On fundamental deformation retracts
and on some related notions

by

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Abstract. The main result of this note is the following

Theorem. If \( Y_1 \supset Y_2 \supset \cdots \) is a sequence of fundamental deformation retracts of
a space \( X \), then the set \( Y = \bigcap_1^\infty Y_n \) is a fundamental deformation retract of \( X \).

Several notions, as unnamed to the F-stability, of the PR-stability and of the
S-stability of a compactum are introduced and also several problems formulated.

§ 1. Introduction. Among notions of the classical homotopy theory,
the notion of the stable space, due to H. Hopf and E. Pannwitz ([4],
p. 435) plays an essential role. A space \( X \) is said to be stable if for every
map \( f: X \to X \) homotopic to the identity map \( 1_X: X \to X \), the relation
\( f(X) = X \) holds true. In particular all closed, compact manifolds are
stable.

A closed set \( Y \subset X \) is said to be a homotopy support (or an H-support)
of \( X \) if there exists a map \( f: X \to X \) homotopic to \( 1_X \) and such that \( f(Y) \subset Y \).
Thus a stable space is the same as spaces \( X \) with any H-support different from \( X \). In particular, every deformation retract of \( X \) is an
H-support of \( X \), but not conversely. The question if every subset being
both a retract and an H-support of \( X \) is a deformation retract of \( X \) re-
mains open. It is clear that

(1.1) If \( Y \) is an H-support of \( X \) and \( Z \) is an H-support of \( Y \) then \( Z \) is
an H-support of \( X \).

If \( X \) does not contain any deformation retract of \( X \) different from \( X \),
then \( X \) is said to be R-stable. Notice that

(1.2) Every stable space is R-stable.

But the converse is not true, even in the case of polyhedra, as it
follows by the following well-known example due to H. Hopf and
E. Pannwitz ([4], p. 435):

(1.3) Example. Let \( L \) be an arc lying on a 2-sphere \( S \) and let \( a, b \)
denote its endpoints. Denote by \( Z \) the curvilinear polyhedron which one
obtains from $S$ by identifying the points $a$ and $b$. By this identification the arc $L$ passes onto a simple closed curve $C$ lying in $Z$. Let $A$ be a disk with the boundary $C$ and with the interior disjoint to $Z$. Setting
$$P = A \cup Z,$$
one gets a curvilinear polyhedron and it is known ([4], p. 447) that $Z$ is an $H$-support of $P$, hence $P$ is not stable. However $P$ is $R$-stable, because
none proper subset of $P$ is a deformation retract of $P$.

The concepts of the stable and of $R$-stable spaces have in the case of spaces with a rather regular local structure (as polyhedra, or as ANR-spaces) a clear intuitive sense. However another situation is when one considers spaces with a more complicated local structure. The aim of this note is to apply the notions of the theory of shape in order to obtain concepts which, for arbitrary compacts, seem to be reasonable substitutes for the classical concepts of the stable and of $R$-stable spaces.

§ 2. F-stable and FR-stable spaces. Let $X$ be a compactum lying in a space $M \in AR(3R)$. A compactum $Y \subset X$ is said to be a fundamental support (or an $F$-support) of $X$ (in $M$) if $X$ contains a fundamental sequence
$$f = (f_k, X, X)_{M, M}$$
such that
$$f_k = (f_k, X, X)_{M, M} \approx \xi_{x, M},$$
where $\xi_{x, M}$ denotes the fundamental identity sequence $\{t, t, t\}_{M, M}$.

