Normal radicals of endomorphism rings of free and projective modules *

by

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Abstract. A radical N in a class of associative rings is normal if, for every Morita context, it satisfies a condition stated in Amitsur's paper "Rings of quotients and Morita contexts", (J. Algebra 17 (1971), pp. 273–298) for the radicals of Baer, Levitzki and Jacobson (see also — M. Jaegermann, "Morita contexts and radicals", Bull. Acad. Polon. Sci. 20 (1972), pp. 619–623 and A. D. Sands, "Radicals and Morita contexts", J. Algebra 24 (1973), pp. 335–345).

For every ring R we denote by R_I the ring of $I\times I$-matrices with only a finite number of non-zero entries from R in each row, by $\langle R_I, b\rangle$ — the ring of matrices with a finite number of non-zero columns, and by $\langle R_I, f\rangle$ — the ring of matrices with only a finite number of non-zero entries in each matrix.

It is proved that for every normal radical N we have: $N(R_I)\subseteq N(R)_I$ with an equality for finite I, $N\langle R_I, b\rangle \subseteq \langle (N(R)_I, b\rangle$ with an equality for supernilpotent normal radicals, and $N\langle R_I, f\rangle = \langle N(R)_I, f\rangle$. Moreover all these radicals are, in some sense, dense in a ring $N(R)_I$. We have strictly similar results for rings of endomorphisms of projective R-modules.

As applications, a short proof of the Ware–Zelmanowitz description of the Jacobson radical of a ring of endomorphisms of a projective module and a new equivalent version of the Koethe problem are given.

Let $\mathcal{N}$ be a radical property in the class of associative rings, and let $N(R)$ denote an $\mathcal{N}$-radical of a ring R. A property $\mathcal{N}$ is called a *normal radical property* if for every Morita context (R, V, W, S) we have

$$(V, N(S)W)\subseteq N(R), \quad \text{or equivalently} \quad [W, N(R)V]\subseteq N(S),$$

where R, S are rings, V is an R-S-bimodule, and W is an S-R-bimodule. For the definition of a Morita context and for the notation we refer to [2] and [5]. As was proved in [5] and [9], many of the classical radicals are normal. In particular, normal are the radicals of Baer, Levitzki and Jacobson (cf. [2]).

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For every ring R we denote by R_I the ring of matrices indexed by I with entries from R (i.e., functions from $I\times I$ to R) with only a finite number of non-zero entries occurring in each row. Operations are defined in the usual manner. By the ring of row-bounded matrices $\langle R_I, b\rangle$ we understand a two-sided ideal of R_I which consists of those matrices which have only a finite number of non-zero columns. The ring of finite matrices $\langle R_I, f\rangle$ contains only those matrices from R_I which contain only a finite number of non-zero entries.

For a normal property $\mathcal{N}$ we shall prove $N(R_I)\subseteq N(R)_I$, $N\langle R_I, b\rangle = \langle R_I, b\rangle \cap N(R_I) \subseteq \langle N(R)_I, b\rangle$ (with equalities for some additional conditions), $N\langle R_I, f\rangle = \langle N(R)_I, f\rangle$ and $N\langle R_I, f\rangle \subseteq N\langle R_I, b\rangle \subseteq N(R_I)$. This generalizes the results of E. M. Patterson [8] and A. D. Sands [9].

In the second part we shall prove analogues of our theorems for rings of endomorphisms of projective modules. As an easy application we shall give a short proof of R. Ware's and J. Zelmanowitz's description of the Jacobson radical of the endomorphism ring of a projective module and show a new equivalent version of the problem of Koethe (see [6], [7], [9]).

The definitions and properties of radicals are to be found in [4]. We shall call a radical-property $\mathcal{R}$ *supernilpotent* if no $\mathcal{R}$-semisimple ring contains non-zero nilpotent ideals. We shall extensively use the fact that a $\mathcal{P}$-radical of an ideal A of a ring R is a $\mathcal{P}$-ideal of R (cf. [3] and [4], Theorem 47) and hence $P(A)\subseteq P(R)$. An ideal always means a two-sided ideal.

Given a ring R, let $R^{\#}$ denote the ring R if R has an identity element, and let $R^{\#}$ denote the usual extension of R to a ring with identity by the ring of integers $\boldsymbol{Z}$ in another case. Now it is convenient to regard the matrix rings described above as the rings of endomorphisms, acting on the right, of a free $R^{\#}$-module F with a basis $\{e_i|\ i\in I\}$. Namely, if a is a matrix then $e_i a = \sum_j a(i,j)e_j$ where $(i,j)\in I\times I$.

For any ring R, we denote by R^+ a ring which has the same additive group as R and zero multiplication.

1. Normal radicals of matrix rings.

Lemma 1.1. *Let $\mathcal{N}$ be a radical property. If $N(\boldsymbol{Z}^{\#})\neq 0$ then the property $\mathcal{N}$ is supernilpotent.*

Proof. If $N(\boldsymbol{Z}^+)\neq 0$ then the additive group of $N(\boldsymbol{Z}^+)$ is an infinite cyclic group. Thus every cyclic group with zero multiplication which is a homomorphic image of $N(\boldsymbol{Z}^+)$ is an $\mathcal{N}$-radical ring.

