Clumps of continua *

by

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Abstract. A nondegenerate collection G of continua is called a *clump* provided that the sum of all the continua of G is a continuum and there exists a continuum C which is a proper subcontinuum of every element of G and is the intersection of each two elements of G.

Various structural conditions which may be imposed upon clumps are studied leading to theorems indicating under what conditions the sum of the elements of a clump of tree-like continua is itself tree-like.

The purpose of this paper is to study conditions under which a continuum which is the sum of tree-like continua is itself tree-like.

A continuum is a closed, compact, and connected subset of a metric space. A mapping is a continuous transformation. A non degenerate collection G of continua is called a *clump* provided that G^* (the sum of all the continua of the collection G) is a continuum and there exists a continuum C, called the *center of* G, such that C is a proper subcontinuum of every element of G and is the intersection of each two elements of G. A radiation is a clump G of arcs with a degenerate center whose point is an end point of each arc of the collection G. A clump (or radiation) G is said to be *decomposable* if, and only if, there exist two proper subcollections H and K of G such that $G = H \cup K$ and H^* and K^* are both closed; otherwise, G is said to be *indecomposable*. Recall that an indecomposable continuum is a nondegenerate continuum which is not the sum of two of its proper subcontinua and note that, if G is a clump of continua, whether decomposable or indecomposable, then G^* is not an indecomposable continuum.

A clump G of continua with center C is said to be *upper semi-continuous* provided that, if $p_1, p_2, p_3, \ldots$ and $q_1, q_2, q_3, \ldots$ are two sequences of points of G^* converging to points p and q, respectively, of $G^* \setminus C$ and, for each i, p_i and q_i belong to the same element of G, then p and q belong to the same element of G. Clearly, any clump H which is a subcollection of an upper semi-continuous clump G is itself upper semi-continuous. A clump G is said to be *fully decomposable* provided that, for each two

* Dedicated to the memory of Ralph B. Bennett.

elements h and k of G, there exists two subcollections H and K of G such that $G = H \cup K$, h is not in K, k is not in H, and H^* and K^* are both closed. Recall that a continuum M is said to be *fully decomposable* provided that, if x and y are two points of M, then M is the sum of two continua, one not containing x and the other not containing y; but note that the sum of the arcs of a fully decomposable radiation need not be fully decomposable.

The author's conversations with Professor W. B. Johnson, while this research was in progress, have been of inestimable value. Indeed, the example described in the proof of Theorem 18 (though he described it differently) is due to Johnson.

It is clear that every finite clump is decomposable. It may appear that every clump is decomposable, but this is not so.

THEOREM 1. *There exists an indecomposable radiation G such that G^* is the one point compactification of the plane.*

Proof. Let $S = E^2 \cup \{\infty\}$, the one point compactification of the plane, E^2, by the ideal point ∞. If p is a point (p_1, p_2) of E^2 with p_1 an odd integer, let g_p be the arc from p to ∞ consisting of ∞, all points of the closed interval pq, where q is the point (q_1, p_2) with $p_1 < q_1 < p_1 + 2$ and $p_2 = \tan(q_1, \pi/2)$, and all points (q_1, q_2) with $q_2 < p_2$. Let G be the collection of all arcs g_p for all points p of E^2 whose abscissa is an odd integer. The intersection of each two elements of G is $\{\infty\}$ and $G^* = S$, hence G is a radiation.

Suppose that G is the sum of two proper subcollections H_1 and H_2 such that H_1^* and H_2^* are closed. Then there exists an integer $i \in \{1, 2\}$, a decreasing sequence $n_1, n_2, n_3, \ldots$ of odd negative integers, and, for each positive integer r, an increasing sequence $m_{r1}, m_{r2}, m_{r3}, \ldots$ of positive integers such that, for each positive integer S, if p is the point (n_r, m_{rs}) of E^2, then $g_p \in H_i$. Thus, for each positive integer r and each positive integer j, each point with abscissa $n_r + 2j$ is in $\overline{H_i^*}$, and, thus, for each such point p, $g_p \in H_i$. But then, $H_i = G$, contrary to the supposition that H_i is a proper subcollection of G.

