A wild Cantor set in E^n with simply connected complement

by

D. G. DeGryse and **R. P. Osborne** (Fort Collins, Colo.)

Abstract. It is well known that the complement of a *tame* Cantor set embedded in E^n, $n \geqslant 3$, is simply connected. The converse of this statement is false. In fact there is an example of a *wild* Cantor set in E^3 whose complement is simply connected, due to A. Kirkor. However, no example was known in dimensions greater than three, since it was not clear how to generalize Kirkor's construction to higher dimensions. Using solid tori we construct a wild Cantor set in E^3, whose complement is simply connected. This construction generalizes to E^n, $n > 3$. This result is then used to construct wild cells and spheres in E^n with simply connected complements.

In 1921, L. Antoine [1] constructed a wild zero-dimensional subset of S^3. This set was proven to be wild by showing that the fundamental group of its complement in S^3 was non-trivial, [3]. In 1951, W. A. Blankenship [2] extended Antoine's construction to n-dimensions giving an example of a compact, zero-dimensional set $A \subset E^n$, whose complement was not simply connected. In 1958, A. Kirkor [5] gave the first known example of a wild zero-dimensional subset of S^3 whose complement was simply connected. This set was constructed by entangling arcs in S^3. Since it is not clear how to generalize Kirkor's construction to n-dimensions, we will give a different construction in E^3 which can be generalized to n-dimensions, $n \geqslant 3$.

A Cantor set in E^n will be called *tame* if there exists a homeomorphism of E^n onto itself which maps the Cantor set into a straight line segment. A Cantor set which is not tame is *wild*.

The following is our principal theorem.

THEOREM. *For $n \geqslant 3$, there exists a wild Cantor set $A^n \subset E^n$ whose complement is simply connected.*

COROLLARY 1. *For $n \geqslant 3$ and $0 < k \leqslant n$, there exists wild k-balls in E^n with simply connected complements.*

COROLLARY 2. *For $n \geqslant 3$, $0 < k \leqslant n-3$, there exists wild k-spheres in E^n with simply connected complements. There also exists wild $(n-2)$-spheres in E^n whose complements have infinite cyclic fundamental groups, and wild $(n-1)$-spheres in E^n whose complementary domains are simply connected.*

The paper will be divided into five parts. In part I the Cantor set $A^3 \subset E^3$ is constructed. In part II, we prove that A^3 is wild by showing that $\pi(T_0^3 - A^3)$ is not what it would be if A^3 were tame (T_0^3 is the canonical solid three-dimensional torus). In part III, we show that the complement of A^3 in E^3 is simply connected. In part IV we generalize the construction and results to A^n in E^n, $n \geqslant 3$, and in part V we shall prove the two corollaries following the principal theorem.

I. The construction of the Cantor set A^3. We first give an intuitive construction of A^3.

DEFINITION 1.1. Let J be a polygonal simple closed curve in E^3. Given $\varepsilon > 0$, denote by $N_\varepsilon(J)$ a regular neighborhood of J in E^3 such that $d(x, J) < \varepsilon$ for all $x \in N_\varepsilon(J)$. J will be called the *central simple closed curve* of $N_\varepsilon(J)$.

The Cantor set A^3 will be the intersection of a decreasing sequence of non-empty compact sets, $A_0^3 \supset A_1^3 \supset A_2^3 \ldots$ The sequence of sets will depend on the two standard models shown in Figure 1.

In Model δ, J^* is the central simple closed curve of the solid unknotted torus T_0^3 in E^3 and T_δ, $T_{\delta'}$ are solid tori embedded in T_0^3 so that they are entangled as shown in the figure. This embedding will be accomplished in such a way that if J^* is cut into two arcs J_1^*, J_2^* at the points t_1, t_2, then given $\delta > 0$ and points $p \in T_\delta$, $p' \in T_{\delta'}$,

(i) $d(p, J_1^*) < \delta$, and

(ii) $d(p', J_2^*) < \delta$.

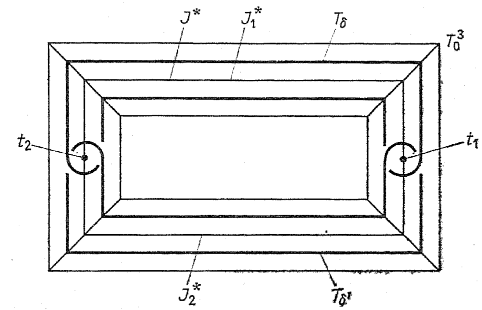
Model δ is considered to be the torus T_0^3 with T_δ and $T_{\delta'}$ as subsets of T_0^3 and will be denoted as the pair $(T_0^3, T_\delta \cup T_{\delta'})$.

In Model F, F^3 is a solid torus embedded in T_0^3 as shown in Figure 1. Model F is considered to be T_0^3 with F^3 as a subset and will be denoted as the pair (T_0^3, F^3).

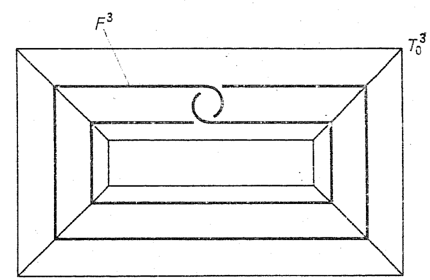
To obtain A^3 we must construct a decreasing sequence of non-empty compact sets. To obtain the first set we will construct a set D_ε^3 in T_0^3 which will be defined in detail later.

To construct D_ε^3 in T_0^3 we will start with Model δ. We will use iterations of a mapping which sends Model δ onto each of the tori in the preceding step. For example, the first step will be the mapping of Model δ onto each of the tori already embedded in T_0^3. These mappings will be continued until each of the tori in the intersection of the images of $T_\delta \cup T_{\delta'}$ in T_0^3 have diameters less than a given $\varepsilon > 0$. It is this part of the construction which will give us the zero-dimensionality of the set A^3.

After the diameters of each torus is less than a given $\varepsilon > 0$, we will then map (T_0^3, F^3) onto each of the tori. This step will be instrumental in proving that the complement of A^3 is simply connected. A^3 is then constructed by iterating these two processes.



Model δ



Model F

Fig. 1

In proving $\pi(E^3 - A^3)$ is trivial and in proving $\pi(T_0^3 - A^3)$ is not what it would be if A^3 were tame, we will use a direct limit argument on the fundamental groups.

Now we will go into the detail of constructing A^3. In constructing A^3 we must be sure that

(i) the diameters of each of the tori become as small as we want, to insure the zero-dimensionality,

(ii) $\pi(T_0^3 - A^3) \neq Z$, to insure the wildness of A^3, and

(iii) $\pi(E^3 - A^3)$ is trivial.

First of all, we will construct a set, D_ε^3, consisting of disjoint solid tori. D_ε^3 will be the intersection of images of $T_\delta \cup T_{\delta'}$ in T_0^3. These tori will have diameters less than ε. D_ε^3 will be used in the definition of A_0^3, the first set in the sequence of decreasing sets used in defining A^3.

