(5.1) PROBLEM. *Does there exist for every continuum X an* S-*stable continuum* $X_0 \in \mathrm{Sh}(X)$?

(5.2) PROBLEM. *Is it true that* S-*stable* FANR-*spaces are the same as* FR-*stable* FANR-*spaces?*

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An open-perfect mapping of a hereditarily disconnected space onto a connected space

by

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Abstract. In this paper the authors construct an example of open-perfect mapping of a hereditarily disconnected space onto a connected space. Both of the spaces are metrizable and separable.

The aim of this paper is to construct the hereditarily disconnected space X and the open-perfect mapping f of X onto the connected space Y. Both of the spaces will be separable metrizable. We should mention here that 0-dimensionality and totally disconnectedness are invariants under open-closed mappings in the class of metrizable spaces. The first of these facts is obvious, the second was proved by Kombarov [3]. The basis of our construction is the well-known Knaster–Kuratowski Broom, the subspace of the plane with dispersion point (see [4], § 46, II). At the end of this paper we shall prove the theorem related to this object.

Terminology and notation are as in [2] and [4]. In particular, the word "mapping" and the symbol $g\colon A \to B$ mean continuous mapping, the symbol $g\colon A \twoheadrightarrow B$ means that $g(A) = B$. By connected space we always understand non-one-point space. For $x \in X$ and $A \subset X$ we write $x \times A$ instead of $\{x\} \times A$.

1. We start from some auxiliary constructions in the Euclidean plane R^2 with the standard metric ϱ. The symbol R denotes the set of all real numbers, N denotes the set of non-negative integers, I is the interval $[-1, 1]$ of reals, P and Q irrationals and rationals of I respectively. For $t \in R$ let $\bar{t} = (t, 0) \in R^2$.

Divide the set P into two disjoint, dense in P sets P^* and Q^* such that $Q^* = \aleph_0$. Let

$$M = P^* \times P \cup Q^* \times Q .$$

For $x = (x_1, x_2) \in R^2$ and real number $\alpha > 0$ we set

$$U(x, \alpha) = \{y \in R^2 | \varrho(y, x') < \alpha , \quad \text{where} \quad x' = (x_1, x_2 \mp \alpha)\} \cup \{x\} ,$$

and

$$K(x, \alpha) = \{y \in R^2 | \varrho(y, x) < \alpha\} .$$

Let $Q^* = \{q_i\}_{i=1}^{\infty}$ be an enumeration of Q^* and let us choose a strictly decreasing, convergent to 0 sequence $\{r_i\}_{i=1}^{\infty}$ of elements of P, such that $U(\bar{q}_i, r_i) \cap U(\bar{q}_j, r_j) = \emptyset$ for $i \neq j$. Denote $J_i = q_i \times [-r_i, r_i]$. Let $\{G_i\}_{i=1}^{\infty}$ be a sequence of open subsets of the plane such that $G_i \supseteq \bar{G}_{i+1}$ and $\bigcap_{i=1}^{\infty} G_i = (I \times 0) \cup \bigcup_{i=1}^{\infty} J_i$. Such a sequence exists, because the set on the right-hand side of the last equality is closed in R^2.

Now fix $k \geqslant 1$. For $i \leqslant k$ let us choose numbers $\alpha_k^i < q_i < \beta_k^i$ from Q and define sets

$$E_k^i = [\alpha_k^i, \beta_k^i] \times [-r_i, 0], \quad F_k^i = [\alpha_k^i, \beta_k^i] \times [0, r_i], \quad D_k^i = E_k^i \cup F_k^i,$$

satisfying conditions:

$$D_k^i \cap D_k^j = \emptyset, \quad D_k^i \subseteq G_k, \quad U(\bar{q}_i, r_i) \cap D_k^j = \emptyset, \text{ for } i \neq j,\ i, j \leqslant k$$

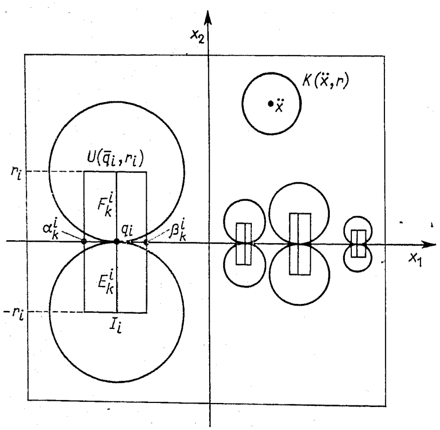
and $D_k^i \subseteq D_{k-1}^i$ for $i \leqslant k-1$. It follows from geometrical considerations that an inductive construction of such numbers and sets is possible.