(The proof given here that G is indecomposable becomes superfluous with our later Theorem 9, but the author feels that its inclusion may shed some light on the structure of indecomposable clumps.)

THEOREM 2. *Suppose that G is a clump of continua in the plane, E^2, with center C, such that no element of G separates the plane, and, if g is an element of G, $g \setminus C$ is connected. Then G is decomposable.*

Proof. Assume that G has more than two elements. Let $D = E^2 \setminus G^*$. Then D is connected, [8, Theorem 122, p. 250]. Let β denote the boundary of D. If β were a subset of an element g of G, then $E^2 \setminus D \subset g$ and $G^* \subset g$, a contradiction. Hence, there exists an arc xy lying, except for its end points x and y, wholly in D such that x and y are points of two different elements of G. Denote by g_x and g_y different elements of G containing x and y respectively. Then $g_x \cup g_y$ does not separate E^2, and $E^2 \setminus (g_x \cup g_y \cup xy)$ is the sum of two mutually exclusive connected open sets D_1 and D_2, [8, Theorem 131, p. 263].

Suppose that D_1 contains a point of G^*. Then there exists an arc pq lying, except for its end points p and q, wholly in $D_1 \cap D$ such that p is a point of $G^* \setminus (g_x \cup g_y)$ and q is a point of the arc xy distinct from x and y. Let g_p denote the element of G containing p. Then, [8, Theorem 131, p. 263], $D_1 \setminus (g_p \cup pq)$ is the sum of two mutually exclusive connected open sets I_1 and I_2 with no point of $g_x \setminus C$ in $\bar{I}_2$. But every element of G is a subset of one of the two point sets $I_1 \cup g_x \cup g_y$ and $I_2 \cup g_x \cup g_y$. Let H_1 be the collection of all elements of G which are subsets of $I_1 \cup g_x \cup g_p$ and H_2 be the collection of all elements of G which are subsets of $I_2 \cup g_y \cup g_p$. Then g_x is not in H_2, g_y is not in H_1, H_1^* and H_2^* are closed, and $G = H_1 \cup H_2$. Thus G is decomposable.

Similarly, if D_2 contains a point of G^*, then G is decomposable. But G^* intersects one of D_1 and D_2, so G is decomposable.

THEOREM 3. *A clump G of continua is upper semi-continuous if, and only if, it is fully decomposable.*

Proof. Suppose that the clump G of continua is upper semi-continuous, C is the centre of G, and h and k are two elements of G. Let $D_1, D_2, D_3, \ldots$ be a sequence of open subsets of G^* such that $\bigcap_{i>0} D_i = C$ and, for each i, $\overline{D_{i+1}} \subset D_i$. For each i, let G_i be the collection of all point sets g' such that, for some element g of G, $g' = g \cap (G^* \setminus D_i)$. For each i, G_i is a collection of mutually exclusive closed and compact point sets filling up $G^* \setminus D_i$. Let $H_1, H_2, H_3, \ldots$ and $K_1, K_2, K_3, \ldots$ be sequences such that, for each i, are subcollections of G_i containing $h \cap (G^* \setminus D_i)$ and $k \cap (G^* \setminus D_i)$ respectively, H_i^* and K_i^* are open relative to G_i^*, $\overline{H_i^*}$, and $\overline{K_i^*}$ are mutually exclusive, $\overline{H_i^*} \subset H_{i+1}^*$ and $\overline{K_i^*} \subset K_{i+1}^*$. Let H denote the collection of all elements of G which do not intersect $\bigcup_{i>0} K_i^*$ and K denote the collection of all elements of G which do not intersect $\bigcup_{i>0} H_i^*$. Clearly, h is not in K, k is not in H, and $G = H \cup K$. Suppose that p is a point of $G^* \setminus H^*$. There is a positive integer i such that p is not in $\bar{D}_i$ and $p \in K_i^*$. Then $K_i^* \cap (G^* \setminus \bar{D}_i)$ is an open set containing p but no point of H^*. Thus, H^* is closed. Similarly, K^* is closed. So G is fully decomposable. Now, suppose that G is a fully decomposable clump of continua with centre C which is not upper semi-continuous, h and k are two elements of G, p and q are points of $h \setminus C$ and $k \setminus C$ respectively, $p_1, p_2, p_3, \ldots$ and $q_1, q_2, q_3, \ldots$ are two sequences of points of G^* converging to p and q respectively such that, for each i, p_i and q_i belong to the same element of G. Now, G is the sum of two collections H and K such that h is not in K, k is not in H, and H^* and K^* are closed. Then $G^*\setminus H^*$ and $G^*\setminus K^*$ are mutually exclusive open subsets of G^* containing q and p respectively and, for some n, $q_n \in G^*-H^*$ and $p_n \in G^*-K^*$. Then the element of G which contains both p_n and q_n is in neither H nor K, a contradiction. Hence, every fully decomposable clump of continua is upper semi-continuous.

