A wild Cantor set in $E^n$ with simply connected complement

by

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Abstract. It is well known that the complement of a tame Cantor set embedded in $E^n$, $n \geq 3$, is simply connected. The converse of this statement is false. In fact there is an example of a wild Cantor set in $E^n$ whose complement is simply connected, due to A. Kirkor. However, no example was known in dimensions greater than three, since it was not clear how to generalize Kirkor's construction to higher dimensions. Using solid tori we construct a wild Cantor set in $E^n$, whose complement is simply connected. This construction generalizes to $E^n$, $n > 3$. This result is then used to construct wild cells and spheres in $E^n$ with simply connected complements.

In 1921, L. Antoine [1] constructed a wild zero-dimensional subset of $S^1$. This set was proven to be wild by showing that the fundamental group of its complement in $S^1$ was non-trivial, [3]. In 1961, W. A. Blankenship [2] extended Antoine's construction to $n$-dimensions giving an example of a compact, zero-dimensional set $A \subset E^n$, whose complement was not simply connected. In 1958, A. Kirkor [5] gave the first known example of a wild zero-dimensional subset of $S^1$ whose complement was simply connected. This set was constructed by entangling arcs in $S^1$. Since it is not clear how to generalize Kirkor's construction to $n$-dimensions, we will give a different construction in $E^n$ which can be generalized to $n$-dimensions, $n > 3$.

A Cantor set in $E^n$ will be called tame if there exists a homeomorphism of $E^n$ onto itself which maps the Cantor set into a straight line segment. A Cantor set which is not tame is wild.

The following is our principal theorem.

Theorem. For $n \geq 3$, there exists a wild Cantor set $A \subset E^n$ whose complement is simply connected.

Corollary 1. For $n \geq 3$ and $0 < k \leq n$, there exists wild $k$-balls in $E^n$ with simply connected complements.
COROLLARY 2. For $n \geq 3, 0 < k < n - 3$, there exists wild $k$-spheres in $\mathbb{F}^n$ with simply connected complements. There also exists wild $(n-2)$-spheres in $\mathbb{F}^n$ whose complements have infinite cyclic fundamental groups, and wild $(n-1)$-spheres in $\mathbb{F}^n$ whose complementary domains are simply connected.

The paper will be divided into five parts. In part I the Cantor set $A^4 \subset \mathbb{F}^3$ is constructed. In part II, we prove that $A^4$ is wild by showing that $\pi(T^3_2 - A^4)$ is not what it would be if $A^4$ were tame ($T^3_2$ is the canonical solid three-dimensional torus). In part III, we show that the complement of $A^4$ in $\mathbb{F}^3$ is simply connected. In part IV we generalize the construction and results to $A^8$ in $\mathbb{F}^5, n > 5$, and in part V we shall prove the two corollaries following the principal theorem.

I. The construction of the Cantor set $A^4$. We first give an intuitive construction of $A^4$.

DEFINITION 1.1. Let $J$ be a polygonal simple closed curve in $\mathbb{F}^3$. Given $\varepsilon > 0$, denote by $N_{\varepsilon}(J)$ a regular neighborhood of $J$ in $\mathbb{F}^3$ such that $d(x, J) < \varepsilon$ for all $x \in N_{\varepsilon}(J)$. $J$ will be called the central simple closed curve of $N_{\varepsilon}(J)$.

The Cantor set $A^4$ will be the intersection of a decreasing sequence of non-empty compact sets, $A^4_0 \supset A^4_1 \supset A^4_2 \cdots$. The sequence of sets will depend on the two standard models shown in Figure 1.

In Model $\delta$, $J^\ast$ is the central simple closed curve of the solid unknotted torus $T^2_2$ in $\mathbb{F}^3$ and $T_1, T_2$ are solid tori embedded in $T^2_2$ so that they are entangled as shown in the figure. This embedding will be accomplished in such a way that if $J^\ast$ is cut into two arcs $J^1_1, J^1_2$ at the points $t_1, t_2$, then given $\delta > 0$ and points $p \in T_1, p' \in T_2$,

(i) $d(p, J^1_1) < \delta$, and

(ii) $d(p', J^1_2) < \delta$.

Model $\delta$ is considered to be the torus $T^2_2$ with $T_1$ and $T_2$ as subsets of $T^2_2$ and will be denoted as the pair $(T^2_2, T_1 \cup T_2)$.

In Model $F$, $F^3$ is a solid torus embedded in $T^2_2$ as shown in Figure 1. Model $F$ is considered to be $T^2_2$ with $F^3$ as a subset and will be denoted as the pair $(T^2_2, F^3)$.

To obtain $A^4$ we must construct a decreasing sequence of non-empty compact sets. To obtain the first set we will construct a set $D^2_2$ in $T^2_2$ which will be defined in detail later.

To construct $D^2_2$ in $T^2_2$ we will start with Model $\delta$. We will use iterations of a mapping which sends Model $\delta$ onto each of the tori in the preceding step. For example, the first step will be the mapping of Model $\delta$ onto each of the tori already embedded in $T^2_2$. These mappings will be continued until each of the tori in the intersection of the images of $T_1 \cup T_2$ in $T^2_2$ have diameters less than a given $\varepsilon > 0$. It is this part of the construction which will give us the zero-dimensionality of the set $A^4$.

After the diameters of each torus is less than a given $\varepsilon > 0$, we will then map $(T^2_2, F^3)$ onto each of the tori. This step will be instrumental in proving that the complement of $A^4$ is simply connected. $A^4$ is then constructed by iterating these two processes.

In proving $\pi(\mathbb{F}^3 - A^4)$ is trivial and in proving $\pi(T^2_2 - A^4)$ is not what it would be if $A^4$ were tame, we will use a direct limit argument on the fundamental groups.

Now we will go into the detail of constructing $A^4$. In constructing $A^4$ we must be sure that

(i) the diameters of each of the tori become as small as we want, to insure the zero-dimensionality,
(ii) \( \pi(T_2 - A) \neq Z \), to insure the wildness of \( A \), and
(iii) \( \pi(B_0 - A) \) is trivial.