Now, let R be an $\mathcal{N}$-semi-simple ring and let I be a nilpotent ideal of R. We may assume $I^2=0$. If some $a\in I$, then a zero-ring on a cyclic group of a is an $\mathcal{N}$-ideal of I, and so $a\in N(I)$. But $N(I)$ is an $\mathcal{N}$-ideal of an $\mathcal{N}$-semi-simple ring R. Hence $a=0$ and $I=0$. This means that the property $\mathcal{N}$ is supernilpotent. ∎

Lemma 1.2. *If A is an ideal of a ring R and $N(R/A)=0$, then $N(A) = N(R)$ for every radical property $\mathcal{N}$.*

Proof. Obvious. ∎

Theorem 1.3. *Let $\mathcal{N}$ be a normal radical property. If $N(\boldsymbol{Z})\neq 0$ then also $N(\boldsymbol{Z}^+)\neq 0$, and so property $\mathcal{N}$ is supernilpotent.*

Proof. Let us assume that $N(\boldsymbol{Z})\neq 0$. We can consider the Morita context

$$(\boldsymbol{Z}, \boldsymbol{Z}, \boldsymbol{Z}, \boldsymbol{Z}^+)$$

with products $(x,y)=0\in \boldsymbol{Z}$ and $[x,y]=xy\in \boldsymbol{Z}^+$, where $x\in\boldsymbol{Z}$ and $y\in\boldsymbol{Z}$. The structures of $\boldsymbol{Z}^+$-modules are determined uniquely. The property $\mathcal{N}$ is normal, and so we have

$$0\neq N(\boldsymbol{Z})^+ = \big(\boldsymbol{Z}N(\boldsymbol{Z})\boldsymbol{Z}\big)^+ = [\boldsymbol{Z}, N(\boldsymbol{Z})\boldsymbol{Z}]\subseteq N(\boldsymbol{Z}^+)\,.$$

Hence $N(\boldsymbol{Z}^+)\neq 0$, and the property is supernilpotent by Lemma 1.1. ∎

Theorem 1.4. *Let $\mathcal{N}$ be a normal radical property. If R is an $\mathcal{N}$-ring, then R^+ is also an $\mathcal{N}$-ring.*

Proof. If the property $\mathcal{N}$ is supernilpotent, then every ring with zero multiplication is $\mathcal{N}$-radical. Thus, by Lemma 1.1, we may assume that $N(\boldsymbol{Z}^+)=0$. Hence $N(\boldsymbol{Z})=0$ by Theorem 1.3.

Since R is an ideal in R and $R^{\#}/R$ is either 0 or $\boldsymbol{Z}$, it follows from Lemma 1.2 that $N(R)=N(R^{\#})$. Similarly $N(R^+)=N(R^{\#+})$. Now let us assume $R=N(R)$. Obviously $R^+=N(R)^+$.

We have the context

$$(R^{\#}, R^{\#}, R^{\#}, R^{\#+})$$

with $(x,y)=0\in R^{\#}$ and $[x,y]=xy\in R^{\#+}$, $x\in R^{\#}$ and $y\in R^{\#}$. $\mathcal{N}$ is a normal property, and so we obtain

$$N(R^{\#})^+ = \big(R^{\#}N(R^{\#})R^{\#}\big)^+ = [R^{\#}, N(R^{\#})R^{\#}]\subset N(R^{\#+})\,.$$

This and the equalities above imply

$$R^+ = N(R)^+ = N(R^{\#})^+ \subset N(R^{\#+}) = N(R^+)\,,$$

but this means $R^+ = N(R^+)$. ∎

Lemma 1.5. *Let $\mathcal{N}$ be a normal radical property and let R be any ring. Then*

(i) $$N(R) = N(R^{\#})\cap R\,,$$

(ii) $$N(R_I) = N(R_I^{\#})\cap R_I\,,$$

(iii) $$\mathcal{N}\langle R_I, b\rangle = \mathcal{N}\langle R_I^{\#}, b\rangle \cap R_I,$$

(iv) $$\mathcal{N}\langle R_I, f\rangle = \mathcal{N}\langle R_I^{\#}, f\rangle \cap R_I.$$

Proof. If R is a ring with an identity element, then $R = R^{\#}$ and all the equalities are obvious.

If $\mathcal{N}(Z^+) \neq 0$, then the property $\mathcal{N}$ is supernilpotent by Lemma 1.1. From Theorem 2 of [5] it follows that then the $\mathcal{N}$-radical of an ideal of some ring is an intersection of the ideal and a radical of the ring. Let us observe that $R, R_I, \langle R_I, b\rangle$ and $\langle R_I, f\rangle$ are ideals of $R^{\#}, R_I^{\#}, \langle R_I^{\#}, b\rangle$ and $\langle R_I^{\#}, f\rangle$, respectively. Moreover, $\mathcal{N}\langle R_I^{\#}, b\rangle \cap \langle R_I, b\rangle = \mathcal{N}\langle R_I^{\#}, b\rangle \cap R_I$ and $\mathcal{N}\langle R_I^{\#}, f\rangle \cap \langle R_I, f\rangle = \mathcal{N}\langle R_I^{\#}, f\rangle \cap R_I$. This and the remarks above immediately imply all the equalities.

Now let us assume that $\mathcal{N}(Z^+) = 0$. One can represent, in an obvious way, all the rings $Z_I^+ = (Z^+)_I$, $\langle Z_I, b\rangle^+ = \langle (Z^+)_I, b\rangle$, $\langle Z_I, f\rangle^+ = \langle (Z^+)_I, f\rangle$ as a subdirect sum of $I \times I$ copies of an $\mathcal{N}$-semisimple ring Z^+. So all these rings are $\mathcal{N}$-semisimple, and by Theorem 1.4, Z, Z_I, $\langle Z_I, b\rangle$ and $\langle Z_I, f\rangle$ are also $\mathcal{N}$-semisimple. Hence Lemma 1.2 implies that $\mathcal{N}(R) = \mathcal{N}(R^{\#})$, $\mathcal{N}(R_I) = \mathcal{N}(R_I^{\#})$, $\mathcal{N}\langle R_I, b\rangle = \mathcal{N}\langle R_I^{\#}, b\rangle$ and $\mathcal{N}\langle R_I, f\rangle = \mathcal{N}\langle R_I^{\#}, f\rangle$, which is more than has been stated in the lemma. ■

THEOREM 1.6. *If $\mathcal{N}$ is a normal radical property, then*

$$\mathcal{N}(R_I) \subseteq \mathcal{N}(R)_I$$

for every ring R.