From the following theorem, it is evident that, in Theorem 2, we could not have concluded that G is fully decomposable.

THEOREM 4. *There is in the plane, E^2, a radiation which is not upper semi-continuous.*

Proof. For each number x, $0 < x \leqslant 1$, but g_x denote the semi-circle in the upper half plane with centre $(x/2, 0)$ and radius $x/2$ and let G_1 denote the collection of all such semi-circles g_x. For each number y, $-1 \leqslant y < 0$, let g_y denote straight line interval from the origin to the point $(1, y)$ and let G_2 denote the collection of all such intervals g_y. Then $G_1 \cup G_2$ is a radiation.

For each positive integer n, let p_n and q_n denote the points of $g_{-(1/n)}$ with abscissae $\frac{1}{3}$ and $\frac{2}{3}$ respectively. Then $p_1, p_2, p_3, \ldots$ converges to the point $(0, \frac{1}{3})$ of $g_{1/3}$ and $q_1, q_2, q_3, \ldots$ converges to the point $(0, \frac{2}{3})$ of $g_{2/3}$. Hence, G is not upper semi-continuous.

THEOREM 5. *There exists in the plane, E^2, an indecomposable clump K of continua whose center is a simple closed curve C such that, if k is in K, $k\setminus C$ is connected and such that K^* is the sum of C and its bounded complementary domain.*

Proof. There exists a homeomorphism h of E^2 onto the interior of a simple closed curve C. For each element g of the collection G described in the proof of Theorem 1, let k_g denote $h(g\setminus\{\infty\}) \cup C$ and let K denote the collection of all such continua k_g. Then K is an indecomposable radiation with center C such that K^* is C plus its interior.

THEOREM 6. *There exists in the plane, E^2, an indecomposable clump G of continua with degenerate center such that no element of G separates E^2.*

Proof. Consider the unit circle S^1 with center at the origin. Let $K_1, K_2, K_3, \ldots$ denote mutually exclusive finite subsets of S^1 such that, for each n, each point of S^1 is at a distance less than $1/n$ from K_n. For each n, let g_n denote the sum of all straight line intervals from the origin to a point of K_n. Let G denote the collection to which G belongs if, and only if, for some n, $g = g_n$ or, there exists a point $p \in S^1\setminus\big(\bigcup_{i>0} K_i\big)$ such that g is the interval from the origin to p. Clearly, G is a clump with center $\{(0, 0)\}$ such that G^* is S^1 plus its interior. Suppose that G is the sum of two proper subcollections H_1 and H_2 such that H_1 contains infinitely many of the continua $g_1, g_2, g_3, \ldots$ Then $\overline{H_1^*} = G^*$. Hence, G is indecomposable.

If G is a clump of continua, the clump H of continua is said to be a *refinement of* G if every element of H is a subcontinuum of some element of G, and H is said to be a *full refinement of* G if H is a refinement of G and $H^* = G^*$.