First of all, we will construct a set, \( D_i \), consisting of disjoint solid tori. \( D_i \) will be the intersection of images of \( T_2 \cup T_p \) in \( T_2 \). These tori will have diameters less than \( \varepsilon \). \( D_i \) will be used in the definition of \( A_i \), the first set in the sequence of decreasing sets used in defining \( A \).

**Theorem 1.2.** Given \( \varepsilon > 0 \), we can construct a set \( D_i \) of solid tori in \( T_2 \) such that

(i) the diameters of each of the tori comprising \( D_i \) are less than \( \varepsilon \),
(ii) these tori are unknotted and unlinked in \( T_2 \), and
(iii) \( \phi_2 : \pi(T_2) \to \pi(T_2 - D_i) \) is a monomorphism, where \( \phi_2 \) is induced by inclusion.

To prove this theorem, we shall use the following lemmas.

**Lemma 1.3.** Let \( J \) be a polygonal simple closed curve in \( E^3 \). Let \( P \) be a plane in \( E^3 \). And let \( \varepsilon > 0 \) be given. Define \( \bar{P} \) to be a “thickening” of \( P \), i.e., \( \bar{P} = \{ x \in E^3 : d(x, P) < \varepsilon \} \). Define \( {\bar{P}}_1 \) and \( {\bar{P}}_2 \) to be the boundary components of \( \bar{P} \). Define \( E_{\bar{P}} \) and \( E_{\bar{P}}_2 \) to be the spaces “above” \( {\bar{P}}_1 \) and “below” \( {\bar{P}}_2 \) respectively, such that \( E_{\bar{P}} \cup E_{\bar{P}}_2 \cup \bar{P} \) is pairwise disjoint but \( E_{\bar{P}} \cup E_{\bar{P}}_2 \cup \bar{P} \) is \( T_2 \). Let \( X(J) \) be a regular neighborhood of \( J \). Then

(i) there exists a minimal set \( M \) consisting of an even number, \( 2k \), of distinct points in \( J \cap P \) such that every pair of points in \( J \cap P \) lies entirely in \( E_{\bar{P}} \cup \bar{P} \) or \( E_{\bar{P}}_2 \cup \bar{P} \), and
(ii) after \( i \) steps of the construction, involving a suitable homeomorphism, mapping \( \varphi \) onto \( \varphi(J) \), each of the tori embedded in \( X(J) \) will lie entirely in \( E_{\bar{P}} \cup \bar{P} \) or \( E_{\bar{P}}_2 \cup \bar{P} \).

**Proof.** Part (i) is easily proved by routine methods. The proof of part (ii) will be by induction on \( k \). For \( k = 0 \) the conclusion is clear. Assume it is true for \( k = m - 1 \). Consider the \( 2m \) distinct points in \( M \). Pick one of them, call it \( x_1 \). Proceed around \( J \) in a clockwise fashion and order the points \( x_1, x_2, \ldots, x_{2m+1} \) as we encounter them. Consider \( x_1 \) and \( x_{2m+1} \). Define our homeomorphism \( \alpha : (T_2, T_2 \cup T_2) \to X(J) \) such that

1. \( \alpha(x_1) = x_1 \),
2. \( \alpha(x_2) = x_{2m+1} \), and
3. \( \alpha(X) = X \).

Such a homeomorphism clearly exists. Since \( x_1 \) is a homeomorphism of a compact set onto a compact set then \( \alpha \) is uniformly continuous. Hence, given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that, given points \( p \in T_2 \) and \( r \in T_2 \) such that \( d(p, r) < \delta \), then \( d(\alpha(p), \alpha(r)) < \varepsilon \). The same holds true for points \( q \in T_2 \) and \( s \in T_2 \).

Now we have two tori \( \alpha(T_2) \) and \( \alpha(T_2 \cup T_2) \) embedded in \( X(J) \) by a suitable homeomorphism. Their central simple closed curves are \( J_1 \) and \( J_2 \) respectively. The minimal sets for each of \( J_1 \) and \( J_2 \) then contains two fewer points than the minimal set for \( J \). The lemma follows inductively.

The set \( D_i \) will be the intersection of a decreasing sequence of non-empty compact sets

\[ C_0 \supset C_1 \supset \ldots \supset C_i \]

Define \( C_0 \) to be \( T_2 \cup T_2 \) with \( \varepsilon \) small enough. By “cutting” \( T_2 \) with planes one at a time and by using the suitable homeomorphisms we can construct \( C_j \),

\[ C_1 \supset \ldots \supset C_i \]

for \( 1 \leq i \leq k \), in such a way that the tori contained in \( \bigcap_{i=0}^{k} C_i \) lie in cubes with diameter less than \( \varepsilon \). \( D_i \) will then be equal to \( C_i \). Notice \( D_i \subset T_2 \).

**Proof of Theorem 1.2.**

(i) It is clear that since \( E^3 \) can be partitioned into cubes with diameter less than \( \varepsilon \), \( D_i \) then be equal to \( C_i \). Notice \( D_i \subset T_2 \).

(ii) To prove this part we need only remark that at each stage of the construction the tori are unknotted and hence unlinked in \( E^3 \). In \( C_i \) we have two tori. It is well known that since the central simple closed curves of \( T_2 \) and \( T_2 \) bound non-singular disks, \( T_2 \) and \( T_2 \) are unknotted in \( E^3 \). Hence the tori in \( C_i \) are unknotted in \( E^3 \). We proceed inductively and arrive at the result.

(iii) The third part will be proven later in the paper (Lemma 2.4).

\( A \) will then consist of the intersection of a decreasing sequence of non-empty compact sets

\[ A_0 \supset A_1 \supset \ldots \]

\( A_0 \) will be defined to be \( D_i \). \( A_1 \) will be the subset of \( A_2 \) equal to the union of the images of \( F_0 \) obtained by homeomorphically mapping \( (T_2, F_2) \) onto each of the tori in \( A_1 \). \( A_2 \) is the subset of \( A_3 \) equal to the union of the images of \( E_i \) obtained by mapping \( (T_2, E_i) \) homeomorphically onto each of the tori in \( A_2 \). Here, \( \theta \) is a function chosen so that when \( (T_2, E_i) \) is mapped onto each of the tori in \( A_1 \), the images of each of the tori in \( E_i \) have diameters less than \( \varepsilon \).