THEOREM 1.2. *Given $\varepsilon > 0$, we can construct a set D_ε^3 of solid tori in T_0^3 such that*

(i) *the diameters of each of the tori comprising D_ε^3 are less than ε,*

(ii) *these tori are unknotted and unlinked in E^3, and*

(iii) *Φ_3: $\pi(\partial T_0^3) \to \pi(T_0^3 - D_\varepsilon^3)$ is a monomorphism, where Φ_3 is induced by inclusion.*

To prove this theorem, we shall use the following lemma.

LEMMA 1.3. *Let J be a polygonal simple closed curve in E^3. Let P be a plane in E^3. And let $\varepsilon > 0$ be given. Define $\tilde{P}$ to be a "thickening" of P, i.e., $\tilde{P} = \{x \in E^3 \mid d(x, P) \leqslant \varepsilon\}$. Define P_+ and P_- to be the boundary components of $\tilde{P}$. Define E_a^3 and E_b^3 to be the spaces "above" P_+ and "below" P_-, respectively, such that E_a^3, E_b^3 and $\tilde{P}$ are pairwise disjoint but $E_a^3 \cup E_b^3 \cup \tilde{P} = E^3$. Let $N_1(J)$ be a regular neighborhood of J. Then*

(i) *there exists a minimal set M consisting of an even number, $2k$ of distinct points in $J \cap P$ such that between any two consecutive points, J lies entirely in $E_a^3 \cup \tilde{P}$ or $E_b^3 \cup \tilde{P}$, and*

(ii) *after i stages of the construction, involving a suitable homeomorphism, mapping Model δ onto $N_1(J)$, each of the tori embedded in $N_1(J)$ will lie entirely in $E_a^3 \cup \tilde{P}$ or $E_b^3 \cup \tilde{P}$.*

Proof. Part (i) is easily proved by routine methods. The proof of part (ii) will be by induction on k. For $k = 0$ the conclusion is clear. Assume it is true for $k = m-1$.

Consider the $2m$ distinct points in M. Pick one of them, call it x_1. Proceed around J in a clockwise fashion and order the points $x_2, x_3, \ldots, x_m, \ldots, x_{2m}$ of M as we encounter them. Consider x_1 and x_{m+1}. Define our homeomorphism α: $(T_0^3, T_\delta \cup T_{\delta'}) \to N_1(J)$ such that

1° $\alpha(t_1) = x_1$,

2° $\alpha(t_2) = x_{m+1}$, and

3° $\alpha(J^*) = J$.

Such a homeomorphism clearly exists. Since α is a homeomorphism of a compact set onto a compact set then α is uniformly continuous. Hence, given $\varepsilon > 0$, there exists $\delta > 0$ such that, given points $p \in T_\delta$ and $r \in J_1^*$ such that $d(p, r) < \delta$, then $d[\alpha(p), \alpha(r)] < \varepsilon$. The same holds true for points $q \in T_{\delta'}$ and $s \in J_2^*$.

Now we have two tori $\alpha(T_\delta)$ and $\alpha(T_{\delta'})$ embedded in $N_1(J)$ by a suitable homeomorphism. Their central simple closed curves are J_1 and J_2 respectively. The minimal sets for each of J_1 and J_2 then contains two fewer points than the minimal set for J. The lemma follows inductively.

The set D_ε^3 will be the intersection of a decreasing sequence of non-empty compact sets

$$C_0^3 \supset C_1^3 \supset \ldots \supset C_k^3 .$$

Define C_0^3 to be $T_\delta \cup T_{\delta'}$ in T_0^3. By "cutting" T_0^3 with planes one at a time and by using the suitable homeomorphisms we can construct C_i^3, $1 \leqslant i \leqslant k$, in such a way that the tori contained in $\bigcap_{i=0}^{k} C_i^3$ lie in cubes with diameter less than ε. D_ε^3 will then be equal to $\bigcap_{i=0}^{k} C_i^3$. Notice $D_\varepsilon^3 \subset T_0^3$.

Proof of Theorem 1.2.

(i) It is clear that since E^3 can be partitioned into cubes with diameter less any given $\varepsilon > 0$ and by Lemma 1.3 that given any $\varepsilon > 0$ the tori in D_ε^3 have diameters less than ε.

(ii) To prove this part we need only remark that at each stage of the construction the tori are unknotted and hence unlinked in E^3. In C_0^3 we have two tori. It is well known that since the central simple closed curves of T_δ and $T_{\delta'}$ bound non-singular disks, T_δ and $T_{\delta'}$ are unknotted in E^3. Hence the tori in C_0^3 are unknotted in E^3. We proceed inductively and arrive at the result.

(iii) The third part will be proven later in the paper (Lemma 2.4).

A^3 will then consist of the intersection of a decreasing sequence of non-empty compact sets

$$A_0^3 \supset A_1^3 \supset A_2^3 \ldots$$

A_0^3 will be defined to be D_1^3. A_1^3 is the subset of A_0^3 equal to the union of the images of F^3 obtained by homeomorphically mapping (T_0^3, F^3) onto each of the tori in A_0^3. A_2^3 is the subset of A_1^3 equal to the union of the images of $D_{\theta(1/2)}^3$ obtained by mapping $(T_0^3, D_{\theta(1/2)}^3)$ homeomorphically onto each of the tori in A_1^3. Here, θ is a function chosen so that when $(T_0^3, D_{\theta(1/2)}^3)$ is mapped onto each of the tori in A_1^3, the images of each of the tori in $D_{\theta(1/2)}^3$ have diameters less than $\frac{1}{2}$.

In general, A_i^3, $i \neq 0$, $i = 2j$, is the subset of A_{i-1}^3 equal to the union of the images of $D_{\theta(1/i)}^3$, obtained by mapping $(T_0^3, D_{\theta(1/i)}^3)$ homeomorphically onto each of the tori in A_{i-1}^3, and A_i^3, i odd, is the subset of A_{i-1} equal to the union of the images of F^3, obtained by mapping (T_0^3, F^3)

homeomorphically onto each of the tori in A^3_{i-1}. We then define A^3 to be $\bigcap_{i=0}^{\infty} A^3_i$.

It is a routine matter to show that A^3 is a Cantor set.

II. A^3 is wild. In this section we will prove that A^3 is wild by first noticing that given a tame Cantor set X, $\pi(T^3_0 - X) = \pi(T^3_0) = Z$, the free group on one generator. Whereas $\pi(T^3_0 - A^3) \neq Z$, hence proving A^3 is wild.

Now to show that $\pi(T^3_0 - A^3) \neq Z$ we will need the following theorem.

All homomorphisms given below are taken to be those induced by inclusion.

