Fixed point theorems in topological spaces

by

I. B. Cirić (Beograd)

Abstract. Let $T: X \to X$ be a mapping of a topological space $X$ into itself. $T$ is called a \textit{strongly non-periodic mapping} if $x \neq Tz$ implies $x \notin \text{cl}(T^n x, T^n z, \ldots)$. In the present paper are given conditions in metric and topological spaces under which these mapping have fixed points.

1. Let $X$ be a topological space and $T: X \to X$ a mapping. A point $x \in X$ is called a \textit{fixed point under $T$} if $T^n x = x$. A point $x \neq X$ is called a \textit{periodic point under $T$} if there exists a positive integer $k > 2$ such that $T^k x = x$, i.e., if $x \in \{T^n x, T^n z, \ldots\}$. For $x \in X$, let

$$O(T^n x) = \{T^n x, T^{n+1} x, \ldots\}, \quad n = 0, 1, 2, \ldots$$

(where it is understood that $T^n x = x$ and $\text{cl}(T^n x) = \text{cl}(T^n z, T^{n+1} x, \ldots)$).

Now we will introduce a notion of a strongly non-periodic mapping.

\textbf{Definition 1.} A mapping $T: X \to X$ is called \textit{strongly non periodic} if for every $x \in X$

$$x \neq Tz \quad \text{implies} \quad x \notin O(T^n z).$$

Let $T$ be a mapping of a metric space $M$ into itself. In [1] an orbitally continuous mapping and a $T$-orbitally complete space are defined as follows. A mapping $T$ is said to be \textit{orbitally continuous} if $u, z \in M$ are such that $u = \lim_{n \to \infty} T^n z$ then $T u = \lim_{n \to \infty} T^n z$. A space $M$ is said to be \textit{T-orbitally complete} if every Cauchy sequence of the form $(T^n x ; i = X)$ converges in $M$. Now the corresponding concept for orbitally continuous mappings in a topological space $X$ will be given.

\textbf{Definition 2.} A mapping $T: X \to X$ is said to be \textit{orbitally continuous} if $u, z \in X$ are such that $u$ is a cluster point of $O(z)$, then $T u$ is a cluster point of $T O(z)$.

In the present paper we investigate strongly non-periodic and orbitally continuous mappings from a topological space into itself, which are not necessarily continuous. We present a result which contains many of results for contractive mappings (mappings which shrink distance in some manner) from a metric space into itself.

1 — Fundamenta Mathematicae LXXXVII
2. We prove the following result.

Theorem 1. Let \( X \) be a topological space and \( T: X \to X \) be a strongly non-periodic and orbitally continuous mapping. If for some \( a_0 \in X \) the set \( O(a_0) \) is compact, then there exists a cluster point \( u \) of \( O(a_0) \) such that \( Tu = u \). Furthermore, if for every \( (u, v) \in X \times X \), \( u \neq v \) implies \( (Tu, Tv) \neq (u, v) \), then \( u \) is a uniquely fixed point of \( T \) under \( T \).

Proof. It is clear that if \( y \in O(a_0) \) then \( Ty \notin O(a_0) \). Now, let \( y \) be a cluster point of \( O(a_0) \). Since \( T \) is orbitally continuous it follows that \( Ty \in \overline{O(a_0)} \). Therefore, we have

\[
T(\overline{O(a_0)}) \subset \overline{O(a_0)}. 
\]

\( \mathcal{F} \) be a family of all nonempty closed subsets of \( O(a_0) \) which \( T \) maps into itself. Since \( T(\overline{O(a_0)}) \subset \overline{O(a_0)} \) the family \( \mathcal{F} \) is not empty. Let \( \mathcal{F} \) be partially ordered by the set inclusion and let \( \mathcal{F} \) be a totally ordered subfamily of \( \mathcal{F} \). Put \( F_0 = \bigcup \{ P: P \in \mathcal{F} \} \). Let \( F_0 \) is closed nonempty subset of \( O(a_0) \) by the compactness of \( O(a_0) \) and it is a lower bound of \( \mathcal{F} \). Using Zorn’s lemma we can find a subset \( \mathcal{G} \) of \( \mathcal{F} \) which is minimal with respect to being nonempty, closed and mapped into itself by \( T \). By the minimality of \( \mathcal{G} \) we have \( T(\overline{G}) = G \).

Let \( u \) be an element in \( \mathcal{G} \) and suppose that \( u \neq Tu \). Then \( u \notin \overline{O(T^\infty u)} \) since \( T \) is strongly non-periodic. The orbital continuity of \( T \) implies that the set \( \overline{O(T^\infty u)} \) is mapped into itself by \( T \) and the minimality of \( \mathcal{G} \) implies that \( \overline{O(T^\infty u)} = 0 \). Since \( u \neq 0 \), one has \( u \notin \overline{O(T^\infty u)} \) which is a contradiction. Therefore, \( u = Tu \).