THEOREM 7. *Suppose that G is a clump of continua such that every refinement of G has a decomposable full refinement, and f is an essential mapping of G^* onto the circle, S^1. Then there exists an element g of G such that H_f is essential.*

Proof. Suppose that, for every element g of G, $f|g$ is inessential.

Let $\mathcal{H}$ denote the collection to which H belongs if, and only if, H is a clump which is a refinement of G and $f|H^*$ is essential. Let $\mathcal{K}$ denote a subcollection of $\mathcal{H}$ such that, (1) if K_1 and K_2 are in $\mathcal{K}$, then one of them is a refinement of the other, and (2) if H is in $\mathcal{H}\setminus\mathcal{K}$, there exists an element K in $\mathcal{K}$ such that neither H nor K refines the other. There exists a collection L each element of which is a continuum such that (1) if $M \in L$ and $K \in \mathcal{K}$, there exists a continuum $k \in K$ such that $M \subset k$ but M is not a proper subset of any other such continuum, and (2) L^* is the intersection of all continua K^* such that $K \in \mathcal{K}$. Then $f|L^*$ is essential.

If L is degenerate, there exists an element g of G such that $L^* \subset g$ and, since $f|L^*$ is essential, then $f|g$ is essential, a contradiction. Then L is nondegenerate. But then, L is a clump which refines G and L is the only full refinement of L. Then there exist two proper subcollections L_1 and L_2 of L such that $L = L_1 \cup L_2$ and L_1^* and L_2^* are closed. If L_1 is degenerate, there exists an element g of G containing L_1^* and, hence, $f|L_1^*$, is inessential. If L_1 is nondegenerate, it is a refinement of L and L_1^* is a proper subcontinuum of L^*, so $f|L_1^*$ is inessential. Hence, $f|L_1^*$, is inessential and, similarly, $f|L_2^*$ is inessential. But $L_1^* \cap L_2^*$ is a continuum, so $f|(L_1^* \cup L_2^*)$ is essential, a contradiction. Thus, there is an element g of G such that $f|g$ is essential.

That the full hypothesis of Theorem 7 is needed can be seen from the example of Lelek and Mohler, [7], of a radiation G such that $\dim(G^*) = 1$ and G^* contains a simple closed curve. Such a continuum G^* can be mapped essentially onto S^1 though no arc can be. The radiation G is not upper semi-continuous.

THEOREM 8. *If G is an upper semi-continuous clump of continua and f is an essential mapping of G^* onto the unit circle, S^1, then there is an element g of G such that $f|g$ is essential.*

Proof. Suppose that G is an upper semi-continuous clump of continua, f is an essential mapping of G^* onto S^1, and, for each $g \in G$,

$f|g$ is inessential. There is a subcollection H of G such that H^* is closed and $f|H^*$ is essential but, if K is a proper subcollection of H and K^* is closed, then $f|K^*$ is inessential. Since H is nondegenerate and upper semi-continuous, it is the sum of two proper subcollections K_1 and K_2 such that K_1^* and K_2^* are closed. But $f|K_1^*$ and $f|K_2^*$ are inessential and $K_1^* \cap K_2^*$ is a continuum, so $f|(K_1^* \cup K_2^*)$ is inessential, a contradiction.

THEOREM 9. *Suppose that G is a clump of continua such that* (1) *G^* is the two dimensional sphere, S^2,* (2) *no element of G separates S^2, and either* (3) *the center of G does not separate S^2 or* (3′) *G has more than two elements. Then G is indecomposable.*

Proof. If (1), (2), and (3′) are true, then (3) is true. Suppose that (1), (2), and (3) are true and G is the sum of two proper subcollections H_1 and H_2 such that H_1^* and H_2^* are closed. Then $H_1 \cap H_2$ is a clump such that no element of $H_1 \cap H_2$ separates S^2 but $(H_1 \cap H_2)^*$ does separate S^2. But this is contrary to [8, Theorem 122, p. 250].