THEOREM 2.1. *Let A and B be open arcwise connected subsets of $A \cup B$ such that $A \cap B$ is arcwise connected. If the homomorphisms $\varphi_1\colon \pi(A \cap B) \to \pi(A)$ and $\varphi_2\colon \pi(A \cap B) \to \pi(B)$ are monomorphisms, then $\psi_1\colon \pi(A) \to \pi(A \cup B)$ and $\psi_2\colon \pi(B) \to \pi(A \cup B)$ are monomorphisms.*

Proof. $\pi(A \cup B)$ is the free product of $\pi(A)$ and $\pi(B)$ with $\varphi_1(\pi(A \cap B))$ and $\varphi_2(\pi(A \cap B))$ amalgamated [6], sec. 4.2.

The following lemmas are immediate consequences of [8], Theorem 2.

LEMMA 2.2. *$\xi\colon \pi(\partial T^3_0) \to \pi(T^3_0 - F^3)$ is a monomorphism.*

LEMMA 2.3. *$\xi^*\colon \pi(\partial F^3) \to \pi(T^3_0 - \mathring{F}^3)$ is a monomorphism.*

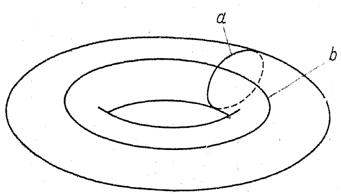


Fig. 2

The two lemmas given below are both proved by similar methods. We prove only the last of them to illustrate the technique.

LEMMA 2.4. *$\eta\colon \pi(\partial T^3_0) \to \pi(T^3_0 - (T_\delta \cup T_{\delta'}))$ is a monomorphism.*

LEMMA 2.5. *$\eta^*\colon \pi(\partial T_\delta) \to \pi(T^3_0 - (\mathring{T}_\delta \cup T_{\delta'}))$ is a monomorphism.*

Proof. Let $\pi(\partial T_\delta) = \langle a, b / [a, b] = 1\rangle$, where a and b are as indicated in Figure 2.

Now consider $a^m b^n \in \pi(\partial T_\delta)$. We show that $\eta^*(a^m b^n) = 1$ if and only if $m = n = 0$. Consider the commutative diagram,

$$\begin{array}{ccc} \pi(\partial T_\delta) & \xrightarrow{\alpha} & \pi(E^3 - \mathring{T}_\delta) \\ & {\scriptstyle\eta^*}\searrow \quad \nearrow & \\ & \pi(T^3_0 - (\mathring{T}_\delta \cup T_{\delta'})) & \end{array}$$

Now $\alpha(a^m b^n)$ is non-trivial unless $m = 0$. It follows that $\eta^*(a^m b^n)$ is non-trivial unless $m = 0$. Next consider the commutative diagram,

$$\begin{array}{ccc} \pi(\partial T_\delta) & \xrightarrow{\alpha^*} & \pi(T^3_0 - \mathring{T}_{\delta'}) \\ & {\scriptstyle\eta^*}\searrow \quad \nearrow & \\ & \pi(T^3_0 - (\mathring{T}_\delta \cup T_{\delta'})) & \end{array}$$

Since $\alpha^*(a^m b^n)$ is non-trivial unless $n = 0$, it follows that $\eta^*(a^m b^n)$ is non-trivial unless $n = 0$.

THEOREM 2.6. *Given any of the sets A^3_i in the construction of A^3, the map $\psi\colon \pi(T^3_0 - A^3_i) \to \pi(T^3_0 - A^3_{i+1})$ is a monomorphism.*

Proof. To prove this we must consider two cases.

Case 1. i is even.

In this case $T^3_0 - A^3_i$ is the complement in T^3_0 of many images of T_δ and $T_{\delta'}$ entangled in the manner set forth in the construction. A^3_{i+1} is the subset of A^3_i obtained by mapping (T^3_0, F^3) homeomorphically onto each of the solid tori in A^3_i.

Set $T^3_0 - A^3_i = T^3_0 - \bigcup_{n=1}^{k} T_n$, where k is the number of tori comprising A^3_i. Now, if we can show that by mapping (T^3_0, F^3) onto one of the T_n's, say T_j, the map

$$\gamma_j\colon \pi\Big(T^3_0 - \bigcup_{n=1}^{k} T_n\Big) \to \pi\Big[T^3_0 - \Big(\bigcup_{\substack{n=1\\ n\neq j}}^{k} T_n \cup F^3_j\Big)\Big]$$

is a monomorphism, then after k such maps we have that $\psi\colon \pi(T^3_0 - A^3_i) \to \pi(T^3_0 - A^3_{i+1})$ is a monomorphism.

Let γ_j be defined by the inclusion and consider the following Van Kampen diagram.

$$\begin{array}{ccc} & \pi(\partial T_j) & \\ {\scriptstyle\xi^*}\swarrow & \downarrow & \searrow{\scriptstyle\xi} \\ \pi\Big(T^3_0 - \bigcup_{n=1}^{k} \mathring{T}_n\Big) & & \pi(T_j - F^3_j) \\ {\scriptstyle\gamma_j}\searrow & \downarrow & \swarrow \\ & \pi\Big[T^3_0 - \Big(\bigcup_{\substack{n=1\\ n\neq j}}^{k} \mathring{T}_n \cup F^3_j\Big)\Big] & \end{array}$$

By Lemmas 2.2 and 2.5 we know that ξ and ξ^* are monomorphisms. Hence, we can apply Theorem 2.1 and conclude γ_j is a monomorphism.

Therefore, $\psi\colon \pi(T_0^3 - A_i^3) \to \pi(T_0^3 - A_{i+1}^3)$ is a monomorphism when i is even.

Case 2. i is odd.

Again, when i is odd, $T_0^3 - A_i^3$ is the complement in T_0^3 of many solid tori. But this time the solid tori are images of F^3 entangled as prescribed in the construction. $T_0^3 - A_{i+1}^3$ is the complement of A_{i+1}^3 in T_0^3, where A_{i+1}^3 is the subset of A_i^3 obtained by mapping (T_0^3, D_s^3) homeomorphically onto each of the tori in A_i^3. Remember that D_s^3 is the union of many homeomorphic images of Model δ.

Consider $T_0^3 - A_i^3$ to be $T_0^3 - \bigcup_{n=1}^{m} T_n$, where m is the number of tori in A_i^3. Now if we can show that by mapping (T_0^3, D_s^3) onto one of the T_n's, say T_j, the map

$$\delta_j\colon \pi\Big(T_0^3 - \bigcup_{n=1}^{m} T_n\Big) \to \pi\Big[T_0^3 - \Big(\bigcup_{\substack{n=1\\ n\neq j}}^{m} T_n \cup D_{s,j}^3\Big)\Big]$$

is a monomorphism, then after m such maps we have that $\psi\colon \pi(T_0^3 - A_i^3) \to \pi(T_0^3 - A_{i+1}^3)$ is a monomorphism.

