$. Hence $M(x)$ serves as a major function for $f(x)$ on $[a, b]$. Similarly we can show that the function

$$m(x) = L_1(x) + L_2(x)$$

where $L_1(x)$ and $L_2(x)$ are the lower integral functions of $f_1(x)$ and $f_2(x)$ respectively is a minor function of $f(x)$ on $[a, b]$. Therefore, the function $f(x)$ has upper and lower LS-integrals on $[a, b]$ and

$$\begin{aligned} &\overline{\int_a^b} f(x)\,d\omega \leqslant \overline{\int_a^b} f_1(x)\,d\omega + \overline{\int_a^b} f_2(x)\,d\omega, \\ &\underline{\int_a^b} f(x)\,d\omega \geqslant \underline{\int_a^b} f_1(x)\,d\omega + \underline{\int_a^b} f_2(x)\,d\omega. \end{aligned} \tag{12}$$

Now suppose that $f_1(x)$ is LS-integrable on $[a, b]$ so that

$$U_1(x) = L_1(x), \quad x \in [a, b]S.$$

Hence $U_1(x)$ is AC-ω on $[a, b]$ and

$$U'_{1\omega}(x) = f_1(x)$$

ω-almost everywhere in $[a, b]$. Denote the upper integral function of $f(x)$ by $U(x)$ and write

$$M(x) = U(x) - U_1(x).$$

Then $M(x)$ is AC-ω below on $[a, b]$ and for ω-almost all points in $[a, b]$

$$M'_\omega(x) = U'_\omega(x) - U'_{1\omega}(x) \geqslant \{f_1(x) + f_2(x)\} - f_1(x) = f_2(x).$$

It follows that $M(x)$ is a major function of $f_2(x)$ on $[a, b]$ and so

$$\overline{\int_a^b} f_2(x)\,d\omega \leqslant \overline{\int_a^b} f(x)\,d\omega - \overline{\int_a^b} f_1(x)\,d\omega. \tag{13}$$

Combining (12) and (13) we get

$$\overline{\int_a^b} f(x)\,d\omega = \overline{\int_a^b} f_1(x)\,d\omega + \overline{\int_a^b} f_2(x)\,d\omega.$$

The proof in the case of lower integral functions is similar.

For the next theorem we require the following notations:

For an unbounded function $f(x)$ defined on $[a, b]$ we introduce the functions $f_n(x)$ and $f_{-n}(x)$ for every natural number n by the rules

$$f_n(x) = \min\{f(x), n\} \quad \text{and} \quad f_{-n}(x) = \max\{f(x), -n\}.$$

THEOREM 5.3. *If the unbounded function $f(x)$ has upper and lower* LS-*integrals on $[a, b]$, then its upper and lower integral functions are* AC-ω *on $[a, b]$ and*

$$\overline{\int_a^b} f(x)\,d\omega = \lim_{n\to\infty} \overline{\int_a^b} f_n(x)\,d\omega = \lim_{n\to\infty} \overline{\int_a^b} f_{-n}(x)\,d\omega,$$

and

$$\underline{\int_a^b} f(x)\,d\omega = \lim_{n\to\infty} \underline{\int_a^b} f_n(x)\,d\omega = \lim_{n\to\infty} \underline{\int_a^b} f_{-n}(x)\,d\omega.$$

Proof. Let $U(x)$ and $L(x)$ be the upper and lower integral function of $f(x)$ on $[a, b]$.

Case (i). We first consider the case when $f(x)$ is bounded below on $[a, b]$. Then the function $f_n(x)$ is bounded on $[a, b]$ and so it has upper integral function $U_n(x)$ which is AC-ω on $[a, b]$. Now consider the sequence of functions

$$\{U_n(x)\}\,, \quad x \in [a, b]S\,.$$

Evidently, the sequence is monotone increasing for $U_{n+1}(x)$ being a major function of $f_n(x)$ for every n,

$$U_{n+1}(x) - U_n(x) \geqslant 0$$

for every $x \in [a, b]S$ and for every value of n. Again, the function $U(x)$ serves as a major function for $f_n(x)$ for every positive integral value of n, and so

$$U(x) - U_n(x) \geqslant 0$$

for every $x \in [a, b]S$ and for every value of n. Let

$$\lim_{n\to\infty} U_n(x) = X(x)\,.$$

Then

(14) $$X(x) \leqslant U(x)\,, \quad x \in [a, b]S\,.$$

Since the difference

$$U_{n+p}(x) - U_n(x)$$

is non-decreasing on $[a, b]S$ for each positive integral value of n and p, it follows that the function

