By Lemma 5 the statement forced is Π_2^1 in constructible parameters and so must be satisfied by every non-constructible path through P. Since there is such a path, this contradicts our original assumption that $<$ is a linear ordering, and establishes the claim.

It is easy to see that the only elements of B invariant under all automorphisms of B are 0 and 1. From this it follows that in $L(S)$ all definable sets are constructible [11], [6, Theorem 6.8]. However it must also be valid that the first non-constructible element in the ordering $<$ is definable and non-constructible.

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On longest paths in connected graphs*

by

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Abstract. It is shown that a connected graph of order $p \geqslant 4$ contains a path of length k, where $1 \leqslant k \leqslant p-1$, if for every integer j with $1 \leqslant j < k/2$, the number of vertices of degree not exceeding j is less than j. Furthermore, a tree of order $p \geqslant 4$ has diameter at least k, where $3 \leqslant k \leqslant p-1$, if the number of vertices of degree one is less than $\{2(p-1)/(k-1)\}$.

A *hamiltonian cycle* (*path*) in a graph G is a cycle (path) containing every vertex of G. Pósa [1] proved that if G is a graph of order $p \geqslant 3$ such that for every integer j with $1 \leqslant j < p/2$, the number of vertices of degree not exceeding j is less than j, then G contains a hamiltonian cycle. In this article, we establish an analogous result for graphs with hamiltonian paths and in fact for graphs containing paths of any specified length.

THEOREM 1. *Let G be a connected graph of order $p \geqslant 4$. Then G contains a path of length k $(1 \leqslant k \leqslant p-1)$ if for every integer j with $1 \leqslant j < k/2$ the number of vertices of degree not exceeding j is less than j.*

Proof. Since G is connected and $p \geqslant 4$, the theorem is true for $k = 1$ and $k = 2$. Henceforth we assume $k \geqslant 3$. Suppose the length of a longest path in G is n where $2 \leqslant n < k$. If P is a longest path in G, let S_P denote $\deg u + \deg v$, where u and v are the endvertices of P. Among all longest paths in G, choose P: $u_0, u_1, \ldots, u_n$ so that S_P is maximum. Suppose $\deg u_0 \leqslant \deg u_n$.

Since P is a longest path, each of u_0 and u_n is adjacent only to vertices of P. If $u_i u_n \in E(G)$, $0 \leqslant i \leqslant n-1$, then $u_0 u_{i+1} \notin E(G)$; for otherwise the cycle

$$C\colon\ u_0, u_1, \ldots, u_i, u_n, u_{n-1}, \ldots, u_{i+1}, u_0$$

of length $n+1$ is present in G. The cycle C cannot contain all vertices of G since $n+1 < p$. Since G is connected, there exists a vertex w not on C adjacent to a vertex of C; however this implies G contains a path of length $n+1$ which is impossible. Hence for each vertex of $\{u_0, u_1, \ldots, u_{n-1}\}$

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adjacent to u_n, there is a vertex of $\{u_1, u_2, \ldots, u_n\}$ not adjacent to u_0. Thus $\deg u_0 \leqslant n - \deg u_n$, so that $\deg u_0 + \deg u_n \leqslant n$. Since $\deg u_0 \leqslant \deg u_n$ and $n < k$ we obtain $\deg u_0 < k/2$, so, by hypothesis, there are fewer than $\deg u_0$ vertices having degree not exceeding $\deg u_0$. However, if $u_0 u_i \in E(G)$, $1 \leqslant i \leqslant n$, then

$$u_{i-1}, u_{i-2}, \ldots, u_0, u_i, u_{i+1}, \ldots, u_n$$

is a longest path in G and from the manner in which u_0 and u_n were chosen, $\deg u_{i-1} \leqslant \deg u_0$. Thus there are at least $\deg u_0$ vertices having degree not exceeding $\deg u_0$. This presents a contradiction so that $k \leqslant n$. Thus G contains a path of length k.