Put $Y_0 = (Q^* \times 0) \cup (M \setminus \bigcup_{i=1}^{\infty} J_i)$. Let us remark that

(1) for any point $y \neq \bar{q}_i$ of Y_0 there exist k_0 and $m_0 \in N$ such that $K\left(y, \frac{1}{m_0}\right) \cap D_k^i = \emptyset$ provided $i \leqslant k$, $k \geqslant k_0$,

(2) if $y = \bar{q}_{i_0}$, then $U(\bar{q}_{i_0}, r_{i_0}) \cap D_k^i = \emptyset$ for $i_0, i \leqslant k$, $i \neq i_0$,

(3) $$Y_0 \cap \bigcap_{k=1}^{\infty} D_k^i = \{\bar{q}_i\}.$$



2. We start the construction of our example by defining the space Y. Let $\lambda_n((y_1, y_2)) = (y_1, y_2 + 2n)$, $Y_n = \lambda_n(Y_0)$, $Y' = \bigcup_{n=0}^{\infty} Y_n$. We define topology in the set $Y = Y' \cup \{\omega\}$, where $\omega \notin Y'$, by taking the family

$$\left\{U\left(y, \frac{1}{m}\right) \cap Y\right\}_{m=1}^{\infty} \quad \text{for} \quad y = (q_i, 2n),$$

$$\left\{K\left(y, \frac{1}{m}\right) \cap Y\right\}_{m=1}^{\infty} \quad \text{for} \quad y \in Y' \text{ and } y \neq (q_i, 2n),\ n \in N,\ i = 1, 2, \ldots$$

and

$$\{Y \setminus \bigcup_{k=0}^{m} Y_k\}_{m=1}^{\infty} \quad \text{for} \quad y = \omega$$

as the neighbourhood basis of y.

The space Y constructed above is regular and has countable basis, hence by the Urysohn theorem it is metrizable and separable. We may also show it is connected; the proof of this fact is similar to the proof that the Knaster–Kuratowski Broom is connected.

Now we define the space X. Let C denote Cantor set. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of all left ends of the contiguous intervals of the set C. Take the space $Z = Y' \times C \times N$ with the product topology and set $Z_n = Y_n \times C \times N$. For $z \in Z$ we write $z = (y_1, y_2, c, j)$, where $(y_1, y_2) \in Y'$, $c \in C$, $j \in N$. Let

$$A_0 = \{z \in Z_0 |\ -1 \leqslant y_2 \leqslant 0,\ j = 0\},$$

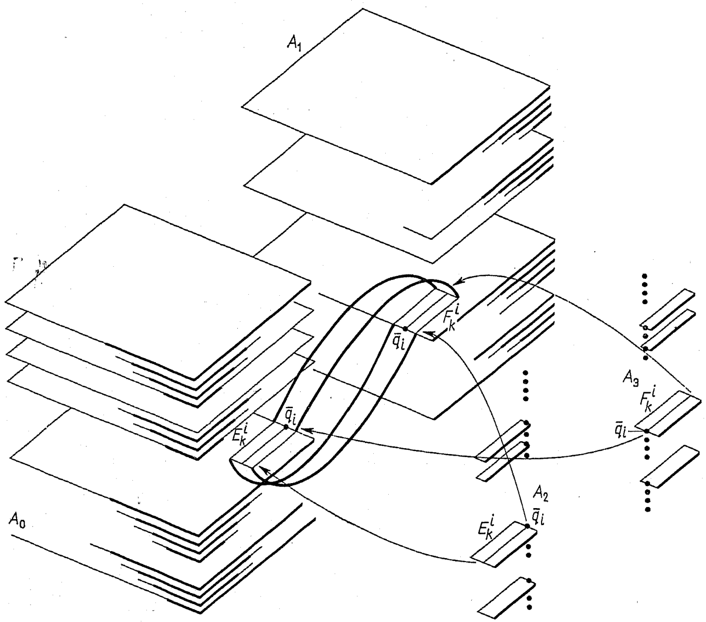
$$A_1 = \{z \in Z_0 |\ \ 0 \leqslant y_2 \leqslant 1,\ j = 1\},$$