A continuum M is said to be *tree-like* provided that, for every positive number ε there exists a mapping f_ε of M onto a finite tree T such that, for each point t of T, $f_\varepsilon^{-1}(t)$ has diameter less than ε. A continuum M is hereditarily unicoherent provided that, for each two points p and q of M, there is only one subcontinuum of M irreducible from p to q. Every tree-like continuum is hereditarily unicoherent. A dendroid is an arcwise connected, hereditarily unicoherent continuum. A λ-dendroid is an hereditarily decomposable, hereditarily unicoherent continuum. Every dendroid and every λ-dendroid is tree-like, [5]. Indeed, one can characterize a dendroid as an arcwise connected tree-like continuum and a λ-dendroid as an hereditarily decomposable tree-like continuum.

THEOREM 10. *If G is a clump of continua, G^* is hereditarily unicoherent, and M is an indecomposable subcontinuum of G^*, then there exists an element g of G such that M is a subcontinuum of g.*

Proof. Suppose the contrary and let C denote the center of G. For each element g of G such that M intersects $g \setminus C$, let h_g denote the continuum $g \cap M$. Let H denote the collection of all such continua h_g. If h_1 and h_2 are in H, then $h_1 \cap C = M \cap C = h_2 \cap C = h_1 \cap h_2$ and, hence, H is a clump with center $M \cap C$. Now $H^* = M$ and the composant intersecting C of M is all of M, a contradiction, [8, Theorems 138 and 139, p. 59].

THEOREM 11. *Suppose that G is a clump of tree-like continua such that $\dim(G^*) = 1$ and every refinement of G has a decomposable full refinement. Then G^* is tree-like.*

Proof. Suppose that G^* is not hereditarily unicoherent. Then there exists an essential mapping f of G^* onto S^1, and, by Theorem 7, an element g of G such that $f|g$ is essential, contrary to [3, Theorem 1]. Thus G^* is hereditarily unicoherent. And, by Theorem 10, if M is an indecomposable continuum lying in G^*, M is a subcontinuum of some element of G and is, therefore, tree-like. Then [5, Theorem 1], G^* is tree-like.

COROLLARY. *If G is a clump of dendroids (λ-dendroids), $\dim(G^*) = 1$, and every refinement of G has a decomposable full refinement, then G^* is a dendroid (λ-dendroid).*

THEOREM 12. *If G is an upper semi-continuous clump of tree-like continua and $\dim(G^*) = 1$, then G^* is tree-like.*

The proof is the same as that for Theorem 11 except that we appeal to Theorem 8 instead of Theorem 7 to establish that (G^*) is hereditarily unicoherent.

COROLLARY. *If G is an upper semi-continuous clump of dendroids (λ-dendroids) and $\dim(G^*) = 1$, then G^* is a dendroid (λ-dendroid).*

THEOREM 13. *If G is a clump of tree-like continua in the plane and $\dim(G^*) = 1$, then G^* is tree-like.*

Proof. If G^* is not tree-like some subcontinuum of G^* separates the plane [1, Theorem 6], and, hence, G^* separates the plane, contrary to [8, Theorem 122, p. 250).

COROLLARY. *If G is a clump of dendroids (λ-dendroids) in the plane and $\dim(G^*) = 1$, the G^* is a dendroid (λ-dendroid).*

THEOREM 14. *If G is a countable clump of hereditarily unicoherent continua, then G^* is hereditarily unicoherent.*