Let us observe that the choice of the space $M \in AR(3R)$ containing $X$ is immaterial, that is,
\[(2.1) \text{If } Y \text{ is an } F \text{-support of } X \text{ in } M \text{ and if } X \subset N \subset AR(3R), \text{ then } Y \text{ is an } F \text{-support of } X \text{ in } N.
\]

In order to see it, consider two maps
$$a : M \to N \quad \text{and} \quad \beta : N \to M$$
such that $a(x) = \beta(x) = x$ for every point $x \in X$. Setting $a_k = a$ and $\beta_k = \beta$ for $k = 1, 2, \ldots$, we get two fundamental sequences
$$a = (a_k, X, X)_{M, N} \quad \text{and} \quad \beta = (\beta_k, X, X)_{M, N}.$$If $Y$ is an $F$-support of $X$ in $M$, then there is a fundamental sequence $f = (f_k, X, X)_{M, M}$ such that $(f_k, X, X)_{M, M} \approx \xi_{x, M}$. Setting
$$g(y) = a_k \beta(y) \quad \text{for every point } y \in N,$$one gets a fundamental sequence $g = (g_k, Y, Y)_{N, N}$. It remains to show that $(g_k, Y, Y)_{N, N}$ is homotopic to $\xi_{x, N}$. In order to do it, consider a neighborhood $W$ of $X$ in $N$. Since $X \subset AR(3R)$ and since $a_k x = x$, there exists a neighborhood $W_k \subset W$ of $X$ in $N$ such that
\[(2.2) \quad a_k \beta(W) \approx \xi(W) \quad \text{in } W.
\]
Now we can select a neighborhood $U$ of $X$ in $M$ such that
\[(2.3) \quad \alpha(U) \subset W_k.
\]Since $(f_k, X, X)_{M, M} \approx \xi_{x, M}$, there exists a neighborhood $U_k \subset U$ of $X$ in $M$ such that
\[(2.4) \quad f_k(U_k) \approx \xi(U_k) \quad \text{in } U \quad \text{for almost all } k.
\]We can assign to $U_k$ a neighborhood $V_k \subset W_k$ of $X$ in $N$ such that $\beta(V_k) \subset U_k$. One infers by $(2.3)$ and $(2.4)$ that
$$a_k \beta(V_k) \approx \xi(V_k) \quad \text{in } W \quad \text{for almost all } k,$$hence $(g_k, Y, Y)_{N, N} = (a_k \beta, X, X)_{M, M} \approx \xi_{x, N}$. Thus the proof of proposition $(2.1)$ is finished.

It follows that in the definition of the $F$-support of $X$ the words "in $M$" are superfluous. We can speak shortly on $F$-supports of $X$.

If there does not exist any $F$-support $Y \neq X$ of $X$, then $X$ is said to be fundamentally stable (or $F$-stable). Let us notice that
\[(2.5) \text{Every } H \text{-support of } X \text{ is an } F \text{-support of } X.
\]In fact, if $f : X \to X$ and $f(X) \subset X$ then there is a map $f : M \to M$ such that $f(x) = f(x)$ for every point $x \in X$. Setting $f_k = f$ for $k = 1, 2, \ldots$, one gets two fundamental sequences
$$f = (f_k, X, X)_{M, M} \quad \text{and} \quad (f_k, X, X)_{M, M}.$$If $f \approx \xi$ then $(f_k, X, X)_{M, M} \approx \xi_{x, M}$ and consequently $Y$ is an $F$-support of $X$. It follows that
\[(2.6) \text{If } X \text{ is } F \text{-stable, then } X \text{ is stable}.
\]Let us observe that
\[(2.7) \text{If } X \subset AR(3R) \text{ and } Y \neq X \text{ is an } F \text{-support of } X, \text{ then there is a compactum } Y' \neq X \text{ being an } H \text{-support of } X.
\]In fact, if $Y$ is an $F$-support of $X$ then there exists a fundamental sequence $f = (f_k, X, Y)_{M, M}$ such that $(f_k, X, Y)_{M, M} \approx \xi_{x, M}$. Since $X \subset AR(3R)$, there exists an open neighborhood $U$ of $X$ in $M$ and a retraction $r : U \to X$. We infer by $Y \neq X$ that there exists a neighborhood $V \subset U$ of $X$ in $M$ such that $r(Y) \neq X$. Moreover, there exists an index $k_0$ such that
\[(2.8) \quad f_{k_0}(X) \approx \xi(X) \quad \text{in } U \quad \text{and} \quad f_{k_0}(X) \subset V.
\]Setting $f(x) = f_k(x)$ for every point $x \in X$, we get a map $f : X \to X$. It follows by $(2.8)$ that $f = f_{k_0}(X) \approx \xi(X) = \xi(X)$ in the set $r(U) = X$. 