The case $k = p-1$ gives a sufficient condition for a graph to possess a hamiltonian path. However, we can obtain a slightly stronger result by employing the previously stated theorem of Pósa. Suppose G is a graph of order $p \geqslant 3$ such that for every integer j with $0 \leqslant j < (p-1)/2$, the number of vertices of degree not exceeding j is less than $j+1$. We construct a graph G^* by adding a new vertex v and new edges joining v to all vertices of G. The graph G has a hamiltonian path if and only if G^* has a hamiltonian cycle. However, G^* satisfies the hypothesis of Pósa's theorem (with $p+1$ in place of p) so that G^* is hamiltonian.

The results of Theorem 1 are, in general, not best possible. However, the result stated in the preceding paragraph is best possible. For example, if $p \geqslant 4$, let G be a connected graph with one cutvertex and the three blocks K_2, K_{n+1}, and K_{p-n-1}, where $1 \leqslant n < (p-1)/2$. Then G satisfies all the hypotheses with one exception: G contains at least $n+1$ vertices of degree not exceeding n. Furthermore, the graph G fails to contain a hamiltonian path.

The simplest connected graphs are the trees. In this case, one can give a sufficient condition for lengths of longest paths in terms of the number of vertices of degree one (endvertices) only. Since there is a unique path between every pair of vertices in a tree, the result can be expressed in terms of its diameter (the length of a longest path for a tree). We denote the diameter of a graph G by $\operatorname{diam} G$.

We begin with two lemmas.

LEMMA 1. *If a nontrivial tree T has m endvertices, where m is even, then there exist paths $P_1, P_2, \ldots, P_{m/2}$ satisfying:*

(1) *Each endvertex of T is an endvertex of exactly one of the paths P_i, $1 \leqslant i \leqslant m/2$,*

(2) $$\bigcup_{j=1}^{m/2} V(P_j) = V(T),$$

(3) $$V(P_i) \cap \Big(\bigcup_{j=1}^{i-1} V(P_j)\Big) \neq \emptyset \quad \textit{for all } i \textit{ with } 2 \leqslant i \leqslant m/2.$$

Proof. We employ induction on m. For $m = 2$, T is a path and the result desired is satisfied by T itself. Assume the result holds for all trees with $m \geqslant 2$ endvertices, and let T be a tree with $m+2$ endvertices. With each endvertex of T, we associate a path of T in the following manner. Let u be an arbitrary endvertex of T. Now, $\deg u = 1$, where u is adjacent to, say, u_1. If $\deg u_1 \geqslant 3$, let P_u be the trivial path: u. Otherwise, $\deg u_1 = 2$ and u_1 is adjacent to u and, say, u_2. If $\deg u_2 \geqslant 3$, let P_u be the path: u, u_1. Otherwise, $\deg u_2 = 2$ and u_2 is adjacent to u_1 and, say, u_3. Continue in this fashion to obtain the path P_u. Since $m+2 \geqslant 4$, we can choose endvertices u and v of T so that $T^* = T - V(P_u) - V(P_v)$ is a tree with m endvertices, each of which is an endvertex of T. By the induction hypothesis, T^* contains a collection of paths $P_1, P_2, \ldots, P_{m/2}$ satisfying conditions (1)-(3). Let u' and v' be the final vertices of P_u and P_v, respectively. Then there exists a $z-w$ path P in T^* where $u'z, wv' \in E(T)$. If we define $P_{(m+2)/2}$ to be the path P_u followed successively by $u'z$, P, wv', and P_v, the result follows.

LEMMA 2. *If a nontrivial tree T has m endvertices, where m is odd, and* $\operatorname{diam} T = k$, *then there exist paths $P_1, P_2, \ldots, P_{(m-1)/2}, P_{(m+1)/2}$ satisfying:*

(1) *$P_{(m+1)/2}$ is of the form P_u (using the notaiton of the previous lemma) for some endvertex u of T and the length of P_u does not exceed $(k/2)-1$,*

(2) *Each endvertex of T other than u is an endvertex of exactly one of the P_i's for $1 \leqslant i \leqslant (m-1)/2$,*

(3) $$V(P_i) \cap \Big(\bigcup_{j=1}^{i-1} V(P_j)\Big) \neq \emptyset \quad \textit{for all } i \textit{ with } 2 \leqslant i \leqslant (m-1)/2.$$

Proof. Since $\operatorname{diam} T = k$, the path $P_{(m+1)/2}$ clearly exists. Let $T^* = T - V(P_{(m+1)/2})$. Then T^* is a tree with $m-1$ endvertices, each of which is an endvertex of T. By applying the previous lemma to T^*, we obtain the desired result.