To show δ_j is a monomorphism it suffices to show that $\mu_j\colon \pi(\partial T_j) \to \pi(T_j - D_{s,j}^3)$ is a monomorphism. An application of the Van Kampen theorem completes the proof. Recall that $D_{s,j}^3$ is the homeomorphic image of D_s^3 and that D_s^3 was constructed by successively mapping the pair $(T_0^3, T_\delta \cup T_{\delta'})$ into its subtori. Again using the Van Kampen theorem and Theorem 2.1 we see that we can reduce our problem to one of showing that $\pi(\partial T_0^3) \to \pi\big(T_0^3 - (T_\delta \cup T_{\delta'})\big)$ and $\pi(\partial T_\delta) \to \pi\big(T_0^3 - (\overset{0}{T_\delta} \cup T_{\delta'})\big)$ are monomorphisms. But this is exactly Lemmas 2.4 and 2.5.

The preceding theorem gives us our inductive step in showing that $\pi(T_0^3 - A^3) \neq Z$ and hence A^3 is wild.

THEOREM 2.7. *$\pi(T_0^3 - A^3) \neq Z$.*

Proof. Since $\pi(T_0^3 - A^3) = \overrightarrow{\lim_{i\to\infty}} \pi(T_0^3 - A_i^3)$, it suffices to show that there exists a subgroup of $\overrightarrow{\lim_{i\to\infty}} \pi(T_0^3 - A_i^3)$ which could not possibly be a subgroup of Z.

It is clear by using Lemma 2.4 that $\pi(\partial T_0^3) \to \pi(T_0^3 - A_0^3)$ is a monomorphism. Since we have also proven that at every stage in the construction, the map $\psi\colon \pi(T_0^3 - A_i^3) \to \pi(T_0^3 - A_{i+1}^3)$ is a monomorphism, then we can say that the inclusion map $\pi(\partial T_0^3) \to \pi(T_0^3 - A^3)$ is a monomorphism.

Since $\pi(\partial T_0^3) = \langle \alpha, \beta \mid \alpha\beta\alpha^{-1}\beta^{-1}\rangle$, the free abelian group on two elements, and $\pi(\partial T_0^3) \to \pi(T_0^3 - A^3)$ is a monomorphism, then $\pi(T_0^3 - A^3)$ has a free abelian subgroup on two elements.

Therefore, $\pi(T_0^3 - A^3) \neq Z$ and hence A^3 is wild in E^3.

III. $E^3 - A^3$ is simply connected.

THEOREM 3.1. *$\pi(E^3 - A^3)$ is trivial.*

Proof. By Theorem 1.2 we have that in the construction of D_s^3 all the tori comprising it are unlinked and unknotted in E^3. Hence

$$\pi(E^2 - A_0^3) = \langle a_1, a_2, \ldots, a_k\rangle$$

the free group on k elements, where k is the number of tori in A_0^3. At the next stage of the construction, we map (T_0^3, F^3) onto each of the tori in A_0^3 homeomorphically. The union of the images of F^3 in A_0^3 is A_1^3.

Consider this mapping and the map $\beta\colon \pi(E^3 - A_0^3) \to \pi(E^3 - A_1^3)$ induced by inclusion. Let $\beta_i\colon \pi(E^3 - T_i) \to \pi(E^3 - F_i^3)$ be the map induced by inclusion. We know that

(a) $\pi(E^3 - T_i) = \langle a_i\rangle$, and

(b) $\pi(E^3 - F_i^3) = \langle b_1, b_2 \mid b_1 b_2^{-1} = 1\rangle$,

where $\pi(E^3 - F_i^3)$ is calculated from Figure 3.

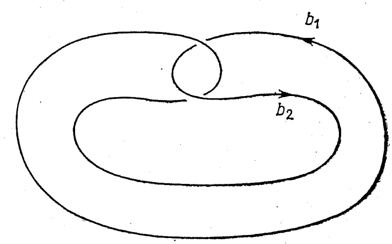


Fig. 3

Now $\beta_i\colon \langle a_i\rangle \to \langle b_1, b_2 \mid b_1 b_2^{-1} = 1\rangle$ and $\beta_i(a_i) = b_1 b_2^{-1} = 1$. Hence β takes all of the generators of $\pi(E^3 - A_0^3)$ to the identity in $\pi(E^3 - A_1^3)$.

Repeating this argument we see that every other map in the direct limit that gives $\pi(E^3 - A^3)$ is the trivial map. Since

$$\pi(E^3 - A^3) = \overrightarrow{\lim_{i\to\infty}} \pi(E^3 - A_i^3)$$

we can use a direct limit argument and conclude that given any generator in one of the groups in the direct limit, that generator is eventually mapped to the identity. Hence $\pi(E^3 - A^3) = \overrightarrow{\lim_{i\to\infty}} (E^3 - A_i^3)$ is trivial.

One should note that the above argument would fail if at any stage of the construction any of the solid tori encountered were knotted. If this happened, the two tori obtained by mapping model δ onto this knotted torus would be linked and the subsequent groups would not be free as asserted above.

This completes the construction of a wild Cantor set A^3 in E^3 whose complement is simply connected.

IV. A generalization to E^n. Using our construction of A^3 in E^3 we will generalize to A^n in E^n by an inductive argument. The following lemma and definition will be useful in the construction.

LEMMA 4.1. *Let $X = X_1 \times X_2 \times X_3$, where X_i is a compact metric space, $i = 1, 2, 3$, and $X_1 = X_2$. Suppose A and B are subsets of $X_2 \times X_3$. Let $h\colon X_2 \times X_3 \twoheadrightarrow A$ be a homeomorphism. Define a "switching map" $s\colon X \to X$ so that $s = s_1 \times s_1^{-1} \times 1$, where $s_1\colon X_1 \twoheadrightarrow X_2$ is a homeomorphism. Then given $\varepsilon > 0$ there exists a $\delta > 0$ such that if* $\operatorname{diam}(A)$ *and* $\operatorname{diam}(B)$ *are less than δ, then* $\operatorname{diam}[(1\times h)s(X_1 \times B)] < \varepsilon$.

The proof of this lemma is a simple point set topology argument.

DEFINITION 4.2. We define T_0^2 to be the unit disk in E^2 and define T_0^n to be $(S^1)^{n-2} \times T_0^2$, where $(S^1)^{n-2}$ denotes the $(n-2)$-fold Cartesian product of the one sphere, S^1. An n-tube is a space homeomorphic with T_0^n.

Henceforth, all unlabeled maps are induced by inclusion.

A^n will be the intersection of a decreasing sequence of non-empty compact sets, $A_0^n \supset A_1^n \supset A_2^n \ldots$, where each A_i^n is the union of disjoint n-tubes. In the construction of A^n we must be sure that

(i) the diameter of each n-tube in A_k^n approaches zero as k goes to infinity,

(ii) A^n is wild in E^n, and

(iii) $E^n - A^n$ is simply connected.