Moreover, \( f(X) = r(X) \cap V(X) \neq X \). Hence the set \( X' = f(X) \) is a required \( H \)-support of \( X \).

It follows by (2.6) and (2.7) that

(2.9) An ANR-space is stable if and only if it is \( F \)-stable.

A simple example of a continuum \( X \) which is stable but not \( F \)-stable is the closure of the diagram of the function

\[
y = \cos \frac{1}{x} + \cos \frac{1}{1-x} \quad \text{for} \quad 0 < x < 1.
\]

Let us recall that a compactum \( Y \subset X \) is said to be a fundamental deformation retract of \( X \) if there exists a fundamental sequence \( r = (r, Y, X)_{M,M} \) such that \( r \circ \bar{r} = Y \) and that \( (r, X, Y)_{M,M} \equiv x \).

Let us add that using the same standard argument as by the proof of proposition (2.1), one knows that in the definition of the fundamental deformation retracts of \( X \) the choice of the space \( M \cdot AR(\mathbb{R}) \) containing \( X \) is immaterial.

It is well known that

(2.10) If \( Y \) is a fundamental deformation retract of \( X \) then \( Sh(X) = Sh(Y) \).

Moreover

(2.11) Every fundamental deformation retract of \( X \) is an \( F \)-support of \( X \), but not conversely.

The question if every set being both a fundamental retract and an \( F \)-support of \( X \) is a fundamental deformation retract of \( X \) remains open.

If there does not exist any fundamental deformation retract \( Y \neq X \) of \( X \), then \( X \) is said to be fundamentally \( H \)-stable (or \( F \)-stable). It is clear that

(2.12) If \( X \) is \( H \)-stable, then \( X \) is \( R \)-stable.

Moreover, (2.11) implies that

(2.13) If \( X \) is \( F \)-stable, then \( X \) is \( F \)-stable.

The following problem remains open:

(2.14) Problem. Does there exist an \( R \)-stable ANR-space which is not \( F \)-stable?

§ 3. Some properties of \( F \)-supports. A preliminary information on shape properties of \( F \)-supports gives the following

(3.1) Theorem. If \( Y \) is an \( F \)-support of \( X \) then \( Sh(X) \leq Sh(Y) \).

Proof. Let \( f = (f, X, Y)_{M,M} \) be a fundamental sequence such that

\[
(f, X, Y)_{M,M} \simeq i_X \cdot M.
\]

Setting \( g = (i, Y, X)_{M,M} \) we get a fundamental sequence \( g \cdot Y \rightarrow X \) satisfying the condition

\[
gf = (f, X, X)_{M,M} = (f, X, X)_{M,M} \simeq i_X \cdot M.
\]

Hence \( Sh(X) \leq Sh(Y) \).

(3.2) Theorem. If \( Z \) is an \( F \)-support of an \( F \)-support \( Y \) of \( X \), then \( Z \) is an \( F \)-support of \( X \).

Proof. Let \( f = (f, X, Y)_{M,M} \) and \( g = (g, Y, Z)_{M,M} \) be fundamental sequences such that \( (f, X, Y)_{M,M} \simeq i_X \cdot M \) and \( (g, Y, Z)_{M,M} \simeq i_Y \cdot M \).

