Fixed point theorems in topological spaces

by

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Abstract. Let $T: X \to X$ be a mapping of a topological space X into itself. T is called a *strongly non-periodic mapping* if $x \neq Tx$ implies $x \bar{\in} \mathrm{cl}\{T^2x, T^3x, \ldots\}$. In the present paper are given conditions in metric and topological spaces under which these mapping have fixed points.

1. Let X be a topological space and $T: X \to X$ a mapping. A point $u \in X$ is called a *fixed point* under T if $Tu = u$. A point $x \in X$ is called a *periodic point* under T if there exists a positive integer $k \geqslant 2$ such that $T^k x = x$, i.e., if $x \in \{T^2x, T^3x, \ldots\}$. For $x \in X$, let

$$O(T^n x) = \{T^n x, T^{n+1}x, \ldots\}, \quad n = 0, 1, 2, \ldots$$

(where it is understood that $T^0 x = x$) and $\bar{O}(T^n x) = \mathrm{cl}\{T^n x, T^{n+1}x, \ldots\}$. Now we will introduce a notion of a strongly non-periodic mapping.

DEFINITION 1. A mapping $T: X \to X$ is called *strongly non periodic* iff for every $x \in X$

$$x \neq Tx \quad \text{implies} \quad x \notin \bar{O}(T^2 x).$$

Let T be a mapping of a metric space M into itself. In [1] an orbitally continuous mapping and a T-orbitally complete space are defined as follows. A mapping T is said to be *orbitally continuous* if $u, x \in M$ are such that $u = \lim_i T^{n_i}x$ then $Tu = \lim_i TT^{n_i}x$. A space M is said to be *T-orbitally complete* if every Cauchy sequence of the form $\{T^{n_i}x: i \in N\}$ converges in M. Now the corresponding concept for orbitally continuous mappings in a topological space X will be given.

DEFINITION 2. A mapping $T: X \to X$ is said to be *orbitally continuous* if $u, x \in X$ are such that u is a cluster point of $O(x)$, then Tu is a cluster point of $T(O(x))$.

In the present paper we investigate strongly non-periodic and orbitally continuous mappings from a topological space into itself, which are not necessarily continuous. We present a result which contains many of results for contractive mappings (mappings which shrink distance in some manner) from a metric space into itself.

2. We prove the following result.

THEOREM 1. *Let X be a topological space and $T\colon X\to X$ be a strongly non-periodic and orbitally continuous mapping. If for some $x_0\in X$ the set $\bar{O}(x_0)$ is compact, then there exists a cluster point u of $O(x_0)$ such that $Tu=u$. Furthermore, if for every $(u,v)\in X\times X$, $u\neq v$ implies $(Tu,Tv)\neq(u,v)$, then u is a unique fixed point in X under T.*

Proof. It is clear that if $y\in O(x_0)$ then $Ty\in O(x_0)$. Now, let y be a cluster point of $O(x_0)$. Since T is orbitally continuous it follows that Ty is a cluster point of $T(O(x_0))=O(Tx_0)\subset O(x_0)$. Therefore, we have $T(\bar{O}(x_0))\subset\bar{O}(x_0)$.

Let $\mathcal{F}$ be a family of all nonempty closed subsets of $\bar{O}(x_0)$ which T maps into itself. Since $T(\bar{O}(x_0))\subset\bar{O}(x_0)$ the family $\mathcal{F}$ is not empty. Let $\mathcal{F}$ be partially ordered by the set inclusion and let $\mathfrak{L}$ be a totally ordered subfamily of $\mathcal{F}$. Put $F_0=\bigcap\{F\colon F\in\mathfrak{L}\}$. F_0 is closed nonempty subset of $\bar{O}(x_0)$ by the compactness of $\bar{O}(x_0)$ and it is a lower bound of $\mathfrak{L}$. Using Zorn's lemma we can find a subset C of $\mathcal{F}$ which is minimal with respect to being nonempty, closed and mapped into itself by T. By the minimality of C we have $\overline{T(C)}=C$.

Let u be an element in C and suppose that $u\neq Tu$. Then $u\notin\bar{O}(T^2u)$ since T is strongly non-periodic. The orbitally continuity of T implies that the set $\bar{O}(T^2u)$ is mapped into itself by T and the minimality of C implies that $\bar{O}(T^2u)=C$. Since $u\in C$, one has $u\in\bar{O}(T^2u)$ which is desired contradiction. Therefore, $u=Tu$.

The last assertion of the theorem is clear. This completes the proof.