Proof. Let F be a free left $R^{\#}$-module with a basis $\{e_i \mid i \in I\}$ and let $w_j \in \mathrm{Hom}(F, R^{\#})$ be a homomorphism such that $e_j w_j = 1$ and $e_i w_j = 0$ for $i \neq j$. Let us consider the Morita context

$$\big(R^{\#}, F, \mathrm{Hom}(F, R^{\#}), R_I^{\#}\big)$$

where $(x, w) = xw \in R_I^{\#}$ and $(*)[w, x] = (*w)x \in R_I^{\#}$, for $x \in F$ and $w \in \mathrm{Hom}(F, R^{\#})$. Since the property $\mathcal{N}$ is normal, we have

$$\big(F, \mathcal{N}(R_I^{\#})\mathrm{Hom}(F, R^{\#})\big) \subseteq \mathcal{N}(R^{\#}).$$

Thus for every matrix $a \in \mathcal{N}(R_I^{\#})$ every entry $a(i,j) = e_i a w_j = (e_i, a w_j)$ belongs to $\mathcal{N}(R^{\#})$. This means that

$$\mathcal{N}(R_I^{\#}) \subseteq \mathcal{N}(R^{\#})_I.$$

But by Lemma 1.5 (ii) and (i) we have $\mathcal{N}(R_I) = \mathcal{N}(R_I^{\#}) \cap R_I$ and $\mathcal{N}(R)_I = \big(\mathcal{N}(R^{\#}) \cap R\big)_I = \mathcal{N}(R^{\#})_I \cap R_I$. Hence

$$\mathcal{N}(R_I) \subseteq \mathcal{N}(R)_I. \ ■$$

An example of the Jacobson radical, which is normal [2], shows us that generally $\mathcal{N}(R_I) \not\subseteq \mathcal{N}(R)_I$. But we have

THEOREM 1.7. *If $\mathcal{N}$ is a normal radical property and I is a finite set, then*

$$\mathcal{N}(R_I) = \mathcal{N}(R)_I$$

for every ring R.

A similar theorem was proved by S. A. Amitsur [1] for right strong and right hereditary supernilpotent radical properties (for the definitions see [5]) and proved again by A. D. Sands [9] by means of Morita contexts. In this result it is possible to omit the assumptions of the supernilpotency and associativity of rings. The last remark is due to J. Krempa.

Proof. Let us consider the same Morita context as in Theorem 1.6. One can easily observe that

$$[\mathrm{Hom}(F, R^{\#}), \mathcal{N}(R^{\#})F] = \mathcal{N}(R^{\#})_I$$

for a finite set I. But by the normality of the property $\mathcal{N}$ we have $[\mathrm{Hom}(F, R^{\#}), \mathcal{N}(R^{\#})F] \subseteq \mathcal{N}(R_I^{\#})$. Hence $\mathcal{N}(R^{\#})_I \subseteq \mathcal{N}(R_I^{\#})$. Lemma 1.5 then implies $\mathcal{N}(R)_I \subseteq \mathcal{N}(R_I)$, and so by Theorem 1.6 we obtain the equality. ■

Let R be some ring and let ε_U be a matrix from $\langle R_I^{\#}, b\rangle$ such that $\varepsilon_U(u, u) = 1$, for u from some finite subset U of I, and $\varepsilon_U(i,j) = 0$, for all other $(i,j) \in I \times I$. We define

$$\langle \mathcal{N}(R_I), b\rangle = \bigcup \mathcal{N}(R_I)\varepsilon_U,$$

where the union is taken over all finite subsets U of I. This means that $\langle \mathcal{N}(R_I), b\rangle$ consists of those matrices from $\langle R_I, b\rangle$ which we can complete to matrices from $\mathcal{N}(R_I)$. In this notation we can formulate

THEOREM 1.8. *If $\mathcal{N}$ is a normal radical property then*

(i) $\mathcal{N}\langle R_I, b\rangle = \langle \mathcal{N}(R_I), b\rangle = \mathcal{N}(R_I) \cap \langle R_I, b\rangle \subseteq \langle \mathcal{N}(R)_I, b\rangle$;

(ii) *if, moreover, the property $\mathcal{N}$ is supernilpotent then*

$$\mathcal{N}\langle R_I, b\rangle = \langle \mathcal{N}(R)_I, b\rangle$$

for every ring R.

Proof. (i) Since $\langle R_I^{\#}, b\rangle$ is an ideal of $R_I^{\#}$, the four-tuple

$$(\langle R_I^{\#}, b\rangle, R_I^{\#}, \langle R_I^{\#}, b\rangle, R_I^{\#}),$$

with multiplications in $R_I^{\#}$ as products, is a Morita context. The property $\mathcal{N}$ is normal. Therefore

$$R_I^{\#}\mathcal{N}(R_I^{\#})\langle R_I^{\#}, b\rangle = \big(R_I^{\#}, \mathcal{N}(R_I^{\#})\langle R_I^{\#}, b\rangle\big) \subseteq \mathcal{N}\langle R_I^{\#}, b\rangle.$$

But the ring $R_I^{\#}$ has identity, and $\langle R_I^{\#}, b\rangle$ contains elements ε_U for every finite subset U of I. Thus

$$\mathcal{N}(R)_I^{\#} \cap \langle R_I^{\#}, b\rangle \subseteq \bigcup \mathcal{N}(R_I^{\#})\varepsilon_U \subseteq R_I^{\#}\mathcal{N}(R_I^{\#})\langle R_I^{\#}, b\rangle$$
$$\subseteq \mathcal{N}\langle R_I^{\#}, b\rangle \subseteq \mathcal{N}(R_I^{\#}) \cap \langle R_I^{\#}, b\rangle,$$

where the union is taken over all finite subsets U of I. The last inclusion holds because $\langle R_I^{\#}, b\rangle$ is an ideal of $R_I^{\#}$.

Particularly, this implies that

$$\langle N(R_I^{\#}), b\rangle \cap R_I = \bigcup N(R_I^{\#})\varepsilon_U \cap R_I \subseteq N(R_I^{\#}) \cap R_I = N(R_I),$$

i.e., that we can complete every matrix from $\langle (R_I^{\#}), b\rangle \cap R_I$ to a matrix from $N(R_I)$, and so $\langle N(R_I^{\#}), b\rangle \cap R_I = \langle N(R_I), b\rangle$.