Proof. Suppose that x and y are two points of G^*, g_x is an element of G containing x, g_y is an element of G containing y, and M is a subcontinum of G^* irreducible from x to y, but M is not a subcontinuum of $g_x \cup g_y$. Then $(g_x \cup g_y) \cap M$ is the sum of two mutually exclusive closed point sets U_x and V_y containing x and y respectively. Let M' denote a subcontinuum of M irreducible from U_x to V_y. Suppose that M' is decomposable, then M' is the sum of two proper subcontinua H_1 and H_2 intersecting U_x and V_y respectively. Let Z be a point of $H_1 \cap H_2$, let H_1' be a subcontinuum of H_1 irreducible from U_x to Z, and let H_2' be a subcontinuum of H_2 irreducible from Z to V_y, and let g_Z denote the element of G containing Z. Then $M' = H_1' \cup H_2'$. Now, the composant containing Z of H_1' is the sum of countably many continua $C_{11}, C_{12}, C_{13}, \ldots$ each containing Z and, for each i, C_{1i} does not intersect $g_x \cup g_y$; and the composant containing Z of H_2' is the sum of countably many continua C_{21}, $C_{22}, C_{23}, \ldots$ each containing Z and, for each i, C_{2i} does not intersect $g_x \cup g_y$. For each i, $C_{1i} \cup C_{2i} \subset g_Z$, for $C_{1i} \cup C_{2i}$ does not intersect K, the center of G, and, hence, if it intersected two elements of G, would be the sum of countably many mutually exclusive closed proper subsets, contrary to [8, Theorem 56, p. 23]. Thus, $Y = \bigcup_{i>0} (C_{1i} \cup C_{2i}) \subset g_Z$ and $\overline{Y} = M' \subset g_Z$.

But $g_x \cup g_y \cup g_Z$ is hereditarily unicoherent, [5, Lemma 2], and, hence, $M' \cap (g_x \cup g_y)$ is a proper subcontinuum of M' intersecting both U_x and V_y, a contradiction. Thus M' is indecomposable. No composant of M' intersects both U_x and V_y, thus, [4, Theorem 3] there is a composant, L, of M' which does not intersect $U_x \cup V_y$. Then L is the sum of countably many continua $C_1, C_2, C_3, \ldots$ having a point p in common. Now, for each i, C_i does not intersect K, so C_i is a subset of g_p, the element of G containing p. Then $Y = \bigcup_{i>0} C_i \subset g_p$ and, thus, $\overline{Y} = M' \subset g_p$. But $g_x \cup g_y \cup g_p$ is hereditarily unicoherent and $M' \cap (g_x \cup g_y)$ is a proper subcontinuum of M' intersecting both U_x and V_y, a contradiction. Thus, $M \subset (g_x \cup g_y)$ if M is an irreducible subcontinuum of G^* from x to y. Now, since $g_x \cup g_y$ is hereditarily unicoherent, there is only one subcontinuum of $g_x \cup g_y$ irreducible from x to y. Thus, if x and y are two points of G^*, there is only one subcontinuum of G^* irreducible from x to y, i.e. G^* is hereditarily unicoherent.

THEOREM 15. *If G is a countable clump of tree-like continua, then G^* is tree-like.*

The proof is the same as that for Theorem 11 except that we appeal to Theorem 14 instead of Theorem 7 to establish that G^* is hereditarily unicoherent.

COROLLARY. *If G is a countable clump of dendroids (λ-dendroids), then G^* is a dendroid (λ-dendroid).*

A rational continuum is a continuum M such that, if p is a point of M and 0 is an open set containing p, then there is a domain, D with respect to M containing p and having a countable boundary with respect to M such that D is a subset of 0.

THEOREM 16. *If G is a clump of continua such that G^* is a rational continuum, then G is countable.*

Proof. If G were uncountable, G^* would contain uncountably many mutually exclusive nondegenerate subcontinua, contrary to [6, Theorem 1.3].

THEOREM 17. *If G is a clump of tree-like continua and G^* is a rational continuum, then G^* is a λ-dendroid.*

Proof. Since G is countable, G^* is tree-like. Since each indecomposable continuum contains uncountably many nondegenerate subcontinua, each rational continuum is hereditarily decomposable, [6]. Then G^* is a λ-dendroid.

No indecomposable radiation can be embedded in the plane. Borsuk, [2], has given an example of a radiation, H, such that H^* cannot be embedded in the plane but every refinement of H is decomposable.

