Homeomorphisms of inverse limits of metric spaces

by

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Abstract. It is shown that for every homeomorphism $f\colon X \to X$ of a completely regular space X there exists an inverse system of metric spaces satisfying some additional conditions such that f is represented by homeomorphisms $f_a\colon X_a \to X_a$ of the spaces of the system.

In [3] (Example 2) J. W. Rogers has shown an example of an inverse limit of polyhedra and a homeomorphism between them such that the homeomorphism cannot be represented by maps between the spaces of the system. An example is also known (see e.g. [2], Example 7) of a compact space which cannot be an inverse system of polyhedra with bonding mappings onto. Thus the question whether every homeomorphism $f\colon X \to X$ of a completely regular space onto itself can be represented by homeomorphisms $f_a\colon X_a \to X_a$ of metric spaces X_a forming an inverse system with bonding maps and canonical projections onto becomes quite natural. In [1] it was proved that every map $f\colon X \to X$ of a completely regular space X may be represented by maps $f_a\colon X_a \to X_a$ of metric spaces X_a forming an inverse system. Using the same methods as in [1], we shall show that if f is a homeomorphism, then the maps f_a can be chosen in such a way that they are homeomorphisms.

In this note we use symbols and notations from [1]. To answer the question it suffices to prove a lemma:

LEMMA. *Let $f\colon (X, \mathfrak{U}) \to (X, \mathfrak{U})$ be a uniform homeomorphism of a uniform space $(X, \mathfrak{U})$. Then there exists a set M, directed with respect to inclusion, of pseudouniformities contained in $\mathfrak{U}$ such that if $a \in M$, then*

(a) $$f^{-1}(a) \subset a, \quad f(a) \subset a,$$

(b) $$\text{weight}\, a \leqslant \aleph_0,$$

(c) $$\bigcup M \textit{ is a base for } \mathfrak{U},$$

and, in addition, if $\dim \mathfrak{U} \leqslant n$ *and* $\operatorname{dweight} \mathfrak{U} \leqslant (\gamma, \tau)$, *then*

(d) $$\dim \alpha \leqslant n\,,$$

(e) $$\operatorname{dweight} \alpha \leqslant (\aleph_0, \tau)\,,$$

(f) $$\operatorname{card} M \leqslant \tau\,.$$

Proof. Without loss of generality assume that $\mathfrak{B}$ is a base of cardinality $\leqslant \gamma$ consisting of coverings of cardinality τ and of order $\leqslant n+1$. Let $P \in \mathfrak{B}$. Define $f^{-0}(P) = f^0(P) = P$ and $f^{-(m+1)}(P) = f^{-1}[f^{-m}(P)]$, $f^{(m+1)}(P) = f[f^m(P)]$. Put $W_1 = \{Q \in \mathfrak{U}\colon Q = f^m(P),\ m = 0, \pm 1, \pm 2, \ldots\}$. Assume that $W_1, \ldots, W_k$ are defined. Define W_{k+1} as a countable family of coverings of $\mathfrak{U}$ such that for every $Q_1, Q_2 \in \bigcup_{i=1}^{k} W_i$ there exists a $Q \in W_{k+1}$ and $Q \succ_* Q_1$, $Q \succ_* Q_2$, $f^m(Q) \in W_{k+1}$ for every $m = 0, \pm 1, \pm 2, \ldots$ It is easy to see that such a family W_{k+1} exists.

Let α_P be the pseudouniformity induced by a base $\bigcup_{k=1}^{\infty} W_k$. It is obvious that the pseudouniformities α_P and a set $M' = \bigcup \{\alpha_P\colon P \in \mathfrak{B}\}$ satisfy the conditions (a)-(f) of the lemma. Hence it is easy to see that for every $\alpha_P, \alpha_{P'} \in M'$ there exists a pseudouniformity $\alpha \supset \alpha_P \cup \alpha_{P'}$ which satisfies the conditions (a), (b), (d), (e). Thus by countable operations we may choose a directed set M which has the required properties.