To construct A^n we will use two basic sets, D_ε^n and F^n in T_0^n.

THEOREM 4.3. *Given $\varepsilon > 0$, we can construct a set D_ε^n in T_0^n consisting of disjoint n-tubes with the following properties:*

(i) *the diameter of each n-tube in D_ε^n is less than ε,*

(ii) $\pi(\partial T_0^n) \to \pi(T_0^n - D_\varepsilon^n)$ *is a monomorphism, and*

(iii) *the n-tubes in D_ε^n are h-unlinkable in T_0^n. (For the definition of h-unlinkable and the proof of* (iii) *see 4.8 and following.)*

Proof. In the previous sections we showed how to construct D_δ^3 in T_0^3 for every choice of $\delta > 0$. Assume inductively that we can construct D_δ^{n-1} for each choice of $\delta > 0$ satisfying (i), (ii), and (iii). D_δ^{n-1} consists of disjoint $(n-1)$-tubes T_i^{n-1}, $i = 1, 2, \ldots, k$. Now $S^1 \times D_\delta^{n-1} \subset S^1 \times T_0^{n-1} = T_0^n$. Let $h_i\colon T_0^{n-1} \twoheadrightarrow T_i^{n-1}$ be a homeomorphism. Define $s\colon S^1 \times S^1 \times T_0^{n-2} \to S^1 \times S^1 \times T_0^{n-2}$ as in Lemma 4.1, then define

$$D_\varepsilon^n = \bigcup_{i=1}^{k} [(1\times h_i)s(S^1 \times D_\delta^{n-1})] \subset T_0^n.$$

By Lemma 4.1 the diameter of each component of D_ε^n will be less than ε if δ is small enough. This establishes (i).

In order to establish (ii), consider the following Van Kampen diagram.

$$\begin{array}{ccc} & \pi(\partial T_i^n) & \\ \swarrow & \downarrow & \searrow \\ \pi\big(S^1 \times (T_0^{n-1} - D_\delta^{n-1})\big) & & \pi\Big((1\times h_i)s\big(S^1 \times (T_0^{n-1} - D_\delta^{n-1})\big)\Big) \\ \searrow & \downarrow & \swarrow \\ & \pi\Big(T_0^n - \big(\bigcup_{\substack{j=1\\ j\neq i}}^{k} T_j^n \cup (1\times h_i)s(S^1 \times D_\delta^{n-1})\big)\Big) & \end{array}$$

Note that $\pi(\partial T_0^{n-1}) \to \pi(T_0^{n-1} - D_\delta^{n-1})$ is a monomorphism for each choice of δ. From this it follows that

$$\pi(\partial T_0^n) = \pi\big(\partial(S^1 \times T_0^{n-1})\big) \to \pi\big(S^1 \times (T_0^{n-1} - D_\delta^{n-1})\big)$$

is a monomorphism. Thus,

$$\pi\big((1\times h_i)s(\partial T_0^n)\big) = \pi(\partial T_i^n) \to \pi\Big((1\times h_i)s\big(S^1 \times (T_0^{n-1} - D_\delta^{n-1})\big)\Big)$$

is a monomorphism. Notice that it also follows inductively that $\pi(\partial T_i^n) \to \pi\big(S^1 \times (T_0^{n-1} - D_\delta^{n-1})\big)$ is a monomorphism. Thus, by Theorem 2.1,

$$\pi\big(S^1 \times (T_0^{n-1} - D_\delta^{n-1})\big) \to \pi\Big(T_0^n - \big(\bigcup_{\substack{j=1\\ j\neq i}}^{k} T_j^n \cup (1\times h_i)s(S^1 \times D_\delta^{n-1})\big)\Big)$$

is a monomorphism. And hence,

$$\pi(\partial T_0^n) \to \pi\Big(T_0^n - \big(\bigcup_{\substack{j=1\\ j\neq i}}^{k} T_j^n \cup (1\times h_i)s(S^1 \times D_\delta^{n-1})\big)\Big)$$

is a monomorphism. It follows inductively that $\pi(\partial T_0^n) \to \pi(T_0^n - D_\varepsilon^n)$ is a monomorphism.

To construct $F^n \subset T_0^n$, we consider F^{n-1} in T_0^{n-1} and define $F^n = S^1 \times F^{n-1} \subset S^1 \times T_0^{n-1} = T_0^n$.

All we need to do now is define the sets $A_0^n \supset A_1^n \supset \ldots$ Similar to the three-dimensional case, we define A_0^n to be D_1^n and in general A_i^n, $i \neq 0$, $i = 2j$ to be the subset of A_{i-1}^n equal to the union of the images of $D_{\delta(1/i)}^n$, obtained by mapping $(T_0^n, D_{\delta(1/i)}^n)$ homeomorphically onto each of the

n-tubes in A^n_{i-1}. And A^n_i, i odd, is the subset of A^n_{i-1} equal to the union of the images of F^m, obtained by mapping (T^m_0, F^m) homeomorphically onto each n-tube in A^n_{i-1}. Then $A^n = \bigcap_{i=0}^{\infty} A^n_i$.

It is a routine matter to show that A^n is a Cantor set.

Now we need to show A^n is wild in E^n.

THEOREM 4.4. *Let X be a tame Cantor set in E^n. Let T^m be an n-tube such that X lies in the interior of T^m. Then $\pi(T^m - X) = \pi(T^m)$.*

The proof of this theorem is easy enough to justify omission.

LEMMA 4.5. *The maps η_n: $\pi(\partial T^n_j) + \pi(T^m_0 - F^m)$, η^*_n: $\pi(\partial T^n_j) \to \pi(T^m_0 - \bigcup_{m=1}^{k} T^{0}_m)$, and Φ^*_n: $\pi(\partial T^n_j) \to \pi(T^m_0 - \bigcup_{m=1}^{p} T^m_m)$ are monomorphisms induced by inclusion.*

This lemma is easy to prove using a standard inductive argument. We will omit the proof.

THEOREM 4.6. *Given any set A^n_i in the construction of A^n, the map ψ: $\pi(T^m_0 - A^n_i) \to \pi(T^m_0 - A^n_{i+1})$ is a monomorphism.*

Proof. Consider two cases.

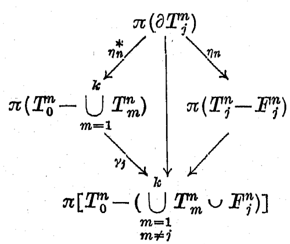
Case 1. i is even.

In this case $A^n_i = \bigcup_{m=1}^{k} T^m_m$, the union of k n-tubes, A^n_{i+1} is the subse of A^n_i equal to the union of the images of F^m, obtained by mapping (T^m_0, F^m) onto each of the n-tubes in A^n_i. If we can show that by mapping (T^m_0, F^m) onto one of the n-tubes in A^n_i, say T^m_j, the map

$$\gamma_j\colon\ \pi(T^m_0 - \bigcup_{m=1}^{k} T^m_m) \to \pi[T^m_0 - (\bigcup_{\substack{m=1\\ m\neq j}}^{k} T^m_m \cup F^m_j)]$$

induced by inclusion, is a monomorphism, then ψ is a monomorphism for i even.