In general, \( A_i, i > 0, i = 2j, \) is the subset of \( A_{i+1} \), equal to the union of the images of \( E_i \) obtained by mapping \( (T_2, E_i) \) homeomorphically onto each of the tori in \( A_i \), and \( A_{i+1}, i \) odd, is the subset of \( A_i \), equal to the union of the images of \( E_i \) obtained by mapping \( (T_2, E_i) \)...
homeomorphically onto each of the tori in $A_{i-1}$. We then define $A^t$ to be
\[ \bigcap_{i=1}^{\infty} A^t_i. \]

It is a routine matter to show that $A^t$ is a Cantor set.

II. $A^t$ is wild. In this section we will prove that $A^t$ is wild by first noticing that given a tame Cantor set $X$, $\pi(T^3_0 - X) = \pi(T^3_0) = Z$, the free group on one generator. Whereas $\pi(T^3_0 - A^t_i) \neq Z$, hence proving $A^t$ is wild.

Now to show that $\pi(T^3_0 - A^t) \neq Z$ we will need the following theorem.

All homomorphisms given below are taken to be those induced by inclusion.

**Theorem 2.1.** Let $A$ and $B$ be open arcwise connected subsets of $A \cup B$ such that $A \cap B$ is arcwise connected. If the homomorphisms $\varphi_i: \pi(A \cap B) \to \pi(A)$ and $\psi_i: \pi(A \cap B) \to \pi(B)$ are monomorphisms, then $\varphi_i: \pi(A) \to \pi(A \cap B)$ and $\psi_i: \pi(B) \to \pi(A \cap B)$ are monomorphisms.

**Proof.** $\pi(A \cup B)$ is the free product of $\pi(A)$ and $\pi(B)$ with $\varphi_i(\pi(A \cap B))$ and $\psi_i(\pi(A \cap B))$ amalgamated [6], sec. 4.2.

The following lemmas are immediate consequences of [8], Theorem 2.

**Lemma 2.2.** $\xi: \pi(T^3_0) \to \pi(T^3_0 - F^2)$ is a monomorphism.

**Lemma 2.3.** $\xi': \pi(T^3_0) \to \pi(T^3_0 - F^2)$ is a monomorphism.

![Diagram of $\pi(T^3_0)$](image)

The two lemmas given below are both proved by similar methods.

**Lemma 2.4.** $\varphi: \pi(T^3_0) \to \pi(T^3_0 - (T^3_0 \cup T^3_0))$ is a monomorphism.

**Lemma 2.5.** $\varphi': \pi(T^3_0) \to \pi(T^3_0 - (T^3_0 \cup T^3_0))$ is a monomorphism.

**Proof.** Let $\pi(T^3_0) = \langle a, b, [a, b] = 1 \rangle$, where $a$ and $b$ are as indicated in Figure 2.

Now consider $a^n b^m \in \pi(T^3_0)$. We show that $\varphi(a^n b^m) = 1$ if and only if $m = n = 0$. Consider the commutative diagram,

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\[
\begin{array}{c}
\pi(T^3_0) \xrightarrow{a} \pi(T^3_0 - T^3_0) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\pi(T^3_0 - (T^3_0 \cup T^3_0)) \\
\end{array}
\]

Now $\varphi(a^n b^m)$ is non-trivial unless $m = 0$. It follows that $\varphi'(a^n b^m)$ is non-trivial unless $m = 0$. Next consider the commutative diagram,

\[
\begin{array}{c}
\pi(T^3_0) \xrightarrow{a} \pi(T^3_0 - T^3_0) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\pi(T^3_0 - (T^3_0 \cup T^3_0)) \\
\end{array}
\]

Since $\varphi(a^n b^m)$ is non-trivial unless $n = 0$, it follows that $\varphi'(a^n b^m)$ is non-trivial unless $n = 0$.

**Theorem 2.6.** Given any of the sets $A^t_i$ in the construction of $A^t$, the map $\psi: \pi(T^3_0 - T^3_0) \to \pi(T^3_0 - T^3_0)$ is a monomorphism.

**Proof.** To prove this we must consider two cases.

Case 1. $i$ is even.

In this case $T^3_0 - A^t_i$ is the complement in $T^3_0$ of many images of $T^3_i$ and $T^3_i$ entangled in the manner set forth in the construction. $A^t_{i-1}$ is the subset of $A^t_i$ obtained by mapping $(T^3_n, F^2)$ homeomorphically onto each of the solid tori in $A^t_i$.

Set $T^3_0 - A^t_i = T^3_0 - \bigcup_{n=1}^{i-1} T^3_n$, where $k$ is the number of tori comprising $A^t_i$. Now, if we can show that by mapping $(T^3_n, F^2)$ onto one of the $T^3_i$'s, say $T^3_i$, the map

\[
\gamma_i: \pi(T^3_0 - \bigcup_{n=1}^{i-1} T^3_n) \to \pi(T^3_0 - (\bigcup_{n=1}^{i-1} T^3_n \cup F^2))
\]

is a monomorphism, then after $k$ such maps we have that $\psi: \pi(T^3_0 - A^t_i) \to \pi(T^3_0 - A^t_{i+1})$ is a monomorphism.

Let $\gamma_i$ be defined by the inclusion and consider the following Van Kampen diagram.
By Lemmas 2.2 and 2.5 we know that $f$ and $f'$ are monomorphisms. Hence, we can apply Theorem 2.1 and conclude $\gamma_1$ is a monomorphism.

Therefore, $\psi: \pi(T_0^2 - A^2_1) \rightarrow \pi(T_0^2 - A^2_{i+1})$ is a monomorphism when $i$ is even.

Case 2. $i$ is odd.