THEOREM 18. *There exists a countable decomposable radiation.*

Proof. For each nonnegative integer n, let T_n be the closed interval in the plane from the origin, 0, to the point $(1, n)$; let b_{n0} be 0 and a_{01} be $(1, 0)$; and, if $n > 0$, let $0 = b_{n0} < a_{n1} < b_{n1} < a_{n2} < b_{n2} < \ldots < a_{nn} < b_{nn} = (1, n)$ be $2n+1$ distinct points, in that order, of T_n. For each n, let $X_n = \bigcup_{i=0}^{n} T_i$. For each positive integer n, let r_n^{n+1} be a retraction of X_n onto X_{n-1} such that, for each positive integer $i \leqslant n$, the intervals $b_{n,i-1}a_{ni}$ and $a_{ni}b_{ni}$ are mapped by r_{n-1}^{n} homeomorphically onto the interval $b_{n-1,0}a_{n-i,1}$ and $r_{n-1}^{n}(a_{ni}) = a_{n-1,i}$.

If i and j are nonnegative integers with $i < j$, let $r_i^j = r_{j-1}^{j} \circ r_{j-2}^{j-1} \circ \ldots \circ r_i^{i+1}$. Let X_∞ denote the inverse limit of the inverse mapping system X_i, r_i^j. For each nonnegative integer n, denote by g_n the subset of X_∞ consisting of the point $(0, 0, 0, \ldots)$ and all points $(x_1, x_2, x_3, \ldots)$ of X_∞ such that $x_k = x_n$ if, and only if, $k \geqslant n$. Let g_ω be the set $[X_\infty \setminus \bigcup_{i \geqslant 0} g_i] \cup \{(0, 0, 0, \ldots)\}$. Clearly, for each i, g_i is an arc (it is homeomorphic to T_i), $g_i \cap g_j = \{(0, 0, 0, \ldots)\}$ if $i \neq j$, and $g_i \cap g_\omega = \{(0, 0, 0, \ldots)\}$. Let G denote the collection consisting of g_ω and all sets g_i for all nonnegative integers i. Then $G^* = X_\infty$ and G is countable.

Suppose that $x = (x_0, x_1, x_2, \ldots)$ is a point of X_∞ and, for some n, $x \neq x_{n+1}$, but x is not in $\bigcup_{i=0}^{n} g_n$. Then there exists an integer $k > n$ such that $x_k \neq x_n$ but, if $n < i < k$, $x_i = x_n$. Then x_k is a point of X_k mapped by r_{k-1}^{k} to x_n and, hence, x_k does not belong to $b_{k0}a_{k1}$, since $k > n+1$. Then, for each integer $S > k$, $x_S = x_{S-1} = x_k$. Then $x \in g_k$. Thus, g_ω is the inverse limit of the increase mapping system

$$\{b_{i0}a_{i1}, r_i^j | b_{j0}a_{j1}\}$$

each coordinate space of which is an arc and each bonding map of which is a homeomorphism. Then g_ω is an arc and G is a radiation.

Suppose that H is an infinite subcollection of G such that H^* is closed. Then there exists an increasing sequence $n_1, n_2, n_3, \ldots$ of nonnegative integers such that, for each i, $g_{n_i} \in H$. Let n be a nonnegative integer and $x = (x_0, x_1, x_2, \ldots)$ be a point of g_n such that $x_n \in (b_{n0}a_{i1} \setminus \{b_{n0}\})$. For every integer i such that $n_i > n$, there exists a point x_{n_i} in $T_{ni} \setminus b_{n_i0}a_{n_i1}$ such that $r_{n_i-1}^{n_i}(x_{n_i}) = x_n$ and a point $y^i = (y_0^i, y_1^i, y_2^i, \ldots)$ in g_{n_i} such that $y_{n_i}^i - x_{n_i}$ and, thus, if $0 \leqslant S \leqslant n_i$, then $y_S^i = x_S$. Then, if $n_j > n$, the sequence $y^j, y^{j+1}, y^{j+2}, \ldots$ converges to x and $x \in H^*$. Thus, for each i, $g_i \in H$. Let $x' = (x_0', x_1', x_2', \ldots)$ be a point of g_ω distinct from $(0, 0, 0, \ldots)$. For every nonnegative integer i, g_i contains a point Z^i whose ith coordinate is x_i'. Then the sequence $Z^0, Z^1, Z^2, \ldots$ converges to x' and $x' \in H^*$. Thus H is G. Then G is indecomposable since it is not the sum of two proper subcollections H_1 and H_2 such that H_1^* and H_2^* are closed.