COROLLARY 1. *Let X be a compact topological space and $T\colon X\to X$ be a strongly non-periodic and orbitally continuous mapping. Then for each $x\in X$ there exists a cluster point u of $O(x)$ such that $Tu=u$. Furthermore, if for every $(u,v)\in X\times X$, $u\neq v$ implies $(Tu,Tv)\neq(u,v)$, then T has a unique fixed point.*

EXAMPLE. The following example shows that in Theorem 1 one cannot delete the requirement that T be orbitally continuous. Let X be the set of reals $0\leqslant x\leqslant 1$ with the usual topology, and let $T\colon X\to X$ be defined by $Tx=\frac{1}{2}x$ for x rational and $x\neq 0$, $T(0)=1$ and $Tx=\frac{1}{3}x$ for x irrational. It is clear that T is strongly non-periodic and X is compact. But T is not orbitally continuous and has not a fixed point. Let now $F\colon X\to X$ be defined by $Fx=Tx$ for $x\neq 0$ and $F(0)=0$. The mapping F is strongly non-periodic and orbitally continuous (but not continuous) and has the fixed point.

In the following corollaries we shall show that contractive mappings on a metric or uniform space are the strongly non-periodic mappings. Some of these contractive conditions are listed below. We suppose that T is a mapping of a metric space M into itself.

(1) (Edelstein [2]). T is said to be *contractive* if for all $x,y\in M$, $x\neq y$, $d(Tx,Ty)<d(x,y)$.

(2) (Kirk [4]). T is said to have *diminishing orbital diameters* if for each $x\in M$ the diameter $\delta(O(x))$ of the orbit $O(x)$ satisfies the property that $0<\delta(O(x))<\infty$ implies $\delta(O(x))>\lim_n\delta(O(T^nx))$.

(3) (Ćirić [1]). T is said to be a *contraction type mapping* if for all $x,y\in M$ there are non-negative numbers $\delta(x,y)$ and $q(x,y)<1$ with $\sup q(x,y)=1$ and such that

$$d(T^nx,T^ny)\leqslant\big(q(x,y)\big)^n\delta(x,y)\,,\qquad n=1,2,\ldots$$

Let now X be a uniform space, $\mathcal{B}$ a basis for the uniformity and $T\colon X\to X$ a mapping.

(4) (Kammerer and Kasriel [3]). T is said to be $\mathcal{B}$-*contractive* if for each $U\in\mathcal{B}$ and $(x,y)\in U$, $x\neq y$, there exists a $W\in\mathcal{B}$ such that $(Tx,Ty)\in W\subset U$ and $(x,y)\notin W$.

COROLLARY 2 (Edelstein [2]). *Let (M,d) be a metric space and $T\colon M\to M$ be contractive. If for some $x_0\in M$ there exists a subsequence $\{T^{m_i}x_0\}$ of the sequence $\{T^nx_0\}$ such that $\lim_iT^{m_i}x_0=u$, then u is a unique fixed point under T and $u=\lim_nT^nx_0$.*

Proof. Let $x\in M$ be arbitrary and suppose that $x\neq Tx$. Then it is impossible that $x=T^kx$ for $k\geqslant 2$. For if so, then $d(x,Tx)=d(T^kx,T^kTx)$ and since T^k is contractive when T is contractive, we have that $d(T^kx,T^kTx)<d(x,Tx)$, which is a contradiction. Also it is impossible that x is a cluster point of $O(T^2x)$. For if so, by routine calculation one can show that then follows $x=Tx$, which is a contradiction with $x\neq Tx$. Therefore, T is strongly non-periodic. It is clear that every contractive mapping is (uniformly) continuous and hence orbitally continuous. Since it follows that $\lim_nT^nx_0=u$, the set $\bar{O}(x_0)$ is compact. If $u\neq v$ then $d(Tu,Tv)<d(u,v)$ implies that $(Tu,Tv)\neq(u,v)\in M^2$. Therefore, all assumptions of Theorem 1 are satisfied.

COROLLARY 3 (Kirk [4]). *Suppose (M,d) is a compact metric space and $T\colon M\to M$ is continuous with diminishing orbital diameters. Then for each $x\in M$, some subsequence $\{T^{m_i}x\}$ of the sequence $\{T^mx\}$ has a limit which is a fixed point of T.*

Proof. Suppose $x\neq Tx$. Then $\delta(O(x))>0$ and by hypothesis $\delta(O(x))>\lim_n\delta(O(T^nx))$. Hence there exists a positive integer m such that $\delta(O(x))>\delta(O(T^mx))=\delta(\bar{O}(T^mx))$ which implies that $O(x)\neq\bar{O}(T^mx)$. Hence $x\notin\bar{O}(T^mx)$. It is impossible that $x\in\bar{O}(T^2x)$. For if so, then $x=T^kx$ for some $k<m$. This implies $x=T^{nk}x$ for every $n=1,2,\ldots$, which is a contradiction with $x\notin\bar{O}(T^mx)$. Therefore, T is a strongly non-periodic mapping.