$$V_n(x) = X(x) - U_n(x)$$

is non-decreasing on $[a, b]S$. In fact if x_1 and x_2 $(> x_1)$ are any two point, of $[a, b]S$ and if p be a positive integer such that

$$X(x_1) < U_{n+p}(x_1) + \varepsilon$$

where $\varepsilon > 0$ is arbitrarily small, then

$$V_n(x_1) = X(x_1) - U_n(x_1) < U_{n+p}(x_1) - U_n(x_1) + \varepsilon$$
$$\leqslant U_{n+p}(x_2) - U_n(x_2) + \varepsilon \leqslant X(x_2) - U_n(x_2) + \varepsilon\,.$$

Since $\varepsilon > 0$ is arbitrary,

$$V_n(x_1) \leqslant V_n(x_2)\,.$$

It follows that the convergence of the sequence $\{U_n(x)\}$ to $X(x)$ is uniform on $[a, b]S$. Hence $X(x) \in \mathfrak{U}_0$ and so $V_n(x)$ is in class $\mathfrak{U}_0$ and is AC-ω below on $[a, b]$. So $X(x)$ is AC-ω below on $[a, b]$. Arguing in a way similar

to that used in the proof of Theorem 3.3, we can show that $X(x)$ is also AC-ω above on $[a, b]$. Now, for ω-almost all points in $[a, b]$

$$X'_\omega(x) \geqslant U'_{n_\omega}(x) \geqslant f_n(x)\,.$$

Since this relation holds for all values of n, we get

$$X'_\omega(x) \geqslant f(x)$$

ω-almost everywhere on $[a, b]$. Hence $X(x)$ is a major function of $f(x)$ which implies that the sign of equality alone is admissible in (14). So the function $U(x)$ is AC-ω on $[a, b]$ and

$$U(x) = \lim_{n\to\infty} U_n(x)\,, \quad x \in [a, b]S\,.$$

Then

(15) $$\int_a^{-b} f(x)\,d\omega = \lim_{n\to\infty} \int_a^{-b} f_n(x)\,d\omega\,.$$

If we consider the sequence $\{L_n(x)\}$ of lower integral functions of $f_n(x)$ and proceed in an analogous way, it will follow that the lower integral function $L(x)$ of $f(x)$ is AC-ω on $[a, b]$ and

(16) $$\int_{-a}^{b} f(x)\,d\omega = \lim_{n\to\infty} \int_{-a}^{b} f_n(x)\,d\omega\,.$$

Case (ii). We now suppose that $f(x)$ is bounded above on $[a, b]$. Since the upper and lower integral functions of the function $-f(x)$ are respectively the negatives of the lower and upper integral functions of $f(x)$, if we proceed with $-f(x)$ as in case (i), we will get that both $U(x)$ and $L(x)$ are AC-ω on $[a, b]$ and

(17) $$\int_a^{-b} f(x)\,d\omega = \lim_{n\to\infty} \int_a^{-b} f_{-n}(x)\,d\omega\,,$$

and

(18) $$\int_{-a}^{b} f(x)\,d\omega = \lim_{n\to\infty} \int_{-a}^{b} f_{-n}(x)\,d\omega\,.$$

Case (iii). We now consider the case when $f(x)$ is neither bounded below nor bounded above on $[a, b]$.

Then $f_n(x)$ is bounded above on $[a, b]$. Since $U(x)$ serves as a major function for $f_n(x)$ and the upper integral function of $|f(x)|$ is a major function for $-f_n(x)$, it follows that the function $f_n(x)$ has upper and lower integral functions, and by case (ii), they are AC-ω on $[a, b]$. If we now proceed as in case (i) we will similarly get that the functions $U(x)$ and $L(x)$ are AC-ω on $[a, b]$ and that the relations (15) and (16) hold in this case also. If $-f(x)$ is considered in place of $f(x)$ we again obtain the same property for $U(x)$ and $L(x)$ and the same relations (17) and (18) as in case (ii). This completes the proof of the theorem.

A function $f(x)$ can be represented by

$$f(x) = f_+(x) - f_-(x)$$

where $f_+(x)$ and $f_-(x)$ are two non-negative functions defined by the rules

$$f_+(x) = \max\{f(x), 0\} \quad \text{and} \quad f_-(x) = \max\{-f(x), 0\}.$$

Now, we prove the following theorem which is more precise than Theorem 5.2.