$$A_{2i} = (\bar{q}_i \times C \times 2i) \cup \bigcup_{k=i}^{\infty} \big((E_k^i \cap Y_0) \times a_k \times 2i\big) \quad \text{and}$$

$$A_{2i+1} = \big(\bar{q}_i \times C \times (2i+1)\big) \cup \bigcup_{k=i}^{\infty} \big((F_k^i \cap Y_0) \times a_k \times (2i+1)\big) \quad \text{for} \quad i = 1, 2, \ldots$$

We introduce the equivalence relation R_0 on the set $X_0' = \bigcup_{j=0}^{\infty} A_j$ by the formula

$$\begin{aligned} z R_0 z' \Leftrightarrow \big[&(z = (y_1, 0, c, 2i) \quad \wedge z' = (y_1, 0, c, 1)) \\ &\vee (z = (y_1, -r_i, c, 2i) \ \wedge z' = (y_1, -r_i, c, 0)) \\ &\vee (z = (y_1, 0, c, 2i+1) \wedge z' = (y_1, 0, c, 0)) \\ &\vee (z = (y_1, r_i, c, 2i+1) \wedge z' = (y_1, r_i, c, 1)) \\ &\vee (z = z')\big],\ i \geqslant 1. \end{aligned}$$



For $n \in N$ let $\tau_n\colon Z_0 \to Z_n$ be the mapping given by

$$\tau_n\big((y_1, y_2, c, j)\big) = \big(y_1, (-1)^n y_2 + 2n, \sigma^n(c), j\big),$$

where σ^n is the nth superposition of $\sigma\colon c \mapsto 1-c$ for $c \in C$. Let $X'_n = \tau_n(X'_0)$. Let us extend the relation R_0 to the equivalence relation R on $X' = \bigcup_{n=0}^{\infty} X'_n \subseteq Z$ (and equipped with the relative topology) by

$$z R z' \Leftrightarrow [z = \tau_n(z_0) \wedge z' = \tau_n(z'_0) \wedge z_0 R_0 z'_0].$$

Let $X'' = X'/R$ be the quotient space, $\varphi\colon X' \twoheadrightarrow X''$ the natural quotient mapping and $p\colon Z \twoheadrightarrow Y'$ the projection. The restriction $g = p|X'\colon X' \to Y'$ is constant on inverses of points under φ, hence it determines a mapping $f\colon X'' \twoheadrightarrow Y'$ such that $g = f\varphi$. Let $X_n = \varphi(X'_n)$ for $n \in N$.

Let the space X be the result of adding to X'' a closed set $\omega \times C$ and taking as a basis of neighbourhoods at the point (ω, c_0) the family $\{X \setminus [\bigcup_{n=0}^{m} X_n \cup q^{-1}(C \setminus U)]\}$, where U is a neighbourhood of c_0 in C, $m \in N$, and $q(x) = c$ for $x = (\omega, c)$ or $x = \varphi\big((y_1, y_2, c, j)\big) \in X''$. The space $X = X'' \cup (\omega \times C)$ is regular and has countable basis; similarly to the

case considered before, it follows from the Urysohn theorem that X is metrizable. We show that X is hereditarily disconnected. Let us remark that the image of $\{a_k\}_{k=1}^{\infty}$ under σ is the set of right ends of the contiguous intervals of the set C. It follows that for each $c \in C$ the set $q^{-1}(c)$ is the union of the point (ω, c) and its open-closed, hereditarily disconnected subspaces, hence it is hereditarily disconnected. Since the mapping q of X into C is continuous, a connected subset of X would be a subset of the hereditarily disconnected space $q^{-1}(c)$ for some $c \in C$; this means that X is hereditarily disconnected.

Let us extend f to a mapping of X onto Y by $f\big((\omega, c)\big) = \omega$.

3. It remains to show that f is open-perfect.

The images of neighbourhoods of points $\varphi\big((q_i, 2n, \sigma^n(a_k), j)\big)$ for $i \leqslant k$ and points different from $\varphi\big((q_i, 2n, c, j)\big)$ are neighbourhoods of images. Thus to show that f is open it suffices to remark that the set A consisting of points of this form is dense in the inverse image $f^{-1}(y)$ of any point $y \in Y$, i.e. $\overline{A \cap f^{-1}(y)} = f^{-1}(y)$.