Furthermore, the inclusions above give us

$$N\langle R_I^{\#}, b\rangle = \langle N(R_I^{\#}), b\rangle = \bigcup N(R_I^{\#})\varepsilon_U = N(R_I^{\#}) \cap \langle R_I^{\#}, b\rangle .$$

Taking intersections with R_I, we obtain by Lemma 1.5 and the remark above

$$N\langle R_I, b\rangle = \langle N(R_I), b\rangle = N(R_I) \cap \langle R_I, b\rangle.$$

This is contained in $\langle N(R)_I, b\rangle$ since $N(R_I) \subseteq N(R)_I$.

(ii). This was proved by A. D. Sands [9]. ∎

We cannot prove (ii) without the assumption of supernilpotency, as is shown by the following example. A ring S has a property $\mathcal{C}$ if the additive group of S is a torsion group. One can check that $\mathcal{C}$ is a normal radical property. By I we denote the set of natural numbers. Let R be a direct sum of all rings $\mathbf{Z}_n$, where $n \in I$. Of course $T(R) = R$. A matrix $\alpha \in \langle R_I, b\rangle = \langle T(R)_I, b\rangle$ such that $\alpha(i, 1)$ is an identity element of $\mathbf{Z}_i$ and all the other entries are zero has an infinite additive rank. Thus $\alpha \notin T\langle R_I, b\rangle$. This means that $T\langle R_I, b\rangle \not\subseteq \langle T(R)_I, b\rangle$.

In the sequel we need the following generalization of a well-known property of the Jacobson radical.

THEOREM 1.9. *If $\mathcal{N}$ is a normal radical property and $e = e^2$ is an idempotent in a ring R, then*

$$N(eRe) = eN(R)e = N(R) \cap eRe .$$

Proof. It is easy to observe that for every subset X of the ring R we have

$$X \cap eRe = eXe . \tag{1.1}$$

Thus the second equality is a special case of (1.1). To prove the first equality let us consider the Morita context (R, Re, eR, eRe) with multiplications in R as products. Since $\mathcal{N}$ is normal, we obtain

$$(Re, N(eRe)eR) = ReN(eRe)eR \subseteq N(R)$$

and

$$[eR, N(R)Re] = eRN(R)Re \subseteq N(eRe) .$$

Hence

$$N(eRe) = e^3N(eRe)e^3 \subseteq eReN(eRe)eRe \subseteq eN(R)e$$

and

$$eN(R)e = e^2N(R)e^2 \subseteq eRN(R)Re \subseteq N(eRe) .$$

This implies

$$N(eRe) = eN(R)e . \;∎$$

THEOREM 1.10. *If $\mathcal{N}$ is a normal radical property then*

$$N\langle R_I, f\rangle = \langle N(R)_I, f\rangle$$

for every ring R.

Proof. From Lemma 1.5 it follows that without loss of generality we may assume that a ring R has identity. Let us write $S = \langle R_I, f\rangle$. For every finite subset U of I let us define a matrix $\varepsilon_U \in S$ putting $\varepsilon(u, u) = 1$, for $u \in U$, and $\varepsilon(i, j) = 0$, for all the other $(i, j) \in I \times I$. We have $\varepsilon_U = \varepsilon_U^2$ and Theorem 1.9 implies that $N(\varepsilon_U S\varepsilon_U) = N(S) \cap \varepsilon_U S\varepsilon_U$. On the other hand, $N(\varepsilon_U S\varepsilon_U) = \varepsilon_U N(R)_I \varepsilon_U = \varepsilon_U\langle N(R)_I, f\rangle\varepsilon_U$, which follows from Theorem 1.7, since U is a finite set and $\varepsilon_U S\varepsilon_U \simeq R_U$. Thus

$$N\langle R_I, f\rangle = N(S) = \bigcup\big(N(S) \cap \varepsilon_U S\varepsilon_U\big)$$
$$= \bigcup \varepsilon_U\langle N(R)_I, f\rangle\varepsilon_U = \langle N(R)_I, f\rangle,$$

where the union is taken over all finite subsets U of I. ∎

Let R be any ring. Recall that a ring R_I acts on the right on a free left $R^{\#}$-module F with a basis $\{e_i|\ i \in I\}$. We shall say that a subset X of R_I is dense in a subring $N(R)_I$ if for every finitely generated submodule G of F and every matrix $\alpha \in N(R)_I$ there exists a $\beta \in X$ such that $G(\alpha - \beta) = 0$.

THEOREM 1.11. *Let $\mathcal{N}$ be a normal radical property. Then, for every ring R,*

$$N\langle R_I, f\rangle \subseteq N\langle R_I, b\rangle \subseteq N(R_I) \subseteq N(R_I),$$

and the radicals $N\langle R_I, f\rangle$, $N\langle R_I, b\rangle$ and $N(R_I)$ are dense in $N(R)_I$.

Proof. The second and the third inclusion were proved in Theorems 1.8 and 1.6. So we have only to prove that $N\langle R_I, f\rangle \subseteq N\langle R_I, b\rangle$ and that the radicals are dense.