Consider the following Van Kampen diagram.



By Lemma 4.5 we know that η_n and η^*_n are monomorphisms. Hence by Theorem 2.1, γ_j is a monomorphism. Therefore, ψ is a monomorphism when i is even.

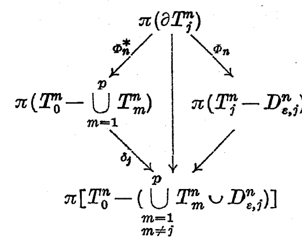
Case 2. i is odd.

In this case again $A^n_i = \bigcup_{m=1}^{p} T^m_m$, the union of p n-tubes. But this time A^n_{i+1} is the subset of A^n_i equal to the union of the images of D^n_ε obtained by mapping (T^n_0, D^n_ε) onto each of the n-tubes in A^n_i. If we can show that by mapping (T^n_0, D^n_ε) onto one of the n-tubes in A^n_i, say T^n_j, the map

$$\delta_j\colon\ \pi(T^n_0 - \bigcup_{m=1}^{p} T^m_m) \to \pi[T^n_0 - (\bigcup_{\substack{m=1\\ m\neq j}}^{p} T^m_m \cup D^n_{\varepsilon,j})],$$

induced by inclusion, is a monomorphism, then ψ is a monomorphism for i odd.

Consider the following Van Kampen diagram:



By Theorem 4.3(ii) and Lemma 4.5, we know that Φ_n and Φ^*_n are monomorphisms. Hence by Theorem 2.1, δ_j is a monomorphism. Therefore, ψ is a monomorphism for i odd.

THEOREM 4.7. $\pi(T^n_0 - A^n) \neq \pi(T^n_0)$.

Proof. Since $\pi(T^n_0 - A^n) = \overrightarrow{\lim_{i\to\infty}}\, \pi(T^n_0 - A^n_i)$ it suffices to show that there exists a subgroup of $\overrightarrow{\lim_{i\to\infty}}\, \pi(T^n_0 - A^n_i)$ which could not possibly be a subgroup of $\pi(T^n_0) = \bigoplus_{i=1}^{n-2} Z$.

We know that Φ_n: $\pi(\partial T^n_0) \to \pi(T^n_0 - A^n_0)$ is a monomorphism by Theorem 4.3(ii). We also know by Theorem 4.6 that at every stage in the construction the map ψ: $\pi(T^n_0 - A^n_i) \to \pi(T^n_0 - A^n_{i+1})$ is a monomorphism. Hence we can say that $\pi(\partial T^n_0) \to \pi(T^n_0 - A^n)$ is a monomorphism.

Since $\pi(\partial T_0^n) = \bigoplus_{i=1}^{n-1} Z$, then $\pi(T_0^n - A^n)$ has a subgroup of the form $\bigoplus_{i=1}^{n-1} Z$. But this implies that $\pi(T_0^n - A^n) \neq \bigoplus_{i=1}^{n-2} Z$.

Therefore, A^n is wild in E^n.

We now consider the problem of showing that $E^n - A^n$ is simply connected. All maps not labeled are induced by inclusion or are the composition of natural quotient maps following maps induced by inclusion.

The concept of geometrically unlinked 3-tubes used in the construction of A^3 in E^3 proved to be very cumbersome to generalize. Because of the difficulties involved we have extracted the algebraic properties necessary to show that when T_0^n is embedded in E^n nicely we have that $E^n - A^n$ is simply connected.

DEFINITION 4.8. Let $\{T_i^n,\ i = 1, 2, \ldots, k\}$ be a collection of disjoint n-tubes in $\mathring{T}_0^n$. Let P be a tree in $T_0^n - \bigcup_{i=1}^{k} \mathring{T}_i^n$ such that $P \cap \partial T_i^n$ is a single point for $i = 0, 1, 2, \ldots, k$. Let $K_i = \ker(\pi(\partial T_i^n) \to \pi(T_i^n))$ and let G_i be a subgroup of $\pi(\partial T_i^n)$ such that $K_i \oplus G_i = \pi(\partial T_i^n)$. Denote by H_0 the smallest normal subgroup of $\pi(T_0^n - \bigcup_{i=1}^{k} \mathring{T}_i^n)$ containing

$$\operatorname{im}\Big(G_0 \to \pi\big(T_0^n - \bigcup_{i=1}^{k} \mathring{T}_i^n\big)\Big).$$

The n-tubes $\{T_i^n,\ i = 1, 2, \ldots, k\}$ are h-unlinkable in T_0^n if for every $i = 1, 2, \ldots, k$,

$$G_i \subset \ker\Big[\pi(\partial T_i^n) \to \pi\big(T_0^n - \bigcup_{j=1}^{k} \mathring{T}_j^n\big)/H_0\Big]$$

and $\pi(P \cup \bigcup_{i=1}^{k} \partial T_i^n) \to \pi(T_0^n - \bigcup_{i=1}^{k} \mathring{T}_i^n)/H_0$ is an epimorphism.

LEMMA 4.9. *If $\{T_i^n,\ i = 1, 2, \ldots, k\}$ is h-unlinkable in T_0^n and $\{T_i^n,\ i = k+1, \ldots, m\}$ is h-unlinkable in T_1^n, then $\{T_i^n,\ i = 2, 3, \ldots, m\}$ is h-unlinkable in T_0^n.*

Proof. Let P_1 be a tree in $T_1^n - \bigcup_{i=k+1}^{m} \mathring{T}_i^n$ such that $P \cup P_1$ is connected and $P_1 \cap \partial T_i^n$ is a single point for $i = k+1, \ldots, m$. Let H_1 be the normal subgroup of $\pi(T_1^n - \bigcup_{i=k+1}^{m} \mathring{T}_i^n)$ generated by $\operatorname{im}\big(G_1 \to \pi(T_1^n - \bigcup_{i=k+1}^{m} \mathring{T}_i^n)\big)$ and let H_0' be the normal subgroup of $\pi(T_0^n - \bigcup_{i=2}^{m} \mathring{T}_i^n)$ generated by $\operatorname{im}\big(G_0 \to \pi(T_0^n - \bigcup_{i=2}^{m} \mathring{T}_i^n)\big)$. It is not difficult to see that since

$$G_j \subset \ker\Big[\pi(\partial T_j^n) \to \pi\big(T_0^n - \bigcup_{i=1}^{k} \mathring{T}_i^n\big)/H_0\Big]$$

for $j = 1, 2, \ldots, k$, and $G_j \to H_1 \to H_0'$ for $j = k+1, \ldots, m$, we have

$$G_j \subset \ker\Big[\pi(\partial T_j^n) \to \pi\big(T_0^n - \bigcup_{i=2}^{m} \mathring{T}_i^n\big)/H_0'\Big] \quad \text{for} \quad j = 2, 3, 4, \ldots, m.$$