Again, when $i$ is odd, $T_0^2 - A^2_i$ is the complement in $T_0^2$ of many solid tori. But this time the solid tori are images of $F^n$ entangled as prescribed in the construction. $T^n_0 - A^n_{i+1}$ is the complement of $A^n_{i+1}$ in $T^n_0$, where $A^n_{i+1}$ is the subset of $A^n_i$ obtained by mapping $(T^n_0, D^n_0)$ homeomorphically onto each of the tori in $A^n_i$. Remember that $D^n_0$ is the union of many homeomorphic images of $\delta T^n_i$.

Consider $T^n_0 - A^n_i$ to be $T^n_0 - \bigcup_{n=1}^m T^n_{n1}$, where $m$ is the number of tori in $A^n_i$. Now if we can show that by mapping $(T^n_0, D^n_0)$ onto each of the $T^n_k$'s, say $T^n_k$, the map

$$\delta_i: \pi(T^n_0 - \bigcup_{n=1}^m T^n_{n1}) \rightarrow \pi(T^n_0 - \bigcup_{n=1}^m T^n_{n1} \cup D^n_{i+1})$$

is a monomorphism, then after $m$ such maps we have that $\psi: \pi(T_0^2 - A^2_1) \rightarrow \pi(T_0^2 - A^2_{i+1})$ is a monomorphism.

To show $\delta_i$ is a monomorphism it suffices to show that $\mu: \pi(\delta T^n_0) \rightarrow \pi(\delta T^n_0')$ is a monomorphism. An application of the Van Kampen theorem completes the proof. Recall that $D^n_0$ is the homeomorphic image of $D^n_0$ and that $D^n_0$ was constructed by successively mapping the pair $(T^n_0, T^n_0 \cup T^n_r)$ into its subtori. Again using the Van Kampen theorem and Theorem 2.1 we see that we can reduce our problem to one of showing that $\pi(\delta T^n_0) \rightarrow \pi(T^n_0 - (T^n_0 \cup T^n_r))$ and $\pi(\delta T^n_0) \rightarrow \pi(T^n_0 - (T^n_0 \cup T^n_r))$ are monomorphisms. But this is exactly Lemmas 2.4 and 2.5.

The preceding theorem gives us an inductive step in showing that $\pi(T_0^2 - A^2) \neq Z$ and hence $A^2$ is wild.

**Theorem 2.7.** $\pi(T_0^2 - A^2) \neq Z$.

**Proof.** Since $\pi(T_0^2 - A^2) = \lim_{\rightarrow} \pi(T_0^2 - A^n_0)$, it suffices to show that there exists a subgroup of $\lim_{\rightarrow} \pi(T_0^2 - A^n_0)$ which could not possibly be a subgroup of $Z$.

It is clear by using Lemma 2.4 that $\pi(\delta T^n_0) \rightarrow \pi(T^n_0 - A^n_0)$ is a monomorphism. Since we have also proven that at every stage in the construction, the map $\psi: \pi(T^n_0 - A^n_0) \rightarrow \pi(T^n_0 - A^n_{i+1})$ is a monomorphism, then we can say that the inclusion map $\pi(\delta T^n_0) \rightarrow \pi(T^n_0 - A^n_0)$ is a monomorphism.

Since $\pi(T_0^2) = \langle a, b | a(a^{-1}b^{-1}) \rangle$, the free abelian group on two elements, and $\pi(T^n_0) \rightarrow \pi(T_0^2 - A^n) = \{0\}$ is a monomorphism, then $\pi(T_0^2 - A^n)$ has a free abelian subgroup on two elements.

Therefore, $\pi(T_0^2 - A^2) \neq Z$ and hence $A^2$ is wild in $F^n$.

**III.** $F^2 - A^2$ is simply connected.

**Theorem 3.1.** $\pi(F^2 - A^2)$ is trivial.

**Proof.** By Theorem 1.2 we have that in the construction of $D^n_0$ all the tori comprising it are unlinked and unknotted in $F^n$. Hence $\pi(F^2 - A^2) = \langle a_1, a_2, \ldots, a_n \rangle$ the free group on $k$ elements, where $k$ is the number of tori in $A^n_i$. At the next stage of the construction, we map $(T^n_0, D^n_0)$ onto each of the tori in $A^n_0$ homeomorphically. The union of the images of $F^n$ in $A^n_0$ is $A^n_0$.

Consider this mapping and the map $\beta: \pi(F^2 - A^n_0) \rightarrow \pi(F^2 - A^n_1)$ induced by inclusion. Let $\beta: \pi(F^2 - A^n_0) \rightarrow \pi(F^2 - A^n_1)$ be the map induced by inclusion. We know that

(a) $\pi(F^2 - T^n_0) = \langle a_1 \rangle$, and
(b) $\pi(F^2 - T^n_0) = \langle b_1, b_2, b_3 \rangle = 1$,

where $\pi(F^2 - T^n_0)$ is calculated from Figure 3.

**Fig. 3**

Now $\beta: \langle a_1 \rangle \rightarrow \langle b_1, b_2, b_3 \rangle = 1$ and $\beta(\langle a_1 \rangle) = b_1 b_2^{-1} = 1$. Hence $\beta$ takes all of the generators of $\pi(F^2 - A^n_0)$ to the identity in $\pi(F^2 - A^n_1)$.

Repeating this argument we see that every other map in the direct limit that gives $\pi(F^2 - A^2)$ is the trivial map. Since

$$\pi(F^2 - A^2) = \lim_{\rightarrow} \pi(F^2 - A^n_0)$$

we can use a direct limit argument and conclude that given any generator in one of the groups in the direct limit, that generator is eventually mapped to the identity. Hence $\pi(F^2 - A^n_0) = \lim_{\rightarrow} F^2$ is trivial.
One should note that the above argument would fail if at any stage of the construction any of the solid tori encountered were knotted. If this happened, the two tori obtained by mapping model 6 onto this knotted torus would be linked and the subsequent groups would not be free as asserted above.

This completes the construction of a wild Cantor set $A^*$ in $F^n$ whose complement is simply connected.