Let $\varepsilon_U \in \langle R_I^{\#}, f\rangle$ be matrices defined as before for all finite subsets U of I. Since $N\langle R_I, f\rangle = \langle N(R)_I, f\rangle$, we have

$$\varepsilon_U N\langle R_I, f\rangle\varepsilon_U = \varepsilon_U\langle N(R)_I, f\rangle\varepsilon_U = \varepsilon_U\langle N(R)_I, b\rangle\varepsilon_U .$$

The last equality holds because U is a finite set. Furthermore

$$\varepsilon_U\langle N(R)_I, b\rangle\varepsilon_U \simeq N(R)_U ;$$

thus Theorem 1.7 gives us $\varepsilon_U\langle N(R)_I, b\rangle\varepsilon_U = N(\varepsilon_U\langle R_I, b\rangle\varepsilon_U)$. Taking into account that $\varepsilon_U\langle R_I, b\rangle\varepsilon_U$ is an ideal of $\varepsilon_U\langle R_I^{\#}, b\rangle\varepsilon_U$, we obtain, by Theorem 1.9 and Lemma 1.5,

$$N(\varepsilon_U\langle R_I, b\rangle\varepsilon_U) \subseteq N(\varepsilon_U\langle R_I^{\#}, b\rangle\varepsilon_U) \cap R_I = N\langle R_I^{\#}, b\rangle \cap \varepsilon_U\langle R_I^{\#}, b\rangle\varepsilon_U \cap R_I$$
$$\subseteq N\langle R_I^{\#}, b\rangle \cap R_I = N\langle R_I, b\rangle.$$

Hence

$$\varepsilon_U N\langle R_I, f\rangle\varepsilon_U = \varepsilon_U\langle N(R)_I, b\rangle\varepsilon_U$$
$$= N(\varepsilon_U\langle R_I, b\rangle\varepsilon_U) \subseteq N\langle R_I, b\rangle,$$

for every finite subset U of I. This implies that

$$\bigcup \varepsilon_U N\langle R_I, f\rangle\varepsilon_U = N\langle R_I, f\rangle \subseteq N\langle R_I, b\rangle,$$

where the union is taken over all finite subsets U of I.

We cannot replace any inclusion in our Theorem by equality, which is shown by the example of the Jacobson radical (cf. [8] and [10]).

To complete the proof we have only to prove that $N\langle R_I, f\rangle$ is a dense set in $N(R)_I$. Let G be a finitely generated submodule of a free $R^{\#}$-module F and let a matrix $a \in N(R)_I \subseteq N(R^{\#})_I$. Obviously a submodule $G + Ga$ is also finitely generated. We fix a finite set of generators and define: an index $i \in I$ belongs to a set U if some generator of $G+Ga$ has a non-zero ith coordinate in $\{e_i| \ i \in I\}$ a basis of F. Let ε_U be again a matrix with $\varepsilon_U(u, u) = 1$ for $u \in U$, and $\varepsilon_U(i, j) = 0$ otherwise. For every $g \in G + Ga$ we have: $g\varepsilon_U = g$. Thus for every $x \in G$ we obtain $xa - x\varepsilon_U a\varepsilon_U = xa - (xa)\varepsilon_U = xa - xa = 0$. This means $G(a - \varepsilon_U a\varepsilon_U) = 0$. Moreover, $\varepsilon_U a\varepsilon_U \in \varepsilon_U N(R)_I\varepsilon_U \subseteq \langle N(R)_I, f\rangle = N\langle R_I, f\rangle$, by Theorem 1.10. Hence $N\langle R_I, f\rangle$ and the remaining radicals are dense subsets of $N(R)_I$. ■

2. Normal radicals of endomorphism rings of projective modules. In the sequel we shall assume that rings have identities and modules are unitary.

A left R-module V is a *direct summand* of an R-module W if and only if there exists a $\Delta \in \mathrm{Hom}_R(W, W)$ such that $\Delta^2 = \Delta$ and $V \simeq W\Delta$. It is easy to observe that the rings $\mathrm{Hom}_R(V, V)$ and $\Delta\mathrm{Hom}_R(W, W)\Delta$ are ring-isomorphic. Then for our purpose we may identify V with $W\Delta$ and $\mathrm{Hom}(V, V)$ with $\Delta\mathrm{Hom}(W, W)\Delta$, putting $a = \Delta a\Delta$ for $a \in \mathrm{Hom}(V, V)$.

A left R-module V is *projective* if and only if V is a direct summand of every such module W that V is an epimorphic image of W. Thus for a projective module V with a set of generators $\{v_i| \ i \in I\}$ there exists a free module F with a basis $\{e_i| \ i \in I\}$ (with the same set of indexes) and $\Delta = \Delta^2 \in \mathrm{Hom}(F, F)$ such that V is isomorphic with F. A couple (F, Δ) will be called a *representation* (with a basis $\{e_i| \ i \in I\}$) of a projective module V. Recall that in such a situation we identify a ring $\mathrm{Hom}(V, V)$ with $\Delta R_I\Delta$. We shall define the ring

$$\mathrm{Hom}_R(V, V, b) = \mathrm{Hom}_R(V, V) \cap \langle R_I, b\rangle$$

for a projective module V. The definition does not depend on the choice of a representation of V. This follows from the

LEMMA 2.1. *If V is a projective module which has a representation (F, Δ) with a basis $\{e_i| \ i \in I\}$ then the following conditions are equivalent:*

(i) $a \in \mathrm{Hom}_R(V, V) \cap \langle R_I, b\rangle$;

(ii) *there exists a finitely generated submodule V' of V such that $Va \subseteq V'$.*

The proof is easy and we leave it to the reader. ■

The opening remarks immediately imply, as a particular case of Theorem 1.9, the following theorem, which is fundamental in this section:

THEOREM 2.2. *Let $\mathcal{N}$ be a normal radical property. If a left R-module V is a direct summand of an R-module W then*

$$N(\mathrm{Hom}_R(V, V)) = \Delta N(\mathrm{Hom}_R(W, W))\Delta$$
$$= \mathrm{Hom}_R(V, V) \cap N(\mathrm{Hom}_R(W, W)).$$

Moreover, if V is a projective R-module with $\{v_i| \ i \in I\}$ as a set of generators then

$$N(\mathrm{Hom}_R(V, V)) = \mathrm{Hom}_R(V, V) \cap N(R_I),$$

for a representation (F, Δ) with a basis $\{e_i| \ i \in I\}$. ■

This gives us a simple way to obtain the Ware–Zelmanowitz theorem [11] on the Jacobson radical of $\mathrm{Hom}(V, V)$. But first we need the following definition. A family of subsets $\{X_t| \ t \in T\}$ of a ring R is called a *right vanishing family* if, given any sequence $x_1, x_2, \ldots$ with $x_k \in X_{t_k}$ for distinct t_k in T, there exists an integer n for which $x_1x_2\ldots x_n = 0$.