If \( U \) is a neighborhood of \( X \) in \( M \) then there exists a neighborhood \( U \cap V(X) \) such that

\[
gs \circ U \simeq i \circ U \quad \text{for almost all} \quad k.
\]

But \( U \) is also a neighborhood of \( Y \) in \( M \). Consequently, there exists a neighborhood \( V \cap U \) of \( Y \) in \( M \) such that

\[
bs \circ V \simeq i \circ V \quad \text{for almost all} \quad k.
\]

Moreover there exists a neighborhood \( U \subset V \cap U \) of \( X \) in \( M \) such that

\[
bs(U) \subset V \quad \text{for almost all} \quad k.
\]

It follows by (3.4) and (3.5) that

\[
bs(U) \simeq i \circ U \quad \text{for almost all} \quad k.
\]

Since \( U \subset U \), one infers by (3.3) that \( bs(U) \simeq i \circ U \) in \( U \) for almost all \( k \), which together with the inclusion \( U \subset U \) and with the relation (3.6) gives

\[
bs(U) \simeq i \circ U \quad \text{in} \quad U \quad \text{for almost all} \quad k.
\]

Hence \( (bs, X, Y)_{M,M} \) is a fundamental sequence homotopic to \( i_X \cdot M \).

Since \( (bs, X, Y)_{M,M} = g \) is a fundamental sequence, we conclude that \( Z \) is an \( F \)-support of \( X \).

By a slight modification of the last proof, one gets the following proposition:

(3.7) If \( Z \) is a fundamental deformation retract of \( Y \) and \( Y \) is a fundamental deformation retract of \( X \), then \( Z \) is a fundamental deformation retract of \( X \).

§ 4. Decreasing sequences of fundamental deformation retracts. Let us recall that a compactum \( X \) is said to be an \( FANR \)-space if for every compact space \( M \) containing \( X \) there exists a compact neighborhood \( U \) of \( X \) in \( M \) such that \( X \) is a fundamental retract of \( U \) (compare [11], p. 66).

The class of all \( FANR \)-spaces is a shape invariant. It contains, in particular,
all ANR-spaces and many other compacts, for instance all plane compacts with finite Betti numbers.

Let us prove the following

4.1 Hypothesis. If \( Y_1 \supseteq Y_2 \supseteq \cdots \) is a sequence of fundamental deformation retracts of a space \( X \times \text{ANR} \), then the set \( Y = \bigcap_{m=1}^{\infty} Y_m \) is a fundamental deformation retract of \( X \).

Proof. Assume that \( X \) lies in the Hilbert cube \( Q \). Then \( X \) has arbitrary small neighborhoods in \( Q \) being ANR-sets. Consequently, there exists a neighborhood \( U \times ANR \) of \( X \) in \( Q \) and a fundamental retraction

\[
\tilde{s} = (s_k, U, X)_{\alpha, \rho, \theta}.
\]

Let \( V_1 \supseteq V_2 \supseteq \cdots \) be a sequence of neighborhoods of \( X \) in \( Q \), lying in \( U \) and shrinking to \( X \). Replacing the sequence \( Y_1, Y_2, \ldots \) by a suitably selected subsequence, we may assume that

\[
V_m \text{ is a neighborhood of } X_m \text{ in } Q \text{ for every } m = 1, 2, \ldots
\]

Let

\[
r_m = (r_m, X, X_m)_{\alpha, \rho, \theta} \text{ be a fundamental deformation retraction; hence}
\]

\[
(r_m, X, X_m)_{\alpha, \rho, \theta} \simeq \tilde{s}_X, \quad \text{for every } m = 1, 2, \ldots
\]

By virtue of (4.4), one sees readily that there exists a decreasing sequence \( U_1 \supseteq U_2 \supseteq \cdots \) of neighborhoods of \( X \) in \( Q \), lying in \( U \) and shrinking to \( X \), such that

\[
\text{For every } m = 1, 2, \ldots \text{ the inclusion } r_m(U_m) \subseteq V_m \text{ holds true for almost all } k.
\]

Moreover, for every \( m = 1, 2, \ldots \) there is a neighborhood \( V_m \subseteq V_m \) of \( X \) in \( Q \) such that

\[
r_m(U_m) \simeq r_m(U_m) \text{ in } V_m \text{ for almost all } k.
\]