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Boolean-valued selectors for families of sets *

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Abstract. Let $\mathfrak{X} = \langle X_\alpha \rangle_{\alpha<\varkappa}$ be a family of sets. We say that $\mathfrak{X}$ has a selector if there is a set S such that $|S \cap X_\alpha = 1$ for every $\alpha < \varkappa$. $\mathfrak{X}$ has partial selectors if for every $\beta < \varkappa$ the family $\mathfrak{X} \upharpoonright \beta = \langle X_\alpha \rangle_{\alpha<\beta}$ has a selector. Let $\mathbf{E}(\varkappa, \varkappa)$ denotes the following statement: *For every family $\mathfrak{X} = \langle X_\alpha \rangle_{\alpha<\varkappa}$ of sets of powers $< \lambda$, if $\mathfrak{X}$ has partial selectors then $\mathfrak{X}$ has a selector.* In this paper we prove a theorem on the invariance of $\mathbf{E}(\varkappa, \varkappa)$ under some generic extensions, namely: *Let $|\mathfrak{B}| = \lambda$, $\mathfrak{B}$ satisfy σ-cc, and $\lambda^{\sigma} < \varkappa$. Moreover, suppose that for each* ZF*-formula Φ with parameters from $\check{V}$ we have $\|\Phi\| \in \{0, 1\}$. Then $\mathbf{E}(\varkappa, \varkappa)$ implies $\|\mathbf{E}(\check{\varkappa}, \check{\varkappa})\| = 1$ in $V^{(\mathfrak{B})}$.*

This paper is a continuation of [3]. For the readers' convenience we repeat the main notions and results of [3].

If $\mathfrak{X} = \langle X_\alpha \rangle_{\alpha<\varkappa}$ is a family of sets then $\mathfrak{X}$ has a *selector* if there is a set S such that $|S \cap X_\alpha| = 1$ for every $\alpha < \varkappa$. We say that $\mathfrak{X}$ has *partial selectors* if for every $\beta < \varkappa$ the family $\mathfrak{X} \upharpoonright \beta = \langle X_\alpha \rangle_{\alpha<\beta}$ has a selector. In [3] the following statement, denoted by $\mathbf{E}(\varkappa, \lambda)$, has been studied: "For every family $\mathfrak{X} = \langle X_\alpha \rangle_{\alpha<\varkappa}$ of sets of powers $<\lambda$ if $\mathfrak{X}$ has partial selectors then $\mathfrak{X}$ has a selector".

The main results of [3] can be presented as follows:

THEOREM. (a) $\mathbf{E}(\varkappa, \varkappa)$ *implies that $\varkappa$ is regular.*
(b) *If $\varkappa$ is weakly compact then $\mathbf{E}(\varkappa, \varkappa)$ holds.*
(c) $\mathbf{E}(\varkappa, \varkappa)$ *implies that $\varkappa$ has the tree property.*
(d) [GCH]. $\mathbf{E}(\varkappa, \varkappa)$ *if and only if $\varkappa$ is weakly compact.*

In this paper we give a theorem about the invariance of the property $\mathbf{E}(\varkappa, \varkappa)$ under some generic extensions. We shall work in the Boolean version of forcing; thus for the readers' convenience we recall the main notions and notations concerning the Boolean-valued universe $V^{(\mathfrak{B})}$. For more information see e.g. [2].

Let $\mathfrak{B}$ be a complete Boolean algebra. We say that $\mathfrak{B}$ *satisfies σ-cc* (*σ-chain condition*) if every family of non-zero disjoint elements of $\mathfrak{B}$ has

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