**IV. A generalization to $F^n$.** Using our construction of $A^*$ in $F^3$ we will generalize to $A^*$ in $F^n$ by an inductive argument. The following lemma and definition will be useful in the construction.

**Lemma 4.1.** Let $X = X_1 \times X_2 \times X_3$, where $X_i$ is a compact metric space, $i = 1, 2, 3$, and $X_1 = X_2$. Suppose $A$ and $B$ are subsets of $X_2 \times X_3$. Let $k: X_2 \times X_3 \rightarrow A$ be a homeomorphism. Define a “switching map” $s: X \rightarrow X$ so that $s = s_k \times s_{i-1} \times 1$, where $s_k: X_2 \rightarrow X_2$ is a homeomorphism. Then given $e > 0$ there exists a $\delta > 0$ such that if $\text{diam}(A)$ and $\text{diam}(B)$ are less than $\delta$, then $\text{diam}([1 \times k]s(X_1 \times B)) < e$.

The proof of this lemma is a simple point set topology argument.

**Definition 4.2.** We define $T^n$ to be the unit disk in $F^n$ and define $T^n_i$ to be $(S^{n-i})_i \times T^{n-i}$, where $(S^{n-i})_i$ denotes the $(n-i)$-fold Cartesian product of the one sphere, $S^1$. An $n$-tube is a space homeomorphic with $T^n_i$.

Henceforth, all unlabeled maps are induced by inclusion. $A^*$ will be the intersection of a decreasing sequence of non-empty compact sets, $A^n_0 \supset A^n_1 \supset A^n_2 \ldots$, where each $A^n_i$ is the union of disjoint $n$-tubes. In the construction of $A^*$ we must be sure that

(i) the diameter of each $n$-tube in $A^n_i$ approaches zero as $k$ goes to infinity,

(ii) $A^*$ is wild in $F^n$, and

(iii) $F^n - A^*$ is simply connected.

To construct $A^n$ we will use two basic sets, $D^n_i$ and $E^n$ in $T^n_i$.

**Theorem 4.3.** Given $e > 0$, we can construct a set $D^n_i$ in $T^n_i$ consisting of disjoint $n$-tubes with the following properties:

(i) the diameter of each $n$-tube in $D^n_i$ is less than $e$,

(ii) $\pi(D^n_i) \rightarrow \pi(T^n_i - D^n_i)$ is a monomorphism, and

(iii) the $n$-tubes in $D^n_i$ are $h$-unknotted in $T^n_i$. (For the definition of $h$-unknotted and the proof of (iii) see 5.8 and following.)

**Proof.** In the previous sections we showed how to construct $D^n_i$ in $T^n_i$ for every choice of $\delta > 0$. Assume inductively that we can construct $D^{n-i} \rightarrow T^{n-i}$ for each choice of $\delta > 0$ satisfying (i), (ii), and (iii). Then $D^{n-i}$ consists of disjoint $(n-1)$-tubes $T^{n-i}_{i, j}$, $i = 1, 2, \ldots, k$. Now $S^i \times D^{n-i} \subset S^i \times T^{n-i}$.

Let $h: T^{n-i}_{i, j} \rightarrow T^{n-i}_{i, j}$ be a homeomorphism. Define $e: S^i \times S^j \times T^{n-i} \rightarrow S^i \times S^j \times T^{n-i}$ as in Lemma 4.1, then define

$$D^n_i = \bigcup_{i=1}^{k} (1 \times h_i)(s(S^i \times D^{n-i}_{i, j})) \subset T^n_i.$$

By Lemma 4.1 the diameter of each component of $D^n_i$ will be less than $e$ if $\delta$ is small enough. This establishes (i).

In order to establish (ii), consider the following Van Kampen diagram.

$\pi(D^n_i) \rightarrow \pi(T^n_i - D^n_i)$

$\pi(T^n_i - (\bigcup_{j=1}^{k} (1 \times h_j)(s(S^i \times D^{n-i}_{i, j})))$
n-tubes in \( A^*_{i+1} \). And \( A^*_i \), \( i \) odd, is the subset of \( A^*_{i-1} \) equal to the union of the images of \( F^n \), obtained by mapping \((T^n_m, F^n)\) homeomorphically onto each n-tube in \( A^*_{i-1} \). Then \( A^* = \bigcup_{n=1}^\infty A^n_0 \).

It is a routine matter to show that \( A^* \) is a Cantor set.

Now we need to show \( A^* \) is wild in \( F^n \).

**Theorem 4.4.** Let \( X \) be a tame Cantor set in \( F^n \). Let \( T^n \) be an n-tube such that \( X \) lies in the interior of \( T^n \). Then \( \pi(T^n - X) = \pi(T^n) \).

The proof of this theorem is easy enough to justify omission.

**Lemma 4.5.** The maps \( \pi_n : \pi(T^n_0) \rightarrow \pi(T^n_0 - F^n) \), \( \pi_* : \pi(T^n_0) \rightarrow \pi(T^n_0 - \bigcup_{n=1}^\infty T^n_m) \), and \( \phi^n : \pi(T^n_0) \rightarrow \pi(T^n_0 - \bigcup_{n=1}^\infty T^n_m) \) are homeomorphisms induced by inclusion.

This lemma is easy to prove using a standard inductive argument. We will omit the proof.

**Theorem 4.6.** Given any set \( A^n \) in the construction of \( A^* \), the map \( \psi : \pi(T^n_0 - A^n_0) \rightarrow \pi(T^n_0 - A^n_{i+1}) \) is a monomorphism.

**Proof.** Consider two cases.

**Case 1.** \( i \) is even.

In this case \( A^n_i = \bigcup_{m=1}^i T^n_m \), the union of \( k \) n-tubes, \( A^n_{i+1} \) is the subset of \( A^n_i \) equal to the union of the images of \( F^n \), obtained by mapping \((T^n_m, F^n)\) onto each of the n-tubes in \( A^n_i \). If we can show that by mapping \((T^n_m, F^n)\) onto one of the n-tubes in \( A^n_i \), say \( T^n_j \), the map

\[
\gamma : \pi(T^n_0 - \bigcup_{m=1}^i T^n_m) \rightarrow \pi(T^n_0 - \bigcup_{m=1}^i T^n_m) \cup F^n_j)
\]

induced by inclusion, is a monomorphism, then \( \psi \) is a monomorphism for \( i \) even.