THEOREM 5.4. *If the function $f(x)$ has upper and lower* LS-*integrals on $[a, b]$, then the positive part $f_+(x)$ and negative part $f_-(x)$ of $f(x)$ have upper and lower* LS-*integrals on $[a, b]$ and*

$$\int_a^{-b} f(x)\,d\omega = \int_a^{-b} f_+(x)\,d\omega - \int_{-a}^{b} f_-(x)\,d\omega,$$

$$\int_{-a}^{b} f(x)\,d\omega = \int_{-a}^{b} f_+(x)\,d\omega - \int_a^{-b} f_-(x)\,d\omega.$$

Proof. By Theorem 5.1, the function $|f(x)|$ has upper and lower LS-integrals on $[a, b]$ and since

$$|f(x)| = f_+(x) + f_-(x),$$

the upper integral function of $|f(x)|$ is a major function of both $f_+(x)$ and $f_-(x)$. Further $m(x) = 0$ is a minor function of both $f_+(x)$ and $f_-(x)$. Hence both the functions $f_+(x)$ and $f_-(x)$ have upper and lower LS-integrals on $[a, b]$.

Let $U(x)$ and $L(x)$ be respectively the upper and lower integral functions of $f(x)$ on $[a, b]$. By Theorem 5.3, both $U(x)$ and $L(x)$ are AC-ω on $[a, b]$ and so by Theorem 3.3 [4] these functions can be represented in the forms

$$U(x) = \alpha(x) - \beta(x), \quad L(x) = \theta(x) - \varphi(x), \qquad x \in [a, b]S,$$

where each of the functions $\alpha(x)$, $\beta(x)$, $\theta(x)$ and $\varphi(x)$ is in class $\mathfrak{U}_0$ and is AC-ω and non-decreasing on $[a, b]$, and

$$\alpha(a+) = \beta(a+) = \theta(a+) = \varphi(a+) = 0.$$

Denote $U'_\omega(x)$ by $u(x)$, and the positive and negative parts of $u(x)$ by $u_+(x)$ and $u_-(x)$ respectively. Then we get (Theorems 3.1 and 3.4 [4])

$$U(x) = \int_a^x u(t)\,d\omega,$$

$$\alpha(x) = \int_a^x u_+(t)\,d\omega, \qquad x \in [a, b]S.$$

$$\beta(x) = \int_a^x u_-(t)\,d\omega,$$

Then

$$f(x) \leqslant U'_\omega(x) = u(x)$$

ω-almost everywhere in $[a, b]$; and so

$$f_+(x) \leqslant u_+(x), \quad f_-(x) \geqslant u_-(x)$$

ω-almost everywhere in $[a, b]$. Hence

(19) $$f_+(x) \leqslant \alpha'_\omega(x), \quad f_-(x) \geqslant \beta'_\omega(x)$$

ω-almost everywhere in $[a, b]$. Evidently, $\alpha(x)$ is a major function of $f_+(x)$ on $[a, b]$ and we have

(20) $$\alpha(b+) \geqslant \int_a^{-b} f_+(x)\,d\omega.$$

Similarly

(21) $$\beta(b+) \leqslant \int_{-a}^{b} f_-(x)\,d\omega.$$

Suppose the sign of equality of (20) does not hold and let

$$N(x) = \begin{cases} 0 & \text{for} \quad x < a, \\ \int_a^{-x} f_+(t)\,d\omega & \text{for} \quad a \leqslant x \leqslant b, \\ N(b-) & \text{for} \quad x > b, \end{cases}$$

and

$$M(x) = N(x) - \beta(x).$$

Then letting $x \to b-$ over S we get

$$M(b-) = N(b-) - \beta(b-) = N(b+) - \beta(b-)$$
$$= \int_a^{-b} f_+(x)\,d\omega - \beta(b-) < \alpha(b-) - \beta(b-),$$

or

$$M(b-) < U(b-) . \tag{22}$$

Now, $M(x)$ belongs to the class $\mathfrak{U}_0$ and is AC-ω on $[a, b]$ and $M(a-) = 0$. Also for ω-almost all points in $[a, b]$

$$M'_\omega(x) \geqslant f_+(x) - \beta'_\omega(x) \geqslant f_+(x) - f_-(x) \quad \text{(by (19))} .$$