We start the proof that f is perfect by showing that for each $j \in N$ the set A_j is closed in Z_0. Let $x^m = (y_1^m, y_2^m, c^m, j)$ for $m \in N$ be any sequence of points of A_j convergent to $x^0 = (y_1^0, y_2^0, c^0, j)$. We may assume $j \geqslant 2$. If $(y_1^m, y_2^m) = \bar{q}_i$ for infinitely many m's, then $x^0 = (q_i, 0, c^0, j) \in A_j$. If there exists $k \in N$ such that infinitely many x^m's belong to the set $(D_k^i \times a_k \times j) \cap A_j$ then $x^0 \in (D_k^i \times a_k \times j) \cap A_j$. If none of these cases holds, then there exists a sequence of indices $k_1 < k_2 < \ldots$ such that $x^{k_m} \in D_{k_m}^i \times a_{k_m} \times j$; hence by (3) the sequence (y_1^m, y_2^m) converges to $\bar{q}_i$ and $x^0 = (q_i, 0, c^0, j) \in A_j$. Since for each $n \in N$ the mapping $\tau_n\colon Z_0 \to Z_n$ is a homeomorphism, the set $\tau_n(A_j)$ is closed in Z_n.

Now we show that the mapping $f_n = f|X_n\colon X_n \twoheadrightarrow Y_n$ is perfect. Since $f_n\varphi|X'_n = g|X'_n$, hence by [1], Chapter 1, § 10, Proposition 5 it suffices to prove that the mapping $g_n = g|X'_n$ is perfect. At first remark that for each $j \in N$ the mapping $g_n|\tau_n(A_j)$ is perfect as the restriction of the perfect (by Kuratowski theorem) projection parallel to the compact axis $p|Z_n$ to a closed set. Therefore the mapping g_n is the combination of the perfect mappings $g_n|\tau_n(A_j)$, where $j \in N$. Thus to finish the proof that the mapping g_n is perfect we need only show that the family $\mathfrak{U} = \{g_n\tau_n(A_j)\}_{j=0}^{\infty}$ is locally finite in Y_n and consists of closed sets. Since $g_n\tau_n(A_{2i} \cup A_{2i+1}) = \lambda_n(D_i^i)$ for $i = 1, 2, \ldots$, hence the closedness of $\mathfrak{U}$ is obvious. Let us take $y \in Y_n$. If $y \neq \lambda_n(\bar{q}_i)$, then by (1) there exists a neighbourhood V of y and $k_0 \in N$ such that $V \cap \lambda_n(D_k^k) = \emptyset$ for $k > k_0$. If $y = \lambda_n(\bar{q}_{i_0})$, then $U\big(\lambda_n(\bar{q}_{i_0}), r_{i_0}\big) \cap \lambda_n(D_k^k) = \emptyset$ for $k > i_0$ by (2). Thus $\mathfrak{U}$ is locally finite.

Since both X and Y are metrizable, to show that f is perfect it suffices to prove that

(4) if $\{x^m\}_{m=1}^{\infty}$ is a sequence of elements of X such that $f(x^m) \to y_0$, then $\{x^m\}_{m=1}^{\infty}$ contains a convergent subsequence.

If $y_0 \in Y'$, then (4) follows from perfectness of the mappings f_n. Hence let us assume that $y_0 = \omega$ and $x^m = \varphi((y_1^m, y_2^m, c^m, j^m))$. Let us choose a subsequence $\{c^{m_k}\}_{k=1}^{\infty}$ of $\{c^m\}_{m=1}^{\infty}$ convergent to some $c^0 \in C$; then $x^{m_k} \to (\omega, c^0)$. This shows (4), and the perfectness of f follows.

4. Remarks. We finish with two remarks.

1° Suppose f is an open-perfect mapping of a Hausdorff space X onto a connected space Y. Then the space of quasi-components of X ([4], § 46, Va) is compact and each of quasi-components is mapped onto Y (Kombarov [3]). If X is hereditarily disconnected, then Y contains no continuum.

The following theorem and corollary give a further information about that situation.