Consider the following Van Kampen diagram.

By Lemma 4.5 we know that \( \phi_n \) and \( \phi_* \) are homeomorphisms. Hence by Theorem 2.1, \( \psi \) is a monomorphism. Therefore, \( \psi \) is a monomorphism for \( i \) even.

**Case 2.** \( i \) is odd.

In this case again \( A^n_i = \bigcup_{m=1}^i T^n_m \), the union of \( k \) n-tubes. But this time \( A^n_{i+1} \) is the subset of \( A^n_i \) equal to the union of the images of \( F^n \) obtained by mapping \((T^n_m, F^n)\) onto each of the n-tubes in \( A^n_i \). If we can show that by mapping \((T^n_m, F^n)\) onto one of the n-tubes in \( A^n_i \), say \( T^n_j \), the map

\[
\delta : \pi(T^n_0 - \bigcup_{m=1}^i T^n_m) \rightarrow \pi(T^n_0 - \bigcup_{m=1}^i T^n_m) \cup F^n_j)
\]

induced by inclusion, is a monomorphism, then \( \psi \) is a monomorphism for \( i \) odd.

Consider the following Van Kampen diagram.

By Theorem 4.3(ii) and Lemma 4.5, we know that \( \phi_n \) and \( \phi_* \) are homeomorphisms. Hence by Theorem 2.1, \( \delta \) is a monomorphism. Therefore, \( \psi \) is a monomorphism for \( i \) odd.

**Theorem 4.7.** \( \pi(T^n_0 - A^n) \neq \pi(T^n_0) \).

**Proof.** Since \( \pi(T^n_0 - A^n) = \lim_{\rightarrow} \pi(T^n_0 - A^n) \) it suffices to show that there exists a subgroup of \( \lim_{\rightarrow} \pi(T^n_0 - A^n) \) which could not possibly be a subgroup of \( \pi(T^n_0) \).

We know that \( \phi_n : \pi(T^n_0) \rightarrow \pi(T^n_0 - A^n) \) is a monomorphism by Theorem 4.3(ii). We also know by Theorem 4.6 that at every stage in the construction the map \( \psi : \pi(T^n_0 - A^n) \rightarrow \pi(T^n_0 - A^n_{i+1}) \) is a monomorphism. Hence we can say that \( \pi(T^n_0) \rightarrow \pi(T^n_0 - A^n) \) is a monomorphism.
Since $\pi(\bigcup_{i=1}^{n} T_i)$ has a subgroup of the form $\bigoplus_{i=1}^{n} Z$, then $\pi(T_n - A^n)$ has a subgroup of the form $\bigoplus_{i=1}^{n} Z$.

But this implies that $\pi(T_n - A^n) \neq \bigoplus_{i=1}^{n} Z$.

Therefore, $A^n$ is wild in $E^n$.

We now consider the problem of showing that $E^n - A^n$ is simply connected. All maps not labeled are induced by inclusion or are the composition of natural quotient maps following maps induced by inclusion.

The concept of geometrically unlinked $3$-tubes used in the construction of $A^n$ in $E^n$ proved to be very cumbersome to generalize. Because of the difficulties involved we have extracted the algebraic properties necessary to show that when $T_k$ is embedded in $E^n$ nicely we have that $E^n - A^n$ is simply connected.

**Definition 4.8.** Let $(T_i, i = 1, 2, ..., k)$ be a collection of disjoint $n$-tubes in $T_k$. Let $P$ be a tree in $T_k - \bigcup_{i=1}^{k} T_i$ such that $P \cap \partial T_i$ is a single point for $i = 0, 1, 2, ..., k$. Let $K_i = \ker(\pi(\partial T_i) \to \pi(T_i))$ and let $G_i$ be a subgroup of $\pi(T_i)$ such that $K_i \cap G_i = \pi(\partial T_i)$. Denote by $H_i$ the smallest normal subgroup of $\pi(T_i - \bigcup_{i=1}^{k} T_i)$ containing

$$\text{im}(G_i \to \pi(T_i - \bigcup_{i=1}^{k} T_i)).$$

The $n$-tubes $(T_i, i = 1, 2, ..., k)$ are $A^n$-unlinked in $T_k$ if for every $i = 1, 2, ..., k$,

$$G_i \subset \ker(\pi(\partial T_i) \to \pi(T_i - \bigcup_{i=1}^{k} T_i) / H_i)$$

and $\pi(P \cup \bigcup_{i=1}^{k} \partial T_i) \to \pi(T_k - \bigcup_{i=1}^{k} T_i) / H_i$ is an epimorphism.

**Lemma 4.9.** If $(T_i, i = 1, 2, ..., k)$ is $A^n$-unlinked in $T_k$ and $(T_i, i = k+1, ..., m)$ is $A^n$-unlinked in $T_k$, then $(T_i, i = 2, 3, ..., m)$ is $A^n$-unlinked in $T_k$.

**Proof.** Let $P_i$ be a tree in $T_k - \bigcup_{i=1}^{k} T_i$ such that $P \cup P_i$ is connected and $P \cap \partial T_i$ is a single point for $i = k+1, ..., m$. Let $H_i$ be the normal subgroup of $\pi(T_i - \bigcup_{i=1}^{k} T_i)$ generated by $\text{im}(G_i \to \pi(T_i - \bigcup_{i=1}^{k} T_i))$ and let $H'_i$ be the normal subgroup of $\pi(T_i - \bigcup_{i=1}^{m} T_i)$ generated by $\text{im}(G_i \to \pi(T_i - \bigcup_{i=1}^{m} T_i))$. It is not difficult to see that since

$$G_i \subset \ker(\pi(\partial T_i) \to \pi(T_i - \bigcup_{i=1}^{m} T_i) / H'_i)$$

for $j = 1, 2, ..., k$, and $G_j \to H_j \to H_j'$ for $j = k+1, ..., m$, we have

$$G_j \subset \ker(\pi(\partial T_i) \to \pi(T_i - \bigcup_{i=1}^{m} T_i) / H'_i)$$

By the Van Kampen theorem $\pi(T_k - \bigcup_{i=1}^{m} T_i)$ is generated by

$$\text{im}(\pi(T_k - \bigcup_{i=1}^{m} T_i) / H_i)$$

and $\pi(T_k - \bigcup_{i=1}^{m} T_i) / H_i$ is an epimorphism.