Recall that N. E. Sexauer and J. E. Warnock proved [10] that a matrix a is from the Jacobson radical $J(R_I)$ of a ring R_I if and only if $\{A_j| \ j \in I\}$ is a right vanishing family of left ideals contained in $J(R)$, where A_j is a left ideal of R generated by the set $\{a(i, j)| \ i \in I\}$.

For a free R-module F with a basis $\{e_i| \ i \in I\}$ let us denote by w_i an R-homomorphism $w_i\colon F \to R$ such that $e_iw_i = 1$ and $e_jw_i = 0$ for $j \neq i$.

THEOREM 2.3. (Ware–Zelmanowitz [11].) *Let V be a projective left R-module and let $a \in \mathrm{Hom}_R(V, V)$. Then the following conditions are equivalent:*

(i) $a \in J(\mathrm{Hom}(V, V))$.

(ii) *There exists a representation (F, Δ) of V with a basis $\{e_i| \ i \in I\}$ such that $\{Va(\Delta w_j)| \ j \in I\}$ is a right vanishing family of left ideals of R contained in $J(R)$.*

(iii) *Given any representation* (F, Δ) *of* V *and any its basis* $\{e_i |\ i \in I\}$, *the family* $\{V\alpha(\Delta w_j)|\ j \in I\}$ *is a right vanishing family of left ideals of* R *contained in* $J(R)$.

Proof. The implication (iii) $\Rightarrow$ (ii) is obvious. So we have only to prove (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iii). Let $\alpha \in J(\mathrm{Hom}(V, V))$. The property of Jacobson is a normal radical property (cf. [2], [5], [9]); thus for every representation (F, Δ) of V with a basis $\{e_i|\ i \in I\}$ we have, by Theorem 2.2, $\alpha = \Delta\alpha\Delta \in \mathrm{Hom}(V, V) \cap J(R_I) \subseteq J(R)_I$. Thus $V\alpha(\Delta w_j) = F(\Delta\alpha\Delta)w_j \subseteq (J(R)F)w_j$ is a left ideal of R contained in $J(R)$. Furthermore, one can easily check that $V\alpha(\Delta w_j)$ is generated by the set $\{\alpha(i,j)|\ i \in I\}$. Hence the Sexauer–Warnock theorem implies that $\{V\alpha(\Delta w_j)\}$ is a right vanishing family.

(ii) $\Rightarrow$ (i). Let $\alpha \in \mathrm{Hom}(V, V)$ and let (F, Δ) be such a representation of V that $\{V\alpha(\Delta w_j)\}$ is a right vanishing family of left ideals of R contained in $J(R)$. Since $\alpha = \Delta\alpha\Delta$, we have $V\alpha(\Delta w_j) = F\Delta\alpha\Delta w_j = F\alpha w_j$ and left ideals of R, $F\alpha w_j \subseteq J(R)$, are generated by suitable sets $\{\alpha(i,j)|\ i \in I\}$. The Sexauer–Warnock theorem gives us

$$\alpha \in \mathrm{Hom}(V, V) \cap J(R)_I$$

but this equals $J(\mathrm{Hom}(V, V))$ as follows from Theorem 2.2. ∎

Theorems 1.6, 1.7 and 2.2 together give us

THEOREM 2.4. *If* $\mathcal{N}$ *is a normal radical property and if* V *is a projective* R*-module then*

$$N(\mathrm{Hom}_R(V, V)) \subseteq \mathrm{Hom}_R(V, N(R)V).$$

Furthermore, if a module V *is finitely generated then*

$$N(\mathrm{Hom}_R(V, V)) = \mathrm{Hom}_R(V, N(R)V).$$

Proof. Let a projective module V have a representation (F, Δ) with a basis $\{e_i|\ i \in I\}$. Then we have

$$N(\mathrm{Hom}(V, V)) = \mathrm{Hom}(V, V) \cap N(R_I) \subseteq \mathrm{Hom}(V, V) \cap N(R)_I$$

with equality for finitely generated modules. Thus it is enough to observe that $\mathrm{Hom}(V, V) \cap N(R)_I = \mathrm{Hom}(V, N(R)V)$. But this is obvious if we consider $\alpha \in \mathrm{Hom}(V, N(R)V)$ as a matrix

$$\Delta\alpha\Delta \in \mathrm{Hom}(F, N(R)F) = N(R)_I. \ ∎$$

To characterise ring $\mathrm{Hom}(V, V, b)$ we need the following

LEMMA 2.5. *Let* $\mathcal{N}$ *be a normal radical property. If* A *is an ideal of a ring* S *and* $a \in A$ *implies* $a \in aA$, *then*

$$N(eAe) = eN(A)e$$

for every idempotent $e = e^2 \in S$.

Proof. Let A be an ideal of a ring S with the required property and let $e = e^2 \in S$. Let us consider the Morita contexts

$$M = (A, Se, eA, eAe) \quad \text{and} \quad M' = (A, Ae, eS, eAe),$$

where all products are multiplications in S. Since $\mathcal{N}$ is a normal property, we obtain from the context M

$$(Se, N(eAe)eA) = SeN(eAe)eA = SN(eAe)A \subseteq N(A), \tag{2.1}$$

and from the context M'

$$[eS, N(A)Ae] = eSN(A)Ae \subseteq N(eAe). \tag{2.2}$$

Since $N(eAe) \subseteq A$ and $N(A) \subseteq A$, by our condition on A we obtain $N(eAe) \subseteq N(eAe)A$ and $N(A) \subseteq N(A)A$. Thus (2.1) implies

$$N(eAe) = e^2N(eAe)e \subseteq eSN(eAe)Ae \subseteq eN(A)e,$$

and (2.2) implies

$$eN(A)e = e^2N(A)e \subseteq eSN(A)Ae \subseteq N(eAe).$$

This means

$$N(eAe) = eN(A)e. \ ∎$$

THEOREM 2.6. *If* $\mathcal{N}$ *is a normal radical property and* (F, Δ) *is a representation of a projective* z*-module* V *then*

$$N(\mathrm{Hom}_R(V, V, b)) = \Delta N(\mathrm{Hom}_R(F, F, b))\Delta$$
$$= N(\mathrm{Hom}_R(F, F, b)) \cap \mathrm{Hom}_R(V, V).$$