Theorem. *If $f\colon X \twoheadrightarrow Y$ is an open, perfect and 0-dimensional mapping of a regular space X onto a connected and locally connected space Y, then for every point $x \in X$ and arbitrary ordinal α the restriction of f to the quasi-component $Q_\alpha(x)$* (¹) *of order α at the point x is open-perfect and $f(Q_\alpha(x)) = Y$.*

Proof. To begin with let us prove the following lemma.

Lemma. *Let $\mathfrak{F}$ be a family of closed subsets of X such that finite intersections of elements of $\mathfrak{F}$ belong to $\mathfrak{F}$ and for every $F \in \mathfrak{F}$ the restriction of f to the set F is open and maps F onto Y. Then the restriction of f to the set $A = \bigcap \mathfrak{F}$ is also open and $f(A) = Y$.*

For the proof of the lemma take a point $x_0 \in A$, $y_0 = f(x_0)$ and an open neighbourhood U of x_0 in the space A. Let V be an open neighbourhood of x_0 in X such that

$$\overline{V} \cap A \subseteq U \quad \text{and} \quad \operatorname{Fr} V \cap f^{-1}(y_0) = \emptyset .$$

Because $y_0 \notin f(\operatorname{Fr} V)$, we can choose an open, connected neighbourhood W of y_0 such that

$$f(\operatorname{Fr} V) \cap W = \emptyset .$$

Denote $V_F = V \cap F$ and by B_F the boundary of V_F in the space F. Then we have

$$\operatorname{Fr} f(V_F) \subseteq f(B_F) \subseteq f(\operatorname{Fr} V),$$

(¹) Let $Q_0(x) = X$ and use transfinite induction to define $Q_\alpha(x)$ for each ordinal α, namely $Q_{\alpha+1}(x)$ is the quasi-component of the space $Q_\alpha(x)$ at the point x and $Q_\beta(x) = \bigcap_{\alpha<\beta} Q_\alpha(x)$ for limit β.

because $f|F$ is open and closed. Thus we obtain $y_0 \in f(V_F)$ and $\operatorname{Fr} f(V_F) \cap W = \emptyset$; it follows by connectedness of W that

$$f(V_F) \supseteq W .$$

We end the proof of the lemma by showing that

$$f(U) \supseteq W .$$

Take an arbitrary $y \in W$. Then for every F we have

$$\emptyset \neq f^{-1}(y) \cap V_F \subseteq f^{-1}(y) \cap \overline{V_F}$$

and the family $\{f^{-1}(y) \cap \overline{V_F}\}_{F \in \mathfrak{F}}$ has the finite intersection property, hence by compactness of $f^{-1}(y)$ also

$$\emptyset \neq f^{-1}(y) \cap \bigcap_{F \in \mathfrak{F}} \overline{V_F} \subseteq f^{-1}(y) \cap \overline{V} \cap A \subseteq f^{-1}(y) \cap U .$$

Now we derive the Theorem from the Lemma by induction. Take an arbitrary $x \in X$. For $\alpha = 0$ we have $Q_\alpha(x) = X$ and the theorem is obvious. Suppose it holds for $\alpha < \beta$. If $\beta = \gamma + 1$ then $Q_{\gamma+1}(x)$ is the intersection of the family $\mathfrak{F}$ of all open-closed subsets of $Q_\gamma(x)$ that contain x. Hence it suffices to use the Lemma. If β is a limit ordinal, then $Q_\beta(x) = \bigcap_{\alpha<\beta} Q_\alpha(x)$ and by virtue of the inductive assumption we can use the Lemma for $\mathfrak{F} = \{Q_\alpha(x)\}_{\alpha<\beta}$.

Corollary. *Let $f\colon X \twoheadrightarrow Y$ be an open, perfect and 0-dimensional mapping of a regular space X onto a connected and locally connected space Y. If there exists a point $y \in Y$ such that $f^{-1}(y)$ is metrizable, then for every $x \in X$ there exists $\alpha < \omega_1$ such that $Q_{\alpha+1}(x) = Q_\alpha(x)$.*