Since $\pi(P \cup \bigcup_{i=1}^{k} \partial T_i) \to \pi(T_k - \bigcup_{i=1}^{m} T_i) / H_i$ and $\pi(P \cup \bigcup_{i=1}^{k} \partial T_i) \to \pi(T_k - \bigcup_{i=1}^{m} T_i) / H_i$ are epimorphisms and

$$\text{im}(H_j \to \pi(T_k - \bigcup_{i=1}^{m} T_i) / H'_i) \subset C H'_i$$

it follows that $\pi(P \cup \bigcup_{i=1}^{k} \partial T_i) \to \pi(T_k - \bigcup_{i=1}^{m} T_i) / H_i$ is an epimorphism.

The following lemma follows easily from the definition of $A^n$-unlinked.

**Lemma 4.10.** If $(T_i, i = 1, 2, ..., k)$ is $A^n$-unlinked in $T_k$, then $(S_{i} \times T_i, i = 1, 2, ..., k)$ is $A^n$-unlinked in $S_{i} \times T_k = T_{k+1}$.

Note that we have previously shown that $(T_0, T_k)$ are $A^n$-unlinked in $T_k$, as is $E^n$ in $T_k$. Using Lemmas 4.9 and 4.10 we can prove

**Theorem 4.11.** $A^n$ consists of $A^n$-unlinked $n$-tubes in $T_k$.

**Lemma 4.12.** Let $H^n$ be the normal subgroup of $\pi(T_k - E^n)$ generated by $\text{im}(G_i \to \pi(T_k - E^n))$. Then $\pi(T_k - E^n) / \pi(T_k - E^n)$ is the trivial homomorphism.

**Proof.** It has already been proven that $\pi(T_k - E^n) / \pi(T_k - E^n)$ is the trivial homomorphism. Suppose now that $\pi(T_k - E^n) / \pi(T_k - E^n)$ is the trivial homomorphism. Since $T_k - E^n = S_{i} \times (T_i - E^n)$ and $\partial T_i = S_{i} \times \partial T_n$ we have that $\pi(T_k - E^n) = \pi(S_{i}) \times \pi(T_k - E^n)$. We note that $H^n$ contains $\text{im}(\pi(S_{i}) \to \pi(T_k - E^n))$ and $H^n$ contains $\text{im}(\pi(T_k - E^n) \to \pi(T_k - E^n))$, thus $H^n$ contains $\text{im}(\pi(T_k - E^n) / \pi(T_k - E^n))$.

**Theorem 4.13.** If $(T_i, i = 1, 2, ..., k)$ is $A^n$-unlinked in $T_k$ and $f_i: T_i - \bigcup_{i=1}^{k} T_i$ are homeomorphisms for $i = 1, 2, ..., k$, then $\pi(T_k - \bigcup_{i=1}^{k} T_i)$
\( \pi(T^n - \bigcup f_i(F^n))/H'_n \) is the trivial homomorphism, where \( H'_n \) is the normal subgroup generated by \( \lim_{H_n \to \pi(T^n - \bigcup f_i(F^n))}. \) (\( H_n \) as defined in 4.8.)

The theorem follows inductively from the previous lemma.

Now let \( h: T^n \to B^n \) be an embedding such that \( G_i \subset \ker(\pi(T^n - \bigcup f_i(F^n)) \to \pi(E^n - h(T^n)).\) Such an embedding can be obtained by rotation of an embedding of \( T^n \) in \( E^n \) about an \((n-2)\)-dimensional hyperplane.

Using Theorems 4.11 and 4.13 we can inductively show \( \pi(T^n - A^n_0) \to \pi(E^n - A^n_m) \) is the trivial homomorphism for \( k \) even. Since \( \pi(E^n - A^n_0) \) is the direct limit of \( \pi(T^n - A^n_0) \to \pi(T^n - A^n_1) \to \pi(T^n - A^n_2) \to \cdots \) it follows that \( \pi(T^n - A^n_k) \to 1 \) and hence \( E^n = A^n_k \) is simply connected.

V. Proof of the corollaries.

**Definition 5.1.** Let \( A \subset E^n \) be a Cantor set. A sequence \( \{A^n_k\} \) of compact subsets of \( E^n \) will be called a defining sequence for \( A \) if \( E^n \) has the following properties:

(i) \( A^n_k \) is a polyhedral \( n \)-manifold in \( E^n \),

(ii) \( A = \bigcap_{n=1}^{\infty} A^n_k \),

(iii) \( \bigcup_{m=1}^{\infty} A^n_k \subset A^n_k \),

(iv) There are finitely many components of \( A^n_k \), each of which has diameter less than \( 1/m \),

(v) \( E^n - A^n_k \) is simply connected for every \( m \).

It is an easy exercise to show that every Cantor set in \( E^n \) has a defining sequence.

**Definition 5.2.** Let \( \{A^n_k\} \) be a defining sequence for \( A \subset E^n \) and suppose that for each \( k \geq 1 \), \( B^n_0 \) is a \( k \)-ball in \( E^n \) with the following properties:

(i) \( B^n_0 \subset E^n - A^n_k \),

(ii) \( B^n_0 \) is a polyhedron in \( E^n \),

(iii) \( B^n_0 \cap \partial A^n_k \) is the union of finitely many disjoint \((k-1)\)-balls, each of which lie in \( A^n_0 - B^n_0 \),

(iv) \( A^n_0 \subset B^n_0 \) is connected,

(v) \( B^n_0 \subset B^n_{k-1} \),

(vi) \( B^n_0 \subset B^n_{k-1} \).