By the Van Kampen theorem $\pi(T_0^n - \bigcup_{i=2}^{m} \mathring{T}_i^n)$ is generated by $\operatorname{im}\big(\pi(T_0^n - \bigcup_{i=1}^{k} \mathring{T}_i^n) \to \pi(T_0^n - \bigcup_{i=2}^{m} \mathring{T}_i^n)\big)$ and $\operatorname{im}\big(\pi(T_1^n - \bigcup_{i=k+1}^{m} \mathring{T}_i^n) \to \pi(T_0^n - \bigcup_{i=2}^{m} \mathring{T}_i^n)\big)$. Since $\pi(P \cup \bigcup_{i=1}^{k} \partial T_i^n) \to \pi(T_0^n - \bigcup_{i=1}^{k} \mathring{T}_i^n)/H_0$ and $\pi(P_1 \cup \bigcup_{i=k+1}^{m} \mathring{T}_i^n) \to \pi(T_1^n - \bigcup_{i=k+1}^{m} \mathring{T}_i^n)/H_1$ are epimorphisms and

$$\operatorname{im}\Big(H_0 \to \pi\big(T_0^n - \bigcup_{i=2}^{m} \mathring{T}_i^n\big)\Big) \subset H_0' \quad \text{and} \quad \operatorname{im}\Big(H_1 \to \pi\big(T_0^n - \bigcup_{i=2}^{m} \mathring{T}_i^n\big)\Big) \subset H_0',$$

it follows that $\pi(P \cup P_1 \cup \bigcup_{i=2}^{m} \partial T_i^n) \to \pi(T_0^n - \bigcup_{i=2}^{m} \mathring{T}_i^n)/H_0'$ is an epimorphism.

The following lemma follows easily from the definition of h-unlinkable.

LEMMA 4.10. *If $\{T_i^n,\ i = 1, 2, \ldots, k\}$ is h-unlinkable in T_0^n, then $\{S^1 \times T_i^n,\ i = 1, 2, \ldots, k\}$ is h-unlinkable in $S^1 \times T_0^n = T_0^{n+1}$.*

Note that we have previously shown that $\{T_\delta^3, T_{\delta'}^3\}$ are h-unlinkable in T_0^3, as is F^3 in T_0^3. Using Lemmas 4.9 and 4.10 we can prove

THEOREM 4.11. *A_k^n consists of h-unlinkable n-tubes in T_0^n.*

LEMMA 4.12. *Let H^n be the normal subgroup of $\pi(T_0^n - F^n)$ generated by $\operatorname{im}\big(G_0 \to \pi(T_0^n - F^n)\big)$. Then $\pi(\partial T_0^n) \to \pi(T_0^n - F^n)/H^n$ is the trivial homomorphism.*

Proof. It has already been proven that $\pi(\partial T_0^3) \to \pi(T_0^3 - F^3)/H^3$ is the trivial homomorphism. Suppose now that $\pi(\partial T_0^{n-1}) \to \pi(T_0^{n-1} - F^{n-1})/H^{n-1}$ is the trivial homomorphism. Since $T_0^n - F^n = S^1 \times (T_0^{n-1} - F^{n-1})$ and $\partial T_0^n = S^1 \times \partial T_0^{n-1}$ we have that $\pi(\partial T_0^n) = \pi(S^1) \times \pi(\partial T_0^{n-1})$. We note that H^n contains $\operatorname{im}\big(\pi(S^1) \to \pi(T_0^n - F^n)\big)$ and H^n contains $\operatorname{im}\big(\pi(\partial T^{n-1}) \to \pi(T_0^n - F^n)\big)$, thus H^n contains $\operatorname{im}\big(\pi(\partial T_0^n) \to \pi(T_0^n - F^n)\big)$.

THEOREM 4.13. *If $\{T_i^n,\ i = 1, 2, \ldots, k\}$ is h-unlinkable in T_0^n and $f_i\colon T_0^n \to T_i^n$ are homeomorphisms for $i = 1, 2, \ldots, k$, then $\pi(T_0^n - \bigcup_{i=1}^{k} \mathring{T}_i^n)$*

$\to\pi\big(T_0^n - \bigcup_{i=1}^{k} f_i(F^n)\big)/H_0''$ *is the trivial homomorphism, where* H_0'' *is the normal subgroup generated by* $\operatorname{im}\Big(H_0\to\pi\big(T_0^n-\bigcup_{i=1}^{k} f_i(F^n)\big)\Big)$. ($H_0$ *as defined in* 4.8.)

The theorem follows inductively from the previous lemma.

Now let $h\colon T_0^n\to E^n$ be an embedding such that $G_0\subset\ker\pi\big(h(\partial T_0^n)\big)\to\pi\big(E^n-h(T_0^n)\big)$. Such an embedding can be obtained by rotation of an embedding of T_0^{n-1} in E^{n-1} about an $(n-2)$-dimensional hyperplane. Using Theorems 4.11 and 4.13 we can show inductively that $\pi(E^n-A_k^n)\to\pi(E^n-A_{k+1}^n)$ is the trivial homomorphism for k even. Since $\pi(E^n-A^n)$ is the direct limit of $\pi(E^n-A_0^n)\to\pi(E^n-A_1^n)\to\pi(E^n-A_2^n)\to\ldots$ it follows that $\pi(E^n-A^n)=1$ and hence E^n-A^n is simply connected.

V. Proof of the corollaries.

DEFINITION 5.1. Let $A\subset E^n$ be a Cantor set. A sequence $\{A_m^n\}$ of compact subsets of E^n will be called a *defining sequence* for A if it has the following properties:

(i) A_m^n is a polyhedral n-manifold in E^n,

(ii) $A=\bigcap_{m=1}^{\infty} A_m^n$,

(iii) $A_{m+1}^n\subset \overset{0}{A}{}_m^n$,

(iv) There are finitely many components of A_m^n, each of which has diameter less than $1/m$,

(v) $E^n-A_m^n$ is connected for every m.

It is an easy exercise to show that every Cantor set in E^n has a defining sequence.

DEFINITION 5.2. Let $\{A_m^n\}$ be a defining sequence for $A\subset E^n$ and suppose that for each $k>1$, B_m^k is a k-ball in E^n with the following properties:

(i) $B_m^k\subset\overline{E^n-A_m^n}$,

(ii) B_m^k is a polyhedron in E^n,

(iii) $B_m^k\cap A_m^n$ is the union of finitely many disjoint $(k-1)$-balls, each of which lie in $\partial A_m^n\cap\partial B_m^k$,

(iv) $A_m^n\cup B_m^k$ is connected,

(v) $\overline{B_{m+1}^k-B_m^k}\subset\overline{A_m^n-A_{m+1}^n}$,

(vi) $B_m^k\subset B_{m+1}^k$.