The last assertion of the theorem is clear. This completes the proof.

Corollary 1. Let \( X \) be a compact topological space and \( T: X \to X \) be a strongly non-periodic and orbitally continuous mapping. Then for each \( a \in X \) there exists a cluster point \( u \) of \( O(a) \) such that \( Tu = u \). Furthermore, if for every \( (u, v) \in X \times X \), \( u \neq v \) implies \( (Tu, Tv) \neq (u, v) \), then \( T \) has a unique fixed point.

Example. The following example shows that in Theorem 1 one cannot delete the requirement that \( T \) be orbitally continuous. Let \( X \) be the set of reals \( 0 \leq x \leq 1 \) with the usual topology, and let \( T: X \to X \) be defined by \( Tx = x^2 \) for \( x \) rational and \( x \neq 0 \), \( T(0) = 1 \) and \( TX = x \) for \( x \) irrational. It is clear that \( T \) is strongly non-periodic and \( X \) is compact. But \( T \) is not orbitally continuous and has not a fixed point. Let now \( P: X \to X \) be defined by \( Px = x^2 \) for \( x \neq 0 \) and \( P(0) = 0 \). The mapping \( P \) is strongly non-periodic and orbitally continuous (but not continuous) and has the fixed point.

In the following corollaries we shall show that contractive mappings on a metric or uniform space are the strongly non-periodic mappings. Some of these contractive conditions are listed below. We suppose that \( T \) is a mapping of a metric space \( M \) into itself.

(1) (Edelstein [2]). \( T \) is said to be contractive if for all \( x, y \in M \), \( x \neq y \),

\[
d(Tx, Ty) < d(x, y). \]

(2) (Kirk [4]). \( T \) is said to have diminishing orbital diameters if for each \( x \in M \) the diameter \( \delta(O(x)) \) of the orbit \( O(x) \) satisfies the property that \( 0 < \delta(O(x)) < \infty \) implies \( \delta(O(x)) \to \lim_{n \to \infty} \delta(O(T^nx)) \).

(3) (Cirić [1]). \( T \) is said to be a contraction type mapping if for all \( x, y \in M \) there are non-negative numbers \( \delta(x, y) \) and \( q(x, y) < 1 \) with \( \sup_{(x, y) \in M} q(x, y) = 1 \) and such that

\[
d(T^nx, T^ny) < q(x, y) \delta(x, y), \quad n = 1, 2, ... \]

Let now \( X \) be a uniform space, \( M \) a basis for the uniformity and \( T: X \to X \) a mapping.

(4) (Kammerer and Kariel [3]). \( T \) is said to be \( M \)-contractive if for each \( U \subseteq M \) and \( (x, y) \in U, x \neq y \), there exists a \( W \subseteq U \) such that \( (T^nx, T^ny) \in W \subseteq U \) and \( (x, y) \notin W \).

Corollary 2. (Edelstein [2]). Let \( (M, d) \) be a metric space and \( T: M \to M \) be contractive. If for some \( x_n \in M \) there exists a subsequence \( \{T^n x_n\} \) of the sequence \( \{T^n x_n\} \) such that \( \lim_{n \to \infty} T^n x_n = u \), then \( u \) is a unique fixed point under \( T \) and \( u = \lim T^n u \).

Proof. Let \( x \in M \) be arbitrary and suppose that \( x \neq Tx \). Then it is impossible that \( x = T^k x \) for \( k \geq 2 \). For if so, then \( d(Tx, Ty) = d(Tx, T^2x) = d(Tx, T^3x) \) and since \( T \) is contractive when \( T \) is contractive, we have \( d(T^nx, T^ny) \to d(x, y) \), which is a contradiction. Also it is impossible that \( x \) is a cluster point of \( O(T^nx) \). For if so, by routine calculation one can show that then follows \( x = Tx \), which is a contradiction with \( x \neq Tx \). Therefore, \( T \) is strongly non-periodic. It is clear that every contractive mapping is (uniformly) continuous and hence orbitally continuous. Since it follows that \( \lim_{n \to \infty} T^n x_n = u \), the set \( O(u) \) is compact. If \( u \neq v \) then \( d(Tx, Ty) < d(u, v) \) implies that \( (Tx, Ty) (u, v) \in M \). Therefore, all assumptions of Theorem 1 are satisfied.

Corollary 3. (Kirk [4]). Suppose \( (M, d) \) is a compact metric space and \( T: M \to M \) is continuous with diminishing orbital diameters. Then for each \( x \in M \), some subsequence \( \{T^n x\} \) of the sequence \( \{T^n x\} \) has a limit which is a fixed point of \( T \).