It follows by (4.6) and (4.7) that there is an increasing sequence of indices \( \alpha(1) < \alpha(2) < \cdots \) such that

\[
r_m(U_m) \subseteq V_m \text{ and } r_m(U_m) \simeq r_m(U_m) \text{ in } V_m \text{ for every } k \geq \alpha(m).
\]

Since \( r_m(U_m) = (r_m, U, X)_{\alpha, \rho, \theta} \), there exists a neighborhood \( \tilde{V}_m \subseteq V_m \) of \( X \) in \( Q \) such that \( \tilde{V}_m \subseteq \tilde{V}_m \) and that

\[
\tilde{V}_m \simeq (\tilde{r}_m, U, X)_{\alpha, \rho, \theta} \text{ in } V_m \text{ for every } m = 1, 2, \ldots
\]

It follows by (4.8) and (4.9) (because \( \tilde{V}_m \subseteq V_m \)) that

\[
r_m, \tilde{V}_m \simeq (\tilde{r}_m, U_m, \tilde{V}_m) \text{ in } V_m \text{ for every } k \geq \alpha(m).
\]

By virtue of (4.4) and (4.5) there exists for every \( m = 1, 2, \ldots \) a neighborhood \( U_m \subseteq U_m \) of \( X \) such that \( \tilde{V}_{m+1} \subseteq U_m \) and a sequence of indices \( \beta(1) < \beta(2) < \cdots \) such that \( \beta(m) \geq \alpha(m) \) for every \( m = 1, 2, \ldots \) and that

\[
r_m(U_m) \subseteq \tilde{V}_m \text{ and } r_m(U_m) \simeq r_m(U_m) \text{ in } V_m \text{ for every } k \geq \beta(m).
\]

Moreover, we infer by (4.3) that there exists a sequence of indices \( \gamma(1) < \gamma(2) < \cdots \) such that \( \gamma(m) \geq \beta(m) \) for every \( m = 1, 2, \ldots \) and that

\[
s_k(U) \simeq s_k(U_m) \text{ in } V_m \text{ for every } k \geq \gamma(m).
\]

Since \( s_k(U) \simeq (s_k, U, X)_{\alpha, \rho, \theta} \), we infer that

\[
s_k(U) \simeq (s_k, U, X)_{\alpha, \rho, \theta} \text{ in } V_m \text{ for every } k \geq \gamma(m).
\]

Setting

\[
r_m = r_m(U_m) \text{ for every } m = 1, 2, \ldots,
\]

we get a sequence of maps \( r_m : Q \to Q \) and we infer by (4.11), (4.12) and by the inequality \( \gamma(m) \geq \beta(m) \) that

\[
r_m(U) \simeq r_m(U_m) \text{ in } V_m \text{ for every } m = 1, 2, \ldots
\]

It follows by (4.13) that

\[
r_m(U) \simeq (s_k, U, X, X)_{\alpha, \rho, \theta} \text{ in } V_m \text{ for every } m = 1, 2, \ldots
\]

From (4.11) and (4.12), and since \( \tilde{V}_{m+1} \subseteq \tilde{V}_m \), we infer that

\[
r_m(U), r_{m+1}(U) \subseteq \tilde{V}_m \text{ for every } m = 1, 2, \ldots
\]

Now let us set

\[
c_m = r_m(U) \text{ for every } m = 1, 2, \ldots
\]

It follows by (4.10) and (4.8) that

\[
c_m \subseteq \tilde{V}_m \text{ in } V_m \text{ for every } m = 1, 2, \ldots,
\]

and

\[
c_m(U) \subseteq \tilde{V}_m \text{ for every } m = 1, 2, \ldots
\]

By virtue of (4.13) and (4.19) one infers that

\[
c_m(U) \subseteq \tilde{V}_m \text{ in } c_m(U_m) \subseteq V_m,
\]

\[
c_m(U) \subseteq \tilde{V}_m \text{ in } c_m(U_m) \subseteq V_m.
\]
Using (4.17) and (4.19), we infer that

\[
\omega_m r_m / U \simeq r_m / U \text{ in } V_m \quad \text{and} \quad \omega_m r_{m+1} / U \simeq r_{m+1} / U \text{ in } V_m
\]

for every \( m = 1, 2, \ldots \).