Then \( \bigcup_{n=1}^m B^n_0 = B^k \) is a \( k \)-ball (see [7]) called an oscillating \( k \)-ball for \( A \) and the sequence \( \{B^n_0\} \) is called a defining sequence for \( B^k \) with respect to \( \{A^n_k\} \).

An oscillating 1-ball (arc) is an arc containing \( A \) in the boundary of an oscillating 2-ball. The existence of oscillating balls for every Cantor set is guaranteed by the results of [7].

It is not difficult to see that \( B^n = \bigcap_{n=1}^m (B^n_0 \cup A^n_0) \).

**Theorem 5.3.** Let \( A \subset E^n \) be a Cantor set and let \( B^k \) be an oscillating \( k \)-ball for \( A \). If \( n > 4 \), then \( \pi(E^n - A) = \pi(E^n - B^k) \).

**Proof.** Let \( \{B^n_0\} \) be a defining sequence for \( B^k \) with respect to \( \{A^n_k\} \). Then \( \pi(E^n - A) = \lim \pi(E^n - A^n_0) \) and \( \pi(E^n - B^k) = \lim \pi(E^n - \partial A^n_0 \cup B^n_0) \). Let \( \lim \{A^n_k\} = \partial A^n_0 \cup B^n_0 \). By our hypotheses, \( G^n_{k-1} \) is an \((n-1)\)-sphere with holes. Now consider the following Van Kampen diagram.

\[
\begin{array}{c}
\pi(G^n_{k-1}) \\
\pi(E^n - (A^n_0 \cup B^n_0)) \\
\pi(E^n - A^n_0) \\
\pi(E^n - A^n_0)
\end{array}
\]

Since \( \pi(G^n_{k-1}) = 1 = \pi(E^n - A^n_0) \) (\( n > 4 \)), it follows that \( \pi(E^n - (A^n_0 \cup B^n_0)) \to \pi(E^n - A^n_0) \) is an isomorphism. Furthermore the diagram

\[
\begin{array}{c}
\pi(E^n - (A^n_0 \cup B^n_0)) \\
\pi(E^n - A^n_0) \\
\pi(E^n - A^n_0)
\end{array}
\]

commutes. (All maps are induced by inclusion.) It follows that

\[
\lim \pi(E^n - (A^n_0 \cup B^n_0)) = \lim \pi(E^n - A^n_0).
\]

Now an oscillating \( k \)-ball, \( B^k \) for \( A \), in \( E^n \) can be obtained by the intersection of a nested sequence of oscillating \( n \)-balls, the complement of each \( n \)-ball having the same fundamental group as \( E^n - A \). Again by direct limits, it follows that \( \pi(E^n - B^k) = \pi(E^n - A) \).

Corollary 1 for \( n > 4 \) follows immediately from Theorem 5.3. For \( n = 3 \) it can be shown that an oscillating ball for a Cantor set with simply connected complement has a simply connected complement.
Examples are known of wild balls in $\mathbb{E}^n$ with simply connected complements [4].

**Definition 5.4.** The boundary of an osculating $k$-ball for $A$ is called an osculating $(k-1)$-sphere.

**Theorem 5.5.** If $S^n$ is an osculating $k$-sphere for $A$, then $\pi_1(\mathbb{E}^n - S^n) = \pi_1(\mathbb{E}^n - S^n)$ for $0 < k < n - 2$ and for $k = n - 2$ there is a short exact sequence $0 \to Z \to \pi_1(\mathbb{E}^n - S^n) \to \pi_1(\mathbb{E}^n - A) \to 1$.

**Proof.** Let $C^n$ be a polyhedral $n$-ball in $\mathbb{E}^n - A$ with the property that $\partial C^n \simeq S^n$ is an unknotted, polyhedral $(k-1)$-sphere in $\partial C^n$. Clearly $\mathbb{E}^n - C^n \simeq B^n$ is an osculating $k$-ball for $A$ in $\mathbb{E}^n$. Furthermore $\partial C^n \simeq S^n$ has the homotopy type of $S^{n-k-1} = S^{n-k-1}$ and clearly $\pi_1(\mathbb{E}^n - (S^n \cup C^n)) = \pi_1(\mathbb{E}^n - S^n)$. Now consider the Van Kampen diagram

$$
\begin{array}{ccc}
\pi(\partial C^n - S^n) & \to & \pi_1(C^n) \\
\downarrow & & \downarrow \\
\pi_1(\mathbb{E}^n - S^n) & \to & \pi_1(\mathbb{E}^n - A)
\end{array}
$$

We have $\pi_1(C^n) = 1$ and $\pi_1(\partial C^n - S^n) = \pi_1(S^{n-k-1}) = 1$ if $n - k - 1 > 1$, i.e., $n - 2 > k$. We see that $\pi_1(\partial C^n - (S^n \cup C^n)) \to \pi_1(\mathbb{E}^n - S^n) \cup C^n$ is an isomorphism or $\pi_1(\mathbb{E}^n - S^n) \to \pi_1(\mathbb{E}^n - B^n)$ is an isomorphism. Thus, by Theorem 5.3, $\pi_1(\mathbb{E}^n - S^n) \to \pi_1(\mathbb{E}^n - A)$ is an isomorphism for $n - k - 1 > 1$.

If $n - 2 = k$, it is not difficult to see that $\pi_1(\partial C^n - S^{n-k}) \to \pi_1(\mathbb{E}^n - (S^{n-k} \cup C^n))$ is a monomorphism and that $\pi_1(\partial C^n - S^{n-k}) \to \pi_1(\mathbb{E}^n - S^{n-k} \cup C^n)$ is trivial. Thus, $1 \to Z \to \pi_1(\mathbb{E}^n - S^{n-k}) \to \pi_1(\mathbb{E}^n - B^{n-2}) \to 1$ is exact. Since $\pi_1(\mathbb{E}^n - B^{n-2}) = \pi_1(\mathbb{E}^n - A)$, the result follows.

References