In our example both of the spaces $X$ and $Y$ are 1-dimensional, hence they can be embedded into 3-dimensional Euclidean space. We do not know if it is possible to construct an example of this kind taking a subspace of the plane as $Y$. We do not know also whether $Y$ would be the Knaster-Kuratowski Broom.

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Added in proof. Recently the second of the authors showed, modifying the present construction, that $Y$ can be taken as a subspace of the plane. We have also proved that if we replace in the construction of Knaster-Kuratowski Broom the rational and irrational numbers of the $x$-axis by two disjoint subsets of irrationals of the second category, then we obtain the space $Y$ with a dispersion point which is not an open-perfect image of any hereditarily disconnected space.  

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The non-existence of $\Sigma^1_1$ well-orderings of the Cantor set

by

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Abstract. It is shown the existence of a $\Sigma^1_1$ well-ordering of the Cantor set implies that all reals are constructible. This is the converse of a theorem of Gödel.

Throughout this paper we assume the existence of a non-constructible real. With that in hand, let us set forth some notation. A finite sequence $s$ is an extension of $t$ if $t$ is an initial subsequence of $s$. A tree $T$ is a set of a finite sequences of 0's and 1's containing every initial subsequence and at least one proper extension of each of its members. For $a$ a function with domain the set of non-negative integers, $\bar{a}(n)$ is the sequence $\langle a(0), a(1), \ldots, a(n-1) \rangle$. A path through the tree $P$ is a function $\alpha$ such that $\bar{a}(n)$ is in $P$ for every $n$. $[P]$ is the set of paths through $P$. It is easily shown that $[P]$ is a closed subset of $2^\mathbb{N}$ and that every closed subset of $2^\mathbb{N}$ is the set of paths through a unique tree. The tree corresponding to a closed set is its code; a closed set with a constructible code is constructibly coded. A closed set is perfect if every sequence in its code has at least two incompatible extensions in the code.

Let $B$ be the Boolean algebra corresponding to forcing with constructibly coded perfect sets ordered by the subset relation. $B$ is a complete Boolean algebra containing the constructible trees as a dense subset. There are several ways to represent $B$; one is as the regular open sets in the space $2^\mathbb{N}$ with the topology generated by the constructibly coded $[P]$'s.

We are going to be using $B$-valued set theory. In that set theory there is a canonical generic function $S$ in $2^\mathbb{N}$. (In the system presented in [6], $S$ is $\langle \bigvee, P \rangle$, $\forall s \in P$ $\lceil \text{length}(s) \leq s \lor \text{length}(s) = 1 \rceil$.) We are also going to be using another Boolean extension of set theory $M$, in which every constructible tree $T$ has a path generic over $V$ with respect to $B$. The Truth Lemma [11] states that for a generic $\varphi$ a formula in the forcing language, $V(\alpha)$ satisfies $\varphi$ iff there is a condition $P$ with $a \epsilon [P]$ and $P \models \varphi$.

In interpreting the forcing language for $V(\alpha)$, $S$ is a name for $a$. Thus if $\varphi(\bar{a})$ is a $\Sigma^1_1$ or $\Pi^1_2$ formula with possible unlisted constructible parameters
and \( a \) is generic, the statement \( \varphi(a) \) iff there is a constructible tree \( P \) with \( a \in P \) and \( P \models \varphi(S) \)" has value one in the model \( M \).

**Lemma 1.** \( \models S \vDash \varphi\left(\alpha_{\mathfrak{L}}^{\mathfrak{L}(\alpha)}\right) \iff \varphi_{M} = 1 \).

**Proof.** See Sacks [8].

**Lemma 2.** If \( \varphi(a) \) is \( \Sigma_{1}^{1} \), and \( P \) is a condition, \( \{ \varphi(a) \} \subseteq \{ a \} \) implies \( P \models \varphi(S) \).

**Proof.** Let \( M \) be a Boolean extension of \( V \) in which every constructible tree has a path generic over \( V \) with respect to \( P \). Suppose that \( [P] \subseteq \{ \varphi(a) \} \) but \( P \) does not force \( \varphi(S) \); then since we are using weak forcing there is a condition \( Q \) extending \( P \) with \( Q \vDash \sim \varphi(S) \). Then \( [Q] \subseteq (a : \varphi(a)) \subseteq \{ Q \} \) and so \( [Q] \cap (a : \varphi(a)) = \{ Q \} \) has a non-constructible element \( \beta \) (via the assumption in the first sentence of this paper). By the absoluteness lemma \( \varphi(\beta) \) is valid in \( M \), and so in \( M [Q] \cap (a : \varphi(a)) \) has a constructively coded perfect subset \( [E] \). (This is the exact statement of the perfect set theorem [5].) Since \( E \) is an extension of \( Q \), \( R \vDash \varphi(S) \); pick any generic \( c \in [E] \) and the contradiction is immediate.