Moreover, (4.18), (4.20) and the inclusion \( U_m \subset U_m' \) imply that

\[
\omega_m r_m / U \simeq \omega m r_{m+1} / U \text{ in } V_m.
\]

It follows by (4.21), (4.22) and (4.23) that

\[
r_m / U \simeq r_{m+1} / U \text{ in } V_m \quad \text{for every } m = 1, 2, \ldots,
\]

hence \( r = \{ r_m, X, Y \}_\infty \) is a fundamental sequence. Moreover, (4.14) implies that \( r_m \subseteq Y \subseteq Y \) and we conclude that \( r \) is a fundamental retraction satisfying (4.16). Hence \( Y \) is a fundamental deformation retract of \( X \) and the proof of Theorem (4.1) is finished.

(4.24) PROBLEM. Does Theorem (4.1) remain true if we replace the hypothesis that \( X \in \mathcal{F} \) by a weaker one that \( X \) is a movable compactum?

(4.25) PROBLEM. Is it true that if \( X \in \mathcal{F} \) is a sequence of fundamental retraction of a space \( X \in \mathcal{F} \), then the set \( Y = \bigcap_{m=1}^\infty Y_m \) is a fundamental retract of \( X \)?

By an example due to C. Cox [34, p. 175] if we replace in this problem the hypothesis that \( X \in \mathcal{F} \) by a weaker one that \( X \) is movable, then the answer would be negative. In fact, consider in the Euclidean 3-space \( E^3 \) a sequence \( T_1, T_2, \ldots \) of topological tori such that \( T_m \) lies in the interior of \( T_{m+1} \) for every \( m = 1, 2, \ldots \) and that the set \( Y = \bigcap_{m=1}^\infty T_m \) is a solenoid of Van Dantzig. Let \( T_m \) denote the boundary of \( T_m \). Then the set

\[
X = Y \cup \bigcup_{m=1}^\infty T_m
\]

is a movable compactum (see [3], p. 140) and

\[
Y_k = Y \cup \bigcup_{m=1}^k T_m
\]

is a retract (hence also a fundamental retract) of \( X \) and \( Y_{k+1} \subset Y_k \) for every \( k = 1, 2, \ldots \) However the set \( Y = \bigcap_{k=1}^\infty Y_k \), being a non-movable compactum, is not a fundamental retract of \( X \).

Using Theorems (3.7) and (4.1), one gets the following

(4.26) COROLLARY. If \( Y \in \mathcal{F} \) is a fundamental deformation retract of \( X \) for \( m = 1, 2, \ldots \), then the set \( Y = \bigcap_{m=1}^\infty Y_m \) is a fundamental deformation retract of \( X \).

Using the well-known Brouwer Reduction Theorem ([5], p. 161), we obtain from Theorem (4.1) the following

(4.27) COROLLARY. For every space \( X \in \mathcal{F} \) there exists an \( F \)-stable compactum \( Y \in \mathcal{F} \) being a fundamental deformation retract of \( X \).

It follows, in particular

(4.28) COROLLARY. Every space \( X \in \mathcal{F} \) contains an \( F \)-stable compactum \( Y \in \mathcal{F} (X) \).

(4.29) PROBLEM. Is it true that for every sequence \( X \in \mathcal{F} \) of \( F \)-supports of a compactum \( X \) the set \( Y = \bigcap_{m=1}^\infty Y_m \) is an \( F \)-support of \( X \)?

(4.30) PROBLEM. Do Corollaries (4.27) and (4.28) remain true if we assume only that \( X \) is a compactum?

§ 5. S-stable compacta. Let us say that a compactum \( X \) is shape-stable (or is \( S \)-stable) if \( \mathcal{F} (X) \neq \mathcal{F} (Y) \) for every compactum \( Y \subseteq X \).

