Homeomorphisms of inverse limits of metric spaces
by
W. Kulpa (Katowice)

Abstract. It is shown that for every homeomorphism \( f: X \to X \) of a completely regular space \( X \) there exists an inverse system of metric spaces satisfying some additional conditions such that \( f \) is represented by homeomorphisms \( f_n: X_n \to X_\alpha \) of the spaces of the system.

In [3] (Example 2) J. W. Rogers has shown an example of an inverse limit of polyhedra and a homeomorphism between them such that the homeomorphism cannot be represented by maps between the spaces of the system. An example is also known (see e.g. [2], Example 7) of a compact space which cannot be an inverse system of polyhedra with bonding mappings onto. Thus the question whether every homeomorphism \( f: X \to X \) of a completely regular space onto itself can be represented by homeomorphisms \( f_n: X_n \to X_\alpha \) of metric spaces \( X_n \) forming an inverse system with bonding maps and canonical projections onto becomes quite natural. In [1] it was proved that every map \( f: X \to X \) of a completely regular space \( X \) may be represented by maps \( f_n: X_n \to X_\alpha \) of metric spaces \( X_n \) forming an inverse system. Using the same methods as in [1], we shall show that if \( f \) is a homeomorphism, then the maps \( f_n \) can be chosen in such a way that they are homeomorphisms.

In this note we use symbols and notations from [1]. To answer the question it suffices to prove a lemma:

Lemma. Let \( f: (X, \mathcal{U}) \to (X, \mathcal{U}) \) be a uniform homeomorphism of a uniform space \((X, \mathcal{U})\). Then there exists a set \( M \), directed with respect to inclusion, of pseuduniformities contained in \( \mathcal{U} \) such that if \( a \in M \), then

1. \( f^{-1}(a) \subset a \),
2. \( f(a) \subset a \),
3. \( \bigcup M \) is a base for \( \mathcal{U} \),

\[ \text{weight } a \leq \kappa \]

1 — Fundamenta Mathematicae, T. LXXXVI
and, in addition, if $\dim \mathcal{U} < n$ and $\text{dweight} \mathcal{U} < (\gamma, \tau)$, then

(d) \hspace{1cm} \dim \alpha_n \leq n,

(e) \hspace{1cm} \text{dweight} \alpha_n < (\kappa_\alpha, \tau),

(f) \hspace{1cm} \text{card} \ M \leq \tau.

Proof. Without loss of generality assume that $S$ is a base of cardinality $\leq \tau$ consisting of coverings of cardinality $\tau$ and of order $\leq n+1$. Let $P \in S$. Define $f^{-1}(P) = f'(P) = P$ and $f^{-m+1}(P) = f^{-m}(P)^-$. Put $W = (Q \in \mathcal{U}; Q = f^{-m}(P), m = 0, \pm 1, \pm 2, \ldots)$. Assume that $W_1, \ldots, W_k$ are defined. Define $W_{k+1}$ as a countable family of coverings of $\mathcal{U}$ such that for every $Q, Q' \in W_k$ there exists a $Q \in W_{k+1}$ and $Q \supseteq Q, Q \supseteq Q'$, $f^{-m}(Q) \subseteq W_{k+1}$ for every $m = 0, \pm 1, \pm 2, \ldots$. It is easy to see that such a family $W_{k+1}$ exists.

Let $\alpha_\mathcal{U}$ be the pseudouniformity induced by a base $\bigcup_{k=1}^{\infty} W_k$. It is obvious that the pseudouniformities $\alpha_P$ and a set $M' = \bigcup \{\alpha_P; P \in S\}$ satisfy the conditions (a)-(f) of the lemma. Hence it is easy to see that for every $\mathcal{U}, \alpha_P, M' \in S$ there exists a pseudouniformity $\alpha \supseteq \alpha_P, \alpha_M$ which satisfies the conditions (a), (b), (d), (e). Thus by countable operations we may choose a directed set $M$ which has the required properties.

Consider the commutative diagram

\[
\begin{array}{ccc}
(X, \mathcal{U}) & \xrightarrow{f} & (X, \mathcal{U}) \\
\downarrow{\alpha_\mathcal{U}} & & \downarrow{\alpha_\mathcal{U}} \\
(X, \alpha) & \xrightarrow{\alpha} & (X, \alpha)
\end{array}
\]

\[
\beta \supseteq \alpha \supseteq \beta, \beta \in M
\]

\[
\begin{array}{ccc}
(\kappa_X, \kappa_0) & \xrightarrow{\kappa} & (\kappa_X, \kappa_0) \\
\downarrow{\kappa} & & \downarrow{\kappa} \\
(\kappa_X, \kappa_1) & \xrightarrow{\kappa} & (\kappa_X, \kappa_1)
\end{array}
\]

The condition (a) of the lemma assures that the maps $f: (X, \alpha) \to (X, \alpha)$ and $f^{-1}: (X, \alpha) \to (X, \alpha)$ are uniform. From Lemma 1 of [1] it follows that $\kappa$ is a functor of the category of pseudouniform spaces onto the category of uniform spaces; hence $f^{-1} = (f^{-1})_\mathcal{U}$, (the uniqueness of $f_\kappa$ and $f^{-1} = f^{-1}_\kappa \in L_\kappa$). Using the same arguments as in [1], we obtain:

**Theorem.** Let $f: (X, \mathcal{U}) \to (X, \mathcal{U})$ be a uniform homeomorphism of a uniform space $(X, \mathcal{U})$. Then there exists an inverse system $S = \{(X, \alpha), \alpha_\mathcal{U}, (X, \mathcal{U}) = \lim S$, which is onto if $\mathcal{U}$ is complete.

2. a uniform embedding $g: (X, \mathcal{U}) \to (X^*, \mathcal{U}^*), (X^*, \mathcal{U}^*) = \lim S$, which is onto if $\mathcal{U}$ is complete.

3. uniform homeomorphisms $f_s: (X, \alpha) \to (X, \alpha), \alpha \in M$, inducing a mapping of the system such that the diagram

\[
\begin{array}{ccc}
(X, \mathcal{U}) & \xrightarrow{f} & (X, \mathcal{U}) \\
\downarrow{\alpha_\mathcal{U}} & & \downarrow{\alpha_\mathcal{U}} \\
(X^*, \mathcal{U}^*) & \xrightarrow{f^*} & (X^*, \mathcal{U}^*)
\end{array}
\]

is commutative.

In addition, if $\dim \mathcal{U} < n$ and $\text{dweight} \mathcal{U} < (\gamma, \tau)$, then $\dim \alpha_n < n$ and $\text{dweight} \alpha_n < (\kappa_\alpha, \tau)$ for every $\alpha \in M$.

**Corollary.** If $f: X \to X$ is a homeomorphism of a completely regular space $X$, then there exists an inverse system $S_f = \{(X, \alpha_\mathcal{U}), M\}$ of metric spaces $X, \dim X < \dim X, \text{card} M = \text{weight} X$ with bonding maps $\alpha_\mathcal{U}$ and canonical projections onto, and there exists a dense embedding $g: X \to X^*, X^* = \lim S_f$ and a family of continuous maps $f_\alpha: X^* \to X$, $\alpha \in M$, inducing a mapping $\tilde{f}: S_f \to S_f$, such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
X^* & \xrightarrow{f^*} & X^*
\end{array}
\]

is commutative.

**References**


**SILESIAN UNIVERSITY**

**Katowice**