Proof. Suppose \( x \neq Tx \). Then \( \delta(O(x)) > 0 \) and by hypothesis \( \delta(O(x)) > 0 \). Hence there exists a positive integer \( m \) such that \( \delta(O(x)) > \delta(O(T^m x)) \) which implies that \( O(x) \neq O(T^m x) \). Hence \( x \notin O(T^m x) \). It is impossible that \( x \in O(T^m x) \). For if so, then \( x = T^nx \) for some \( k \leq m \). This implies \( x = T^nx \) for all \( n \in 1, 2, ... \), which is a contradiction with \( x \notin O(T^m x) \). Therefore, \( T \) is a strongly non-periodic mapping.
Corollary 4. Let $T: M \to M$ be an orbitally continuous mapping of a metric space $M$ into itself and let $M$ be $T$-orbitally complete. If $T$ is a contraction type mapping then $T$ has a unique fixed point $u \in M$ and $u = \lim_n T^n x$ for every $x \in M$.

Proof. Let $x \neq T x$ and suppose that $x = T^n x$, $k > 2$. Then $0 < d(x, T x) = d(T^{n-1} x, T^k x) \leq \delta(x, T x)$ for every $n = 1, 2, \ldots$, is a contradiction with $q(x, T x) < 1$. Also it is impossible that $x$ is a cluster point of $O(T^n x)$ because $O(T^n x)$ has the unique cluster point $u$ for which we have $u = T u$. Therefore, $T$ is strongly non-periodic. It is clear that $O(x)$ is compact for every $x \in M$.

Let now $(X, \mathcal{U})$ be a uniform space and let $\mathcal{B}$ be a basis for the uniformity. $\mathcal{B}$ is said to be ample if $(x, y) \in U \in \mathcal{B}$ implies that there exists a $W \in \mathcal{B}$ for which $(x, y) \in W \subset \bigcap U$. A space $X$ is $U$-chainable, $U \in \mathcal{B}$, if for every $x, y \in X$ there are a finite set of points $u_0 = x, u_1, \ldots, u_n = y$ in $X$ such that $(u_{i-1}, u_i) \in U$, $i = 1, 2, \ldots, n$. This terminology is identical with that used by W. Kammerer and R. Kaaria in [3].

Corollary 5. Let $(X, \mathcal{U})$ be a compact Hausdorff uniform space, $\mathcal{B}$ an open ample basis for $\mathcal{U}$ and $T: X \to X$ be a $\mathcal{B}$-contractive mapping of $X$ into itself. If $X$ is $U$-chainable, $U \in \mathcal{B}$, then there is a unique fixed point $u \in X$ and $\lim_n T^n x = u$ for every $x \in X$.

Proof. Since it can be shown that $T$ has a unique fixed point $u \in X$ and $\lim_n T^n x = u$ for every $x \in X$, we see that $\mathcal{B}$-contractive mappings on $U$-chainable spaces are strongly non-periodic mappings.

Before we prove the following results we recall first some terminologies.

Let $S$ be a bounded subset of a metric space $M$. We denote by $\alpha(S)$ the infimum of all $\varepsilon > 0$ for which $S$ has a finite $\varepsilon$-net and by $\beta(S)$ the infimum of all $\varepsilon > 0$ such that $S$ admits a finite covering consisting of subsets with diameter less than $\varepsilon$. Clearly $\alpha(S) \leq \beta(S) \leq 2\alpha(S)$ and $\alpha(S) = \beta(S) = 0$ iff $S$ is totally bounded. Let $T: M \to M$ be a mapping which is not necessarily continuous. $T$ is said to be condensing if for every bounded $S \subset M$ such that $\alpha(S) > 0$, we have $\alpha(T(S)) < \alpha(S)$. $T$ is said to be $\mathcal{B}$-condensing if for every bounded $S \subset M$, such that $\beta(S) > 0$, we have $\beta(T(S)) < \beta(S)$.

Theorem 2. Let $T: M \to M$ be an orbitally continuous, strongly non-periodic and $\mathcal{B}$-condensing mapping. If $M$ is $T$-orbitally complete and for some $x_0 \in M$ the set $O(x_0)$ is bounded, then $T$ has a fixed point $u \in M$ and $\lim_n T^n x_0 = u$ for some sequence $(T^n x_0) \subset O(x_0)$.

Proof. Since $O(x_0)$ is bounded, $T$ condensing and $\alpha(T(O(x_0))) = \alpha(O(x_0))$, it follows that $\alpha(O(x_0)) = 0$. The set $O(x_0)$ is compact since $O(x_0)$ is of the form $(T^n x_0 : n \in N)$ and $M$ is a $T$-orbitally complete metric space. Now we may apply Theorem 1.