For instance, each closed manifold is \( S \)-stable. Also every continuum \( X \subseteq E^3 \) decomposing the space \( E^3 \) and being the common boundary of each component of the set \( E^3 \setminus X \) is \( S \)-stable. Also the polyhedron \( P \) of H. Hopf and E. Pannwitz, mentioned in § 1, is \( S \)-stable (though it is not stable), because one sees easily that the shape of every compactum \( X \subseteq P \) differs from \( \mathcal{F} (P) \).

It is clear that every \( S \)-stable compactum is \( F \)-stable, but the converse is not true. In fact, if \( X \) is the union of all circles \( C_k, k = 1, 2, \ldots \), given in the plane \( E^2 \) by the equations

\[
(a - k)^2 + k^2 = k^2,
\]

then \( X \) is \( F \)-stable (it is even \( F \)-stable), but it is not \( S \)-stable. Moreover one sees readily that \( X \) does not contain any \( S \)-stable compactum \( X \in \mathcal{F} (X) \). However there exists an \( S \)-stable plane continuum \( X_0 \) such that \( \mathcal{F} (X) = \mathcal{F} (X_0) \). In fact, this property has each continuum \( X_0 \) decomposing \( E^2 \) into \( k \) regions and being the common boundary of each of these regions. Let us also observe that for Cantor discontinuum \( D \) there does not exist any \( S \)-stable compactum \( D \in \mathcal{F} (D) \).

The following problems remain open:
(5.1) Problem. Does there exist for every continuum $X$ an $S$-stable continuum $X_p \in \text{Sh}(X)$?

(5.2) Problem. Is it true that $S$-stable FANR-spaces are the same as $FB$-stable FANR-spaces?

References

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An open-perfect mapping of a hereditarily disconnected space onto a connected space

by

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Abstract. In this paper the authors construct an example of open-perfect mapping of a hereditarily disconnected space onto a connected space. Both of the spaces are metrizable and separable.

The aim of this paper is to construct the hereditarily disconnected space $X$ and the open-perfect mapping $f$ of $X$ onto the connected space $Y$. Both of the spaces will be separable metrizable. We should mention here that 0-dimensionality and totally disconnectedness are invariants under open-closed mappings in the class of metrizable spaces. The first of these facts is obvious, the second was proved by Komborov [3]. The basis of our construction is the well-known Knaster-Kuratowski Broom, the subspace of the plane with dispersion point (see [4], § 46, II). At the end of this paper we shall prove the theorem related to this object.

Terminology and notation are as in [2] and [4]. In particular, the word “mapping” and the symbol $g: A \rightarrow B$ mean continuous mapping, the symbol $g: A \rightarrow B$ means that $g(A) = B$. By connected space we always understand non-one-point space. For $x \in X$ and $A \subseteq X$ the symbol $x \times A$ instead of $(x) \times A$.

1. We start from some auxiliary constructions in the Euclidean plane $\mathbb{E}^2$ with the standard metric $\rho$. The symbol $\mathbb{E}$ denotes the set of all real numbers, $\mathbb{N}$ denotes the set of non-negative integers, $I$ is the interval $[-1, 1]$ of reals, $P$ and $Q$ irrationals and rationals of $I$ respectively. For $t \in \mathbb{E}$, let $l = (t, 0) \in \mathbb{E}^3$.

Divide the set $P$ into two disjoint, dense in $P$ sets $P'$ and $Q'$ such that $Q' = \mathbb{N}$. Let $M = P' \times P \cup Q' \times Q$.

For $a = (a_1, a_2) \in \mathbb{E}^2$ and real number $a > 0$ we set

$U(x, a) = \{y \in \mathbb{E}^2 | \rho(y, x') < a, \quad \text{where} \quad x' = (a_1, a_2 + a) \cup (a)\}$,

and

$K(x, a) = \{y \in \mathbb{E}^2 | \rho(y, x) < a\}$.