THEOREM 2. *Let T be a tree of order $p \geqslant 4$ and let k be a fixed integer with $3 \leqslant k \leqslant p-1$. If $\operatorname{diam} T < k$, then T has at least $\{2(p-1)/(k-1)\}$ endvertices.*

Proof. Let m denote the number of endvertices of T. We consider two cases depending on the parity of m.

Case 1. m is even. By Lemma 1, there exist paths $P_1, P_2, \ldots, P_{m/2}$ with the three required properties. Since $\operatorname{diam} T \leqslant k-1$, we have $|V(P_i)| \leqslant k$ for $i = 1, 2, \ldots, m/2$. We wish to find an upper bound for p. Since $|V(P_1)| \leqslant k$, $|V(P_2)| \leqslant k$, and $V(P_2) \cap V(P_1) \neq \emptyset$, we have

$$|V(P_1) \cup V(P_2)| \leqslant k + (k-1).$$

Moreover, since $V(P_3)\cap\left(\bigcup_{j=1}^{2}V(P_j)\right)\neq\emptyset$ and $|V(P_3)|\leqslant k$, it follows that $|V(P_1)\cup V(P_2)\cup V(P_3)|\leqslant k+(k-1)+(k-1)$. Continuing in this fashion, we obtain $|\bigcup_{j=1}^{m/2}V(P_j)|\leqslant(m/2)(k-1)+1$. Since $V(T)=\bigcup_{j=1}^{m/2}V(P_j)$, we have $p\leqslant(m/2)(k-1)+1$ so that $m\geqslant 2(p-1)/(k-1)$. Thus $m\geqslant\{2(p-1)/(k-1)\}$ since m is integral.

Case 2. m is odd. By Lemma 2, there exist paths $P_1, P_2, \ldots, P_{(m+1)/2}$ with the four required properties. Since $\operatorname{diam}T\leqslant k-1$, $|V(P_i)|\leqslant k$ for $i=1,2,\ldots,(m-1)/2$ and $|V(P_{(m+1)/2})|\leqslant(k-1)/2$. As above, we obtain

$$|\bigcup_{j=1}^{(m-1)/2}V(P_j)|\leqslant\left(\frac{m-1}{2}\right)(k-1)+1$$

and hence

$$|\bigcup_{j=1}^{(m+1)/2}V(P_j)|\leqslant\left(\frac{m-1}{2}\right)(k-1)+1+\frac{(k-1)}{2}.$$

Since $V(T)=\bigcup_{j=1}^{(m+1)/2}V(P_j)$, we have $p\leqslant(m/2)(k-1)+1$ so that

$$m\geqslant 2(p-1)/(k-1).$$

Thus $m\geqslant\{2(p-1)/(k-1)\}$ since m is integral.

The previous theorem may be restated as follows.

THEOREM 2′. *Let T be a tree of order $p\geqslant 4$ and let k be a fixed integer with $3\leqslant k\leqslant p-1$. If T has fewer than $\{2(p-1)/(k-1)\}$ endvertices, then* $\operatorname{diam}T\geqslant k$.

This result is best possible in the following sense. Given an odd integer $k\geqslant 3$ and integer $m\geqslant 2$, there exists an integer $p\geqslant 4$ and tree T of order p and diameter $k-1$ with $3\leqslant k\leqslant p-1$, such that T has m endvertices, where $m=\{2(p-1)/(k-1)\}$. The tree obtained by replacing each edge of the (star) graph $K(1,m)$ with a path of length $(k-1)/2$ serves as such an example.

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