Thus

$$N(\mathrm{Hom}_R(V, V, b)) = N(\mathrm{Hom}_R(V, V)) \cap \mathrm{Hom}_R(V, V, b)$$
$$\subseteq \mathrm{Hom}_R(V, N(R)V, b)$$

and the last inclusion is an equality if the property $\mathcal{N}$ *is supernilpotent.*

Proof. Let (F, Δ) be a representation of V and let X be some subset of a ring R. Since $\alpha = \Delta\alpha\Delta \in \mathrm{Hom}(V, V)$ belongs to $\mathrm{Hom}(F, F, b)$ if and only if $F\alpha = F\Delta\alpha = V\alpha$ is contained in a finitely generated submodule of a module $F \cap V = V$ and $F\alpha = F\alpha\Delta \subset XF\Delta = XV$ for $\alpha \in \Delta\ \mathrm{Hom}(f, X\Gamma)\Delta$, we have

$$\mathrm{Hom}(V, XV, b) = \Delta\,\mathrm{Hom}(F, XF, b)\Delta$$

for every subset X of R. In particular, we obtain

$$\mathrm{Hom}(V, N(R)V, b) = \Delta\,\mathrm{Hom}(F, N(R)F, b)\Delta \tag{2.3}$$

and

$$\mathrm{Hom}(V, V, b) = \Delta\,\mathrm{Hom}(F, F, b)\Delta. \tag{2.4}$$

Now we shall prove

$$N(\mathrm{Hom}(V,V,b)) = \Delta N(\mathrm{Hom}(F,F,b))\Delta\,.$$

Let $\{e_i|\ i\in I\}$ be a basis of F. Denote by A an ideal $\mathrm{Hom}(F,F,b) = \langle R_I, b\rangle$ of a ring $S=\mathrm{Hom}(F,F)=R_I$. If $a\in A$ then U is a finite set of such $j\in I$ that there exists an $i\in I$ and $a(i,j)\neq 0$. Then $a = a\varepsilon_U$, where $\varepsilon_U(u,u)=1$, for $u\in U$, and $\varepsilon_U(i,j)=0$ otherwise. Then $\varepsilon_U\in A$. The ring S, the ideal A and $\Delta=\Delta^2\in S$ satisfy the assumptions of Lemma 2.5. Hence, by (2.4), Lemma 2.5 and (1.1)

$$\begin{aligned}(2.5)\quad N(\mathrm{Hom}(V,V,b)) &= N(\Delta\,\mathrm{Hom}(F,F,b)\Delta) = \Delta N(\mathrm{Hom}(F,F,b))\Delta\\ &= N(\mathrm{Hom}(F,F,b))\cap \mathrm{Hom}(V,V)\,.\end{aligned}$$

Theorem 1.8 gives us

$$\begin{aligned}(2.6)\quad N(\mathrm{Hom}(F,F,b)) &= N\langle R_I,b\rangle = N(R_I)\cap\langle R_I,b\rangle\\ &= N(\mathrm{Hom}(F,F))\cap \mathrm{Hom}(F,F,b)\subseteq \langle N(R)_I, b\rangle\\ &= \mathrm{Hom}(F,N(R)F,b)\,.\end{aligned}$$

Now, applying successively (2.5), (2.6), Theorem 2.2 and (2.3), we obtain

$$\begin{aligned}(2.7)\quad N(\mathrm{Hom}(V,V,b)) &= N(\mathrm{Hom}(F,F))\cap\mathrm{Hom}(V,V)\cap\mathrm{Hom}(F,F,b)\\ &= N(\mathrm{Hom}(V,V,b))\cap\mathrm{Hom}(F,F,b)\\ &= N(\mathrm{Hom}(V,V))\cap\mathrm{Hom}(V,V,b)\\ &\subseteq \Delta\,\mathrm{Hom}(F,N(R)F,b)\Delta = \mathrm{Hom}(V,N(R)V,b).\end{aligned}$$

For supernilpotent normal properties the inclusion in (2.6) is an equality; thus in this case we have only equalities also in (2.7). ■

THEOREM 2.7. *If $\mathcal{N}$ is a normal radical property and V is a projective R-module then*

$$N(\mathrm{Hom}_R(V,V,b))\subseteq N(\mathrm{Hom}_R(V,V))\subseteq \mathrm{Hom}_R(V,N(R)V)$$

and $N(\mathrm{Hom}_R(V,V,b))$ is dense in $\mathrm{Hom}_R(V,N(R)V)$ in the following sense: for every finitely generated submodule G of V and every $a\in\mathrm{Hom}_R(V,N(R)V)$ there exists a $\beta\in N(\mathrm{Hom}_R(V,V,b))$ such that $G(a-\beta)=0$.

Proof. Let (F,Δ) be a representation of V. The first part of the Theorem we obtain from Theorem 1.11 multiplying by Δ the corresponding inclusions from the left and the right side and using (2.5) and Theorems 2.2 and 2.4.

Now, let G be a finitely generated submodule of $V\subseteq F$, and let $a=\Delta a\Delta\in\mathrm{Hom}(V,N(R)V)\subseteq\mathrm{Hom}(F,N(R)F)$. By Theorem 1.11 there exists such a $\beta\in N(\mathrm{Hom}(F,F,b))$ that $G(a-\beta)=0$. Since $G\subseteq V$, we have $G\Delta=G$ and $G(a-\Delta\beta\Delta)=G(\Delta a\Delta-\Delta\beta\Delta)=G\Delta(a-\beta)\Delta=G(a-\beta)\Delta=0$. So $\Delta\beta\Delta\in\Delta N(\mathrm{Hom}(F,F,b))\Delta = N(\mathrm{Hom}(V,V,b))$ (cf. (2.5)) is the required homomorphism and $N(\mathrm{Hom}(V,V,b))$ is dense in $\mathrm{Hom}(V,N(R)V)$.