**Lemma 2** has a converse of sorts which we shall call Lemma 3 even though it is not used in anything that follows. Lemmas 2 and 3 between say that for \( \varphi \) a \( \Sigma_{1}^{1} \) formula, \( [P] \subseteq \{ \varphi(a) \} \) and \( P \models \varphi(S) \) bear roughly the same relation to each other as strong and weak forcing.

Using the Kondo-Addison Uniformization Theorem [7], say \( \Sigma_{1}^{1} \) set \( A \) can be written as the domain of a \( P_{1}^{\infty} \) function \( f_{A} \). Furthermore, in \( ZF \) set theory it can be proven that \( f_{A} \) is a function and \( A \) is its domain.

**Definition.** A \( P_{1}^{\infty} \) set is large if \( A \) has a perfect subset; a \( \Sigma_{1}^{1} \) set \( A \) is large if \( f_{A} \) has a large graph.

Note that the statement "\( A \) is large" is \( \Sigma_{1}^{1} \). Furthermore if \( A \) is large it has a perfect subset [4], but not necessarily vice versa. In the presence of a non-constructible real, the perfect set theorem [3] states that \( A \) is large iff it has a non-constructible element.

**Lemma 3.** If \( \varphi \) is \( \Sigma_{1}^{1} \) and \( P \vDash \varphi(S) \) then \( \{ a : \varphi(a) \} \) is large.

**Proof.** Again let \( M \) be a Boolean extension of \( V \) in which every constructible tree has a generic path. In \( M \) is it valid that \( [P] \cap (a : \varphi(a)) \) has a non-constructible element; any generic path through \( P \) will do. Thus it is also valid in \( M \) that \( [P] \cap (a : \varphi(a)) \) is large. Being a \( \Sigma_{1}^{1} \) statement, it is true in \( V \), completing the proof of the Lemma.

**Lemma 4.** If \( \varphi(a) \) is \( \Pi_{1}^{\infty} \), then every non-constructible path through \( P \) satisfies \( \varphi \).

**Proof.** Otherwise \( [P] \cap \{ a : \sim \varphi(a) \} \) would have a non-constructible element, and hence a constructively coded perfect subset, violating Lemma 2.

The class \( L(a) \) of sets hereditarily constructible from \( a \) is often defined to be the denotation of certain ranked terms \( \tau(a, a_{1}, ..., a_{n}) \) where the \( a_{i} \) are ordinals. These terms are such that within any transitive model for Kripke-Platek set theory containing \( a \) and each \( a_{i} \), \( \tau(a, a_{1}, ..., a_{n}) \) has the same value as it has in the universe. If \( t \) is a well-ordering of integers, let \( |t| \) be its order type.

**Lemma 5.** If \( t_{1}, ..., t_{n} \) are well-orderings of integers, the predicate \( \beta \models \tau(a, a_{1}, ..., a_{n}) \iff |t_{1}| \leq \beta \iff |t_{2}| \leq \beta \leq |t_{n}| \).

**Proof.** \( \beta \models \tau(a, a_{1}, ..., a_{n}) \) iff it is true in any or all countable transitive models for Kripke-Platek set theory containing the parameters. This is in turn equivalent to its being true in any or all well-founded models for Kripke-Platek set theory containing surrogates for the parameters. This latter condition is easily seen to be \( \Delta_{1}^{1} \).

In order to illustrate how these lemmas can be used to elucidate perfect set forcing, let us give a new proof of an old theorem from [8].

**Theorem 1.** The statement "\( S \) has minimal degree of constructibility" has value one.

**Proof.** Suppose otherwise. Then there is a term \( \tau \) and ordinals \( a_{1}, ..., a_{n} \) and a condition \( P \) such that \( P \vDash L(\tau(S, a_{1}, ..., a_{n})) \cap \tau(S, a_{1}, ..., a_{n}) \in L \).

Since \( \varphi_{\mathfrak{L}}^{\mathfrak{L}}(\text{Lemma 1}) \), we may assume that \( a_{1}, ..., a_{n} \) are all constructively countable. Therefore by Lemma 5 and the well-known theorem that \( a \in L(\beta) \) if and only if \( a \in L(\tau(S, a_{1}, ..., a_{n})) \cap \tau(S, a_{1}, ..., a_{n}) \in L \) is \( P_{1}^{\infty} \) in constructible parameters. From Lemma 4 every non-constructible member of \( [P] \) satisfies \( \tau \). Let \( a_{n} \) be a non-constructible element of \( [P] \) and let \( \beta_{a} = \tau(a_{1}, ..., a_{n}) \). The set \( \{ a : \tau(a) \bar{=} \beta_{a} \} \subseteq \beta_{a} \) in \( \beta_{a} \) and constructible parameters, non-empty, and has no element in \( L(\beta_{a}) \), contradicting the absoluteness Lemma. Thus our original assumption is false and the theorem is proven.