COROLLARY 4 ([1]). *Let $T\colon M\to M$ be an orbitally continuous mapping of a metric space M into itself and let M be T-orbitally complete. If T is a contraction type mapping then T has a unique fixed point $u\in M$ and $u=\lim_n T^n x$ for every $x\in M$.*

Proof. Let $x\neq Tx$ and suppose that $x=T^k x$, $k\geqslant 2$. Then

$$0<d(x,Tx)=d(T^{nk}x,T^{nk}Tx)\geqslant\big(q(x,Tx)\big)^n\delta(x,Tx)$$

for every $n=1,2,\ldots$, is a contradiction with $q(x,Tx)<1$. Also it is impossible that x is a cluster point of $O(T^2x)$, because $O(T^2x)$ has the unique cluster point u for which we have $u=Tu$. Therefore, T is strongly non-periodic. It is clear that $\bar{O}(x)$ is compact for every $x\in M$.

Let now $(X,\mathfrak{U})$ be a uniform space and let $\mathfrak{B}$ be a basis for the uniformity. $\mathfrak{B}$ is said to be *ample* if $(x,y)\in U\in\mathfrak{B}$ implies that there exists a $W\in\mathfrak{B}$ for which $(x,y)\in W\subset\overline{W}\subset U$. A space X is *U-chainable*, $U\in\mathfrak{B}$, if for every $x,y\in X$ there are a finite set of points $u_0=x,u_1,\ldots$ $\ldots,u_n=y$ in X such that $(u_{i-1},u_i)\in U$, $i=1,2,\ldots,n$. This terminology is identical with that used by W. Kammerer and R. Kasriel in [3].

COROLLARY 5 (Kammerer and Kasriel [3]). *Let $(X,\mathfrak{U})$ be a compact Hausdorff uniform space, $\mathfrak{B}$ be an open ample basis for $\mathfrak{U}$ and let $T\colon X\to X$ be a $\mathfrak{B}$-contractive mapping of X into itself. If X is U-chainable, $U\in\mathfrak{B}$, then there is a unique fixed point $u\in X$ and $\lim_n T^n x=u$ for every $x\in X$.*

Proof. Since it can be shown that T has a unique fixed point $u\in X$ and $\lim_n T^n x=u$ for every $x\in X$, we see that $\mathfrak{B}$-contractive mappings on U-chainable spaces are strongly non-periodic mappings.

Before we prove the following results we recall first some terminologies.

Let S be a bounded subset of a metric space M. We denote by $\alpha(S)$ the infimum of all $\varepsilon>0$ for which S has a finite ε-net and by $\beta(S)$ the infimum of all $\varepsilon>0$ such that S admits a finite covering consisting of subsets with diameter less than ε. Clearly $\alpha(S)\leqslant\beta(S)\leqslant 2\alpha(S)$ and $\alpha(S)=\beta(S)=0$ iff S is totally bounded. Let $T\colon M\to M$ be a mapping which is not necessarily continuous. T is said to be *condensing* if for every bounded $S\subset M$ such that $\alpha(S)>0$, we have $\alpha\big(T(S)\big)<\alpha(S)$. T is said to be *densifying* if for every bounded $S\subset M$, such that $\beta(S)>0$, we have $\beta\big(T(S)\big)<\beta(S)$.

THEOREM 2. *Let $T\colon M\to M$ be an orbitally continuous, strongly non-periodic and condensing mapping. If M is T-orbitally complete and for some $x_0\in M$ the set $O(x_0)$ is bounded, then T has a fixed point $u\in M$ and $\lim_i T^{n_i}x_0=u$ for some sequence $\{T^{n_i}x_0\}\subset O(x_0)$.*

Proof. Since $O(x_0)$ is bounded, T condensing and $\alpha\big(T(O(x_0))\big)=\alpha\big(O(x_0)\big)$, it follows that $\alpha(O(x_0))=0$. The set $\bar{O}(x_0)$ is compact since $O(x_0)$ is of the form $\{T^n x_0\colon n\in N\}$ and M is a T-orbitally complete metric space. Now we may apply Theorem 1.

The proof of the following theorem is similar to previous and is omitted.

THEOREM 3. *Let $T\colon M\to M$ be an orbitally continuous, strongly non-periodic and densifying mapping. If M is T-orbitally complete and for some $x_0\in M$ the set $O(x_0)$ is bounded, then T has a fixed point $u\in M$ and $\lim_i T^{n_i}x_0=u$ for some sequence $\{T^{n_i}x_0\}\subset O(x_0)$.*

References

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