Remark. One can define

$$\mathrm{Hom}_R(V,V,f) = \mathrm{Hom}_R(V,V)\cap\langle R_I,f\rangle$$

for a projective module V with a representation (F,Δ) with the basis $\{e_i|\ i\in I\}$. It is easy to see that this definition does not depend on the choice of (F,Δ). The technique described here allows us to prove for a radical $N(\mathrm{Hom}(V,V,f))$ analogues of Theorems 1.10 and 1.11. But instead of Lemma 2.5 one has to use the following

LEMMA 2.8. *Let A be a subring of a ring S and let $e=e^2\in S$ be such that $AeA\subseteq A$. If $\mathcal{N}$ is a normal radical property and every element a belonging to A belongs also to AaA, then*

$$N(eAe) = eN(A)e\,.$$

Outline of the proof. From the Morita context (A, Ae, eA, eAe) we have

$$(Ae, N(eAe)eA) = AeN(eAe)eA\subseteq N(A)$$

and

$$[eA, N(A)Ae] = eAN(A)Ae\subseteq N(eAe)\,.$$

Using the properties of A, we obtain the required equality. ■

One can check that the rings $S=\mathrm{Hom}(F,F)$, $A=\mathrm{Hom}(F,F,f)\subseteq S$ and an idempotent $\Delta=\Delta^2\in S$ satisfy the assumptions of Lemma 2.8, and so one can prove the required results.

3. Remark on the problem of Koethe. We shall say that a ring S is a $\mathcal{K}$-*radical ring* if S is a nil-ring. It is an open problem whether every left $\mathcal{K}$-ideal of a ring R is contained in $K(R)$, i.e., whether $\mathcal{K}$ is a strong radical property. This is the problem of Koethe [6]. We shall give an equivalent description of this problem.

THEOREM 3.1. *The following conditions are equivalent.*

(i) *For every ring R with identity and every projective left R-module V we have*

$$K(\mathrm{Hom}_R(V,V))\subseteq\mathrm{Hom}_R(V,K(R)V);$$

(ii)

$$K(eRe)\subset K(R)$$

for every ring R with identity and every $e=e^2\in R$;

(iii) *the problem of Koethe has a positive solution.*

Proof. Implication (iii) ⇒ (i) is an immediate consequence of Theorem 2.4 because if $\mathcal{K}$ is strong then $\mathcal{K}$ is a normal property (cf. [5] Theorem 1, or [9] Theorem 1).

We shall prove (i) ⇒ (ii). If R is a ring with identity and $e = e^2 \in R$ then Re is a projective R-module with the representation (R, e) and $\mathrm{Hom}(Re, Re) = eRe$. Thus

$$K(eRe) = K(\mathrm{Hom}(Re, Re)) \subseteq \mathrm{Hom}(Re, K(R)Re) \subseteq \mathrm{Hom}(R, K(R)R) = K(R).$$

To prove (ii) ⇒ (iii) let us consider a nil-ring A and let us put $R = A^{\#}$. Of course $A = K(R)$. We write $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R_2$. It is easy to see that $e = e^2$ and that a ring eA_2e isomorphic with A is a $\mathcal{K}$-radical of a ring eR_2e isomorphic with R. Hence

$$A_2 = R_2eA_2eR_2 = R_2\cdot K(eR_2e)\cdot R_2 \subseteq R_2K(R_2)R_2 \subseteq K(R_2).$$

This means that a matrix ring A_2 is nil for every nil ring A. In this case the problem of Koethe has a positive solution, as was proved by J. Krempa [7] and A. D. Sands [9]. ■

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On shapes of topological spaces

by

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Abstract. A new approach to shapes of topological spaces and its applications will be given.

The notion of shape was originally introduced by K. Borsuk [1] for the case of compact metric spaces. Since then, this notion has been extended to the case of compact Hausdorff spaces by S. Mardešić and J. Segal [12] (cf. also W. Holsztyński [7]) and to the case of metric spaces by K. Borsuk [2] and R. H. Fox [4]. More recently the notion has been extended to the case of arbitrary topological spaces by S. Mardešić [11].

In this note we shall discuss shapes of topological spaces in the sense of Mardešić from another point of view.

For any category $\mathfrak{C}$, let us denote by Ob $\mathfrak{C}$ the class of all objects of $\mathfrak{C}$, and by $f \in \mathfrak{C}(X, Y)$ we mean that f is a morphism from X to Y in $\mathfrak{C}$.

1. Let $\mathfrak{H}$ be the homotopy category of topological spaces. Its objects are topological spaces and its morphisms are homotopy classes of continuous maps; the homotopy class of a continuous map $f\colon X \to Y$ will be denoted as usual by $[f]$. Let $\mathfrak{W}$ be the full subcategory of $\mathfrak{H}$ whose objects are all topological spaces having the homotopy type of a CW complex. Throughout this paper, by an ANR we shall mean an ANR for the class of metrizable spaces. The following result is known (cf. Mardešić [11]).

LEMMA 1.1. *For a space X the following conditions are equivalent.*

(a) *X has the homotopy type of a CW complex.*

(b) *X has the homotopy type of a simplicial complex with the weak topology (or with the metric topology).*

(c) *X has the homotopy type of an* ANR.

DEFINITION 1.2. Let $\{X_\alpha, [p_{\alpha\alpha'}], A\}$ be an inverse system in the category $\mathfrak{H}$ or $\mathfrak{W}$; that is, A is a directed set, continuous maps $p_{\alpha\alpha'}\colon X_{\alpha'} \to X_\alpha$ are defined for any α, α' with $\alpha < \alpha'$, and $[p_{\alpha\alpha'}][p_{\alpha'\alpha''}] = [p_{\alpha\alpha''}]$ if $\alpha < \alpha' < \alpha''$. We shall say that an inverse system $\{X_\alpha, [p_{\alpha\alpha'}], A\}$ in $\mathfrak{H}$ or