Consider the commutative diagram

$$\begin{array}{ccccc}
(X, \mathfrak{U}) & \xrightarrow{f} & (X, \mathfrak{U}) & \xrightarrow{f^{-1}} & (X, \mathfrak{U}) \\
{\scriptstyle 1_X}\big\downarrow & & {\scriptstyle 1_X}\big\downarrow{\scriptstyle \text{onto}} & & \big\downarrow{\scriptstyle 1_X} \\
(X, \alpha) & \xrightarrow{f} & (X, \alpha) & \xrightarrow{f^{-1}} & (X, \alpha) \\
{\scriptstyle h_\alpha}\big\downarrow & & {\scriptstyle h_\alpha}\big\downarrow{\scriptstyle \text{onto}} & & \big\downarrow{\scriptstyle h_\alpha} \\
(hX, h\alpha) & \xrightarrow{f_\alpha} & (hX, h\alpha) & \xrightarrow{(f^{-1})_\alpha} & (hX, h\alpha) \\
{\scriptstyle \pi^\alpha_\beta}\big\downarrow & & {\scriptstyle \pi^\alpha_\beta}\big\downarrow{\scriptstyle \text{onto}} & & \big\downarrow{\scriptstyle \pi^\alpha_\beta} \\
(hX, h\beta) & \xrightarrow{f_\beta} & (hX, h\beta) & \xrightarrow{(f^{-1})_\beta} & (hX, h\beta)
\end{array}
\qquad \mathfrak{U} \supset \alpha \supset \beta,\ \alpha, \beta \in M$$

The condition (a) of the lemma assures that the maps $f\colon (X, \alpha) \to (X, \alpha)$ and $f^{-1}\colon (X, \alpha) \to (X, \alpha)$ are uniform. From Lemma 1 of [1] it follows that h is a functor of the category of pseudouniform spaces onto the category of uniform spaces; hence $f_\alpha^{-1} = (f^{-1})_\alpha$, (the uniqueness of f_α and $f_\alpha f_\alpha^{-1} = f_\alpha^{-1} f_\alpha = 1_X$). Using the same arguments as in [1], we obtain:

THEOREM. *Let* $f\colon (X, \mathfrak{U}) \to (X, \mathfrak{U})$ *be a uniform homeomorphism of a uniform space* $(X, \mathfrak{U})$. *Then there exists*

1. *an inverse system* $S = \{(X_\alpha, \alpha), \pi^\alpha_\beta, M\}$, $\operatorname{card} M = \operatorname{weight} \mathfrak{U}$, $\operatorname{weight} \alpha \leqslant \aleph_0$ *with bonding maps* π^α_β *and canonical projections onto,*

2. *a uniform embedding* $g\colon (X, \mathfrak{U}) \to (X^*, \mathfrak{U}^*)$, $(X^*, \mathfrak{U}^*) = \varprojlim S$, *which is onto if* $\mathfrak{U}$ *is complete,*

3. *uniform homeomorphisms* $f_\alpha\colon (X_\alpha, \alpha) \to (X_\alpha, \alpha)$, $\alpha \in M$, *inducing a mapping of the system such that the diagram*

$$\begin{array}{ccc}
(X, \mathfrak{U}) & \xrightarrow{f} & (X, \mathfrak{U}) \\
{\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
(X^*, \mathfrak{U}^*) & \xrightarrow{f^*} & (X^*, \mathfrak{U}^*)
\end{array}
\qquad f^* = \varprojlim f_\alpha$$

is commutative.

In addition, if $\dim \mathfrak{U} \leqslant n$ *and* $\operatorname{dweight} \mathfrak{U} \leqslant (\gamma, \tau)$, *then* $\dim \alpha \leqslant n$ *and* $\operatorname{dweight} \alpha \leqslant (\alpha_0, \tau)$ *for every* $\alpha \in M$.

COROLLARY. *If* $f\colon X \to X$ *is a homeomorphism of a completely regular space* X, *then there exists an inverse system* $S_f = \{X_\alpha, \pi^\alpha_\beta, M\}$ *of metric spaces* X_α, $\dim X_\alpha \leqslant \dim X$, $\operatorname{card} M = \operatorname{weight} X$, *with bonding maps* π^α_β *and canonical projections onto, and there exist a dense embedding* $g\colon X \to X^*$, $X^* = \varprojlim S_f$ *and a family of continuous maps* $f_\alpha\colon X_\alpha \to X_\alpha$, $\alpha \in M$, *inducing a mapping* $\bar{f}\colon S_f \to S_f$ *such that the diagram*

$$\begin{array}{ccc}
X & \xrightarrow{f} & X \\
{\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
X^* & \xrightarrow{f^*} & X^*
\end{array}
\qquad f^* = \varprojlim \bar{f}$$

is commutative.

References

[1] W. Kulpa, *Maps and inverse systems of metric spaces*, Fund. Math. (to appear)

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[3] J. W. Rogers, *Inducing approximations homotopic to maps between inverse limits*, Fund. Math. (to appear).

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