Proof. Take an arbitrary $x \in X$. For each ordinal α denote $F_\alpha = f^{-1}(y) \cap Q_\alpha(x)$. Thus we obtain the well ordered family $F_0 = f^{-1}(y) \supseteq F_1 \supseteq \ldots \supseteq F_\alpha \supseteq F_{\alpha+1} \supseteq \ldots$ of closed subsets of the compact, metrizable space $f^{-1}(y)$. Hence there exists ordinal $\alpha < \omega_1$ such that $F_\alpha = F_{\alpha+1}$ ([4], § 24, II, Theorem 2). We shall show that this is required ordinal. By Theorem the restriction $f|Q_\alpha(x) = g$ is an open-closed mapping onto Y. By Kombarov's remark it follows that $g^{-1}(y)$ intersects each quasi-component of the space $Q_\alpha(x)$. Suppose that $Q_\alpha(x) \setminus Q_{\alpha+1}(x) \ni x'$. Then $Q_{\alpha+1}(x') \cap g^{-1}(y) \neq \emptyset$ and $g^{-1}(y) = Q_\alpha(x) \cap f^{-1}(y) = F_\alpha = F_{\alpha+1} = Q_{\alpha+1}(x) \cap f^{-1}(y) = Q_{\alpha+1}(x) \cap g^{-1}(y)$. Hence $g^{-1}(y) \subseteq Q_{\alpha+1}(x)$ and we obtain a contradiction: $Q_{\alpha+1}(x) \cap Q_{\alpha+1}(x') \neq \emptyset$.

In particular, from the above remarks follows that a hereditarily disconnected space cannot be mapped onto locally connected space by open-perfect mapping.

2° In our example both of the spaces X and Y are 1-dimensional, hence they can be embedded into 3-dimensional Euclidean space. We do not know if it is possible to construct an example of this kind taking a subspace of the plane as Y. We do not know also whether Y would be the Knaster–Kuratowski Broom.

We are deeply gratefull to Professor R. Engelking for suggesting the problem. We are also indebted to Doctor J. Krasinkiewicz for interesting discussions about the subject of this paper.

Added in proof. Recently the second of the authors showed, modifying the present construction, that Y can be taken as a subspace of the plane. We have also proved that if we replace in the construction of Knaster–Kuratowski Broom the rational and irrational numbers of the x-axis by two disjoint subsets of irrationals of the second category, then we obtain the space Y with a dispersion point which is not an open-perfect image of any hereditarily disconnected space.

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The non-existence of Σ_2^1 well-orderings of the Cantor set

by

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Abstract. It is shown the existence of a Σ_2^1 well-ordering of the Cantor set implies that all reals are constructible. This is the converse of a theorem of Gödel.

Throughout this paper we assume the existence of a non-constructible real. With that in hand, let us set forth some notation. A finite sequence s is an extension of t if t is an initial subsequence of s. A tree is a set of finite sequences of 0's and 1's containing every initial subsequence and at least one proper extension of each of its members. For α a function with domain the set of non-negative integers, $\bar{\alpha}(n)$ is the sequence $\langle\alpha(0), \alpha(1), \ldots, \alpha(n-1)\rangle$. A path through the tree P is a function α such that $\bar{\alpha}(n)$ is in P for every n. $[P]$ is the set of paths through P. It is easily shown that $[P]$ is a closed subset of 2^N and that every closed subset of 2^N is the set of paths through a unique tree. The tree corresponding to a closed set is its code; a closed set with a constructible code is constructibly coded. A closed set is perfect iff every sequence in its code has at least two imcompatible extensions in the code.

Let B be the Boolean algebra corresponding to forcing with constructibly coded perfect sets ordered by the subset relation. B is a complete Boolean algebra containing the constructible trees as a dense subset. There are several ways to represent B; one is as the regular open sets in the space $2^N - L$ with the topology generated by the constructibly coded $[P]$'s.

We are going to be using B-valued set theory. In that set theory there is a canonical generic function S in 2^N. (In the system presented in [6], S is $\{\langle\check{n}, P\rangle: \forall s \in P\ [\text{length}\ (s) \leqslant n \vee s_n = 1]\}$.) We are also going to be using another Boolean extension of set theory M, in which every constructible tree P has a path generic over V with respect to B. The Truth Lemma [11] states that for α generic and φ a formula in the forcing language, $V(\alpha)$ satisfies φ iff there is a condition P with $\alpha \in [P]$ and $P \Vdash \varphi$. In interpreting the forcing language for $V(\alpha)$, S is a name for α. Thus if $\varphi(x)$ is a Σ_2^1 or Π_2^1 formula with possible unlisted constructible parameters