Then $\overline{\bigcup_{m=1}^{\infty} B_m^k}=B^k$ is a k-ball (see [7]) called an *osculating k-ball* for A and the sequence $\{B_m^k\}$ is called a *defining sequence* for B^k with respect to $\{A_m^n\}$. An osculating 1-ball (arc) is an arc containing A in the boundary of an osculating 2-ball. The existence of osculating balls for every Cantor set is guaranteed by the results of [7].

It is not difficult to see that $B^k=\bigcap_{m=1}^{m}(B_m^k\cup A_m^n)$.

THEOREM 5.3. *Let $A\subset E^n$ be a Cantor set and let B^k be an osculating k-ball for A. If $n\geqslant 4$, then $\pi(E^n-A)=\pi(E^n-B^k)$.*

Proof. Let $\{B_m^n\}$ be a defining sequence for the osculating n-ball B^n with respect to $\{A_m^n\}$. Then $\pi(E^n-A)=\varinjlim\pi(E^n-A_m^n)$ and $\pi(E^n-B^n)=\varinjlim\pi\big(E^n-(A_m^n\cup B_m^n)\big)$. Let $Q_m^{n-1}=\overline{\partial B_m^n-\partial A_m^n}$. By our hypotheses, Q_m^{n-1} is an $(n-1)$-sphere with holes. Now consider the following Van Kampen diagram.

$$\begin{array}{ccc} & \pi(Q_m^{n-1}) & \\ \swarrow & & \searrow \\ \pi\big(E^n-(A_m^n\cup B_m^n)\big) & & \pi(B_m^n) \\ \searrow & & \swarrow \\ & \pi(E^n-A_m^n) & \end{array}$$

Since $\pi(Q_m^{n-1})=1=\pi(B_m^n)$ $(n\geqslant 4)$, it follows that $\pi\big(E^n-(A_m^n\cup B_m^n)\big)\to\pi(E^n-A_m^n)$ is an isomorphism. Furthermore the diagram

$$\begin{array}{ccc} \pi\big(E^n-(A_m^n\cup B_m^n)\big) & \longrightarrow & \pi\big(E^n-(A_{m+1}^n\cup B_{m+1}^n)\big) \\ \updownarrow & & \updownarrow \\ \pi(E^n-A_m^n) & \longrightarrow & \pi(E^n-A_{m+1}^n) \end{array}$$

commutes. (All maps are induced by inclusion.) It follows that

$$\varinjlim\pi\big(E^n-(A_m^n\cup B_m^n)\big)=\varinjlim\pi(E^n-A_m^n)\,.$$

Now an osculating k-ball, B^k for A, in E^n can be obtained by the intersection of a nested sequence of osculating n-balls, the complement of each n-ball having the same fundamental group as E^n-A. Again by direct limits, it follows that $\pi(E^n-B^k)=\pi(E^n-A)$.

Corollary 1 for $n\geqslant 4$ follows immediately from Theorem 5.3. For $n=3$ it can be shown that an osculating ball for a Cantor set with simply connected complement has a simply connected complement.

Examples are known of wild balls in E^3 with simply connected complements [4].

DEFINITION 5.4. The boundary of an osculating k-ball for A is called an *osculating* $(k-1)$-*sphere*.

THEOREM 5.5. *If S^k is an osculating k-sphere for A, then $\pi(E^n-S^k) = \pi(E^n-A)$ for $0<k<n-2$ and for $k=n-2$ there is a short exact sequence* $1\to Z\to\pi(E^n-S^{n-2})\to\pi(E^n-A)\to 1$.

Proof. Let C^n be a polyhedral n-ball in E^n-A with the property that $\partial C^n \cap S^k$ is an unknotted, polyhedral $(k-1)$-sphere in ∂C^n. Clearly $\overline{S^k-C^n} = B^k$ is an osculating k-ball for A in E^n. Furthermore $\partial C^n - S^k$ has the homotopy type of $S^{n-1-(k-1)-1} = S^{n-k-1}$ and clearly $\pi\big(E^n-(S^k\cup \overset{0}{C}{}^n)\big) = \pi(E^n-S^k)$. Now consider the Van Kampen diagram

$$\begin{array}{ccccc} & & \pi(\partial C^n - S^k) & & \\ & \swarrow & \downarrow & \searrow & \\ \pi\big(E^n-(S^k\cup \overset{0}{C}{}^n)\big) & & \downarrow & & \pi(C^n) \\ & \searrow & \downarrow & \swarrow & \\ & & \pi\big((E^n-S^k)\cup C^n\big) & & \end{array}$$

We have $\pi(C^n)=1$ and $\pi(\partial C^n-S^k)=\pi(S^{n-k-1})=1$ if $n-k-1>1$, i.e., $n-2>k$. We see that $\pi\big(E^n-(S^k\cup \overset{0}{C}{}^n)\big)\to\pi\big((E^n-S^k)\cup C^n\big)$ is an isomorphism or $\pi(E^n-S^k)\to\pi(E^n-B^k)$ is an isomorphism. Thus, by Theorem 5.3, $\pi(E^n-S^k)\to\pi(E^n-A)$ is an isomorphism for $n-k-1>1$. If $n-2=k$, it is not difficult to see that $\pi(\partial C^n-S^{n-2})\to\pi\big(E^n-(S^{n-2}\cup \overset{0}{C}{}^n)\big)$ is a monomorphism and that $\pi(\partial C^n-S^{n-2})\to\pi\big((E^n-S^{n-2})\cup C^n\big)$ is trivial. Thus, $1\to Z\to\pi(E^n-S^{n-2})\to\pi(E^n-B^{n-2})\to 1$ is exact. Since $\pi(E^n-B^{n-2}) = \pi(E^n-A)$, the result follows.

References

[1] L. Antoine, *Sur l'homéomorphisme de deux figures et de leurs voisinages*, J. Math. Pures Appl. 4 (1921), pp. 221–325.

[2] W. A. Blankenship, *Generalization of a construction of Antoine*, Ann. of Math. (2) 53 (1951), pp. 276–297.

[3] R. P. Coelho, *On the groups of certain linkages*, Portugaliae Math. 6 (1947), pp. 57–65.

[4] R. H. Fox and E. Artin, *Some wild cells and spheres in three-dimensional space*, Ann. of Math. 49 (1948), pp. 979–990.

[5] A. Kirkor, *Wild 0-dimensional sets and the fundamental group*, Fund. Math. 45 (1958), pp. 228–236.

[6] W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory*, New York 1966.

[7] R. P. Osborne, *Embedding Cantor sets in a manifold*, Part II, Fund. Math. 65 (1969), pp. 147–151.

[8] J. H. C. Whitehead and M. H. A. Newman, *On the group of a certain linkage*, Quart. J. Math. (2) 8 (1937), pp. 14–21. [Vol. II].

Reçu par la Rédaction le 30. 5. 1972