**Theorem 2.** If there is a non-constructible real, there is no \( \Sigma_{1}^{1} \) well-ordering of \( 2^{\mathfrak{L}} \).

**Proof.** Suppose otherwise that \( \beta \) is a \( \Sigma_{1}^{1} \) formula which well-orders \( 2^{\mathfrak{L}} \).

We claim that the Boolean value of \( \# \in L(S) \) is \( \omega \) in \( 2^{\mathfrak{L}} \). First note that by writing down completely \( \bar{<} \text{ well-orders} \omega \), we see that it is of the form \( \varphi \land \bigvee \alpha \beta \models \beta[a = \beta \land a < \beta \land a < \beta] \) where \( \varphi \) is \( \Pi_{1}^{\infty} \). Two applications of the Absoluteness Lemma reveal that since \( \varphi \) is true in \( V \), it is valid in \( V(\mathfrak{L}) \) and hence valid in \( L(S) \). So the only way our claim can be false that for terms \( \tau_{1}, \tau_{2}, \tau_{3}, \text{ and constructively countable ordinals } a_{1}, ..., a_{n} \) and a condition \( P \) the following is satisfied:

\[
P \vDash \tau_{1}(S, a_{1}, ..., a_{n}) \neq \tau_{2}(S, a_{1}, ..., a_{n}) \land \tau_{3}(S, a_{1}, ..., a_{n}) \neq \tau_{4}(S, a_{1}, ..., a_{n})
\]

— Fundamenta Mathematicae, T. LXXVII
By Lemma 5 the statement forced is $H^2$ in constructible parameters and so must be satisfied by every non-constructible path through $P$. Since there is such a path, this contradicts our original assumption that $< \bot$ is a linear ordering, and establishes the claim.

It is easy to see that the only elements of $B$ invariant under all automorphisms of $B$ are $0$ and $1$. From this it follows that in $L(\mathcal{B})$ all definable sets are constructible [11], [6, Theorem 6.8]. However it must also be valid that the first non-constructible element in the ordering $< \bot$ is definable and non-constructible.

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On longest paths in connected graphs *

by Linda Lesnaiak (Kalamazoo, Mich.)

Abstract. It is shown that a connected graph of order $p > 4$ contains a path of length $k$, where $1 < k < p-1$, if for every integer $j$ with $1 < j < k/2$, the number of vertices of degree not exceeding $j$ is less than $j$. Furthermore, a tree of order $p > 4$ has diameter at least $k$, where $3 < k < p - 1$, if the number of vertices of degree one is less than $(2(p-1)/(k-1))$.

A hamiltonian cycle (path) in a graph $G$ is a cycle (path) containing every vertex of $G$. Pósa [1] proved that if $G$ is a graph of order $p > 3$ such that for every integer $j$ with $1 < j < p/2$, the number of vertices of degree not exceeding $j$ is less than $j$, then $G$ contains a hamiltonian cycle. In this article, we establish an analogous result for graphs with hamiltonian paths and in fact for graphs containing paths of any specified length.

**THEOREM 1.** Let $G$ be a connected graph of order $p > 4$. Then $G$ contains a path of length $k$, $1 < k < p-1$, if for every integer $j$ with $1 < j < k/2$, the number of vertices of degree not exceeding $j$ is less than $j$.

Proof. Since $G$ is connected and $p > 4$, the theorem is true for $k = 1$ and $k = 2$. Henceforth we assume $k > 3$. Suppose the length of a longest path in $G$ is $n$ where $2 < n < k$. If $P$ is a longest path in $G$, let $S_P$ denote $\text{deg}_u \cap \text{deg}_v$, where $u$ and $v$ are the endpoints of $P$. Among all longest paths in $G$, choose $P$: $u_0, u_1, \ldots, u_n$ so that $S_P$ is maximum. Suppose $\text{deg}_u \leq \text{deg}_w$.

Since $P$ is a longest path, each of $u_0$ and $u_n$ is adjacent only to vertices of $P$. If $u_n u_0 \in E(G)$, $0 < i < n - 1$, then $u_0 u_{i+1} \notin E(G)$, for otherwise the cycle $C$: $u_0, u_1, \ldots, u_{i+1}, u_i, u_{i+1}, \ldots, u_{n-1}, u_0$ of length $n+1$ is present in $G$. The cycle $C$ cannot contain all vertices of $G$ since $n+1 < p$. Since $G$ is connected, there exists a vertex $w$ not on $C$ adjacent to a vertex of $C$; however this implies $G$ contains a path of length $n+1$ which is impossible. Hence for each vertex of $\{u_0, u_1, \ldots, u_{n-1}\}$

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