Thus, $M(x)$ is a major function of $f(x)$ on $[a, b]$ which contradicts (22). Hence

$$\alpha(b+) = \int_a^{-b} f_+(x)\,d\omega .$$

Similarly we can show

$$\beta(b+) = \int_{-a}^{b} f_-(x)\,d\omega ,$$

$$\theta(b+) = \int_{-a}^{b} f_+(x)\,d\omega , \quad \text{and}$$

$$\varphi(b+) = \int_a^{-b} f_-(x)\,d\omega .$$

Therefore

$$\int_a^{-b} f(x)\,d\omega = \int_a^{-b} f_+(x)\,d\omega - \int_{-a}^{b} f_-(x)\,d\omega ,$$

$$\int_{-a}^{b} f(x)\,d\omega = \int_{-a}^{b} f_+(x)\,d\omega - \int_a^{-b} f_-(x)\,d\omega .$$

THEOREM 5.5. *If the upper and lower* LS-*integrals of a function $f(x)$ on $[a, b]$ are not equal, then $f(x)$ is not ω-measurable on $[a, b]$.*

Proof. We first consider the case when $f(x)$ is non-negative on $[a, b]$. If possible, let $f(x)$ be ω-measurable on $[a, b]$. Then the functions $f_n(x)$ defined by

$$f_n(x) = \min\{f(x), n\} , \quad n = 1, 2, \ldots ,$$

are bounded and ω-measurable on $[a, b]$. So, (LS) $\int_a^b f_n(x)\,d\omega$ exists for all n. Since $f(x)$ is not summable (LS) on $[a, b]$, we get

$$\lim_{n\to\infty} \int_a^b f_n(x)\,d\omega = +\infty . \tag{23}$$

The function

$$m_n(x) = (\mathrm{LS}) \int_a^x f_n(t)\,d\omega , \quad x \in [a, b]$$

belongs, for every fixed n to class $\mathfrak{U}_0$ and is AC-ω on $[a, b]$. Further,

$$m_n(a-) = 0 \quad \text{and} \quad m'_{n\omega}(x) = f_n(x) \leqslant f(x)$$

ω-almost everywhere on $[a, b]$. Thus, $m_n(x)$ serves as a minor function for $f(x)$ on $[a, b]$ and so for all values of n

$$\int_a^b f_n(x)\,d\omega \leqslant \int_{-a}^{b} f(x)\,d\omega . \tag{24}$$

From (23) and (24) we get

$$\int_{-a}^{b} f(x)\,d\omega = \infty$$

which contradicts the fact that the lower integral must be a finite number. Hence $f(x)$ is not ω-measurable on $[a, b]$.

Now let $f(x)$ be a function without any restriction on its sign and let $f_+(x)$ and $f_-(x)$ be the positive and negative parts of $f(x)$. By Theorem 5.4, at least one of the functions f_+ and f_- has upper and lower LS-integrals on $[a, b]$ which are not equal. Since both the functions f_+ and f_- are non-negative, at least one of them is not ω-measurable. Therefore $f(x)$ is not ω-measurable on $[a, b]$. This completes the proof of the theorem.

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Second order forcing, algebraically closed structures, and large cardinals

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Abstract. We investigate four generalizations of the notion "algebraically closed" to a model theoretic context involving weak second order logic. Whereas in first order logic the existence of algebraically closed structures of various sorts is proved by a natural transfinite induction, in our context it is necessary to assume the existence of large cardinals in order to prove the corresponding existence theorems for structures. The present article is devoted to a relatively precise description of the relationship between large cardinals and structures of the special types alluded to. In the final section some examples are discussed.

1. Introduction. If Σ is a class of similar structures, several inequivalent formulations of the notion of an "algebraically closed" structure relative to the class Σ and the appropriate first order language have been studied [4, 11, 13]. The concepts introduced for this purpose may be extended to more powerful languages, such as modal, higher order, or infinitary languages; a detailed treatment of some aspects of the last case is given in [3].

We wish to discuss the algebraically closed structures relative to a second order language L which permits quantification over all non-empty subsets S of the domain of a given structure m such that the cardinality of S is less than the cardinality of the domain of m and to find necessary and sufficient conditions for the existence of such structures. For example, concerning the analogue of Robinson's class G_Σ of the *infinitely generic* structures [10] we will prove:

THEOREM. *The following are equivalent*:

1. *Every increasing function from the ordinals to the ordinals which is continuous at limit ordinals has a regular fixed point.*

2. *For any class Σ of similar structures which is inductive* (*i.e. closed under unions of chains*). *G_Σ is model-consistent with Σ* (*i.e. if m is in Σ then m has an extension m' in G_Σ*).

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