Proximity classes of uniformities

by

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Abstract. If a uniformity $\mathfrak{U}$ on a set X is such that there exists a uniformly discrete subset of X of an infinite cardinality $\mathfrak{m}$, then there exist at least $\exp\exp\mathfrak{m}$ uniformities below $\mathfrak{U}$ belonging to distinct proximity classes and each of the proximity classes has at least $\exp\exp\mathfrak{m}$ distinct uniformities. In a case if the uniform space $(X, \mathfrak{U})$ has a dense subspace of the cardinality $\mathfrak{m}$, then there exist exactly $\exp\exp\mathfrak{m}$ such uniformities. If uniformities $\mathfrak{U}$ and $\mathfrak{V}$ on sets X and Y are such that there exist uniformly discrete sets in X and Y of an infinite cardinality $\mathfrak{m}$ then there exist at least $\exp\exp\mathfrak{m}$ uniformities $\mathfrak{U}_a$ and $\mathfrak{V}_a$ belonging to the proximity classes of $\mathfrak{U}$ and $\mathfrak{V}$, respectively, such that the proximity classes of the products $\mathfrak{U}_a \times \mathfrak{V}_a$ are distinct for distinct indices a.

1. The uniformities considered in this paper are of the covering type (see [2]). All the topological spaces are completely regular. If we consider a family of uniformities on a set, then we understand that the uniformities of this family are compatible with a given topology.

Let $\mathfrak{U}$ be a uniformity on a set X. A filter $\mathcal{F}$ on X is said to be a *Cauchy filter* in $\mathfrak{U}$ iff for every covering $P \in \mathfrak{U}$ we have $P \cap \mathcal{F} \neq \emptyset$. Every Cauchy filter $\mathcal{F}$ is contained in a unique minimal Cauchy filter, which is induced by a base $\{\mathrm{st}(A, P)\colon A \in \mathcal{F}, P \in \mathfrak{U}\}$. If two minimal Cauchy filters have a common limit, then they are equal. This enables us to construct a completion $i\colon (X, \mathfrak{U}) \subset (\tilde{X}, \tilde{\mathfrak{U}})$, where the set $\tilde{X}$ is the set of all minimal Cauchy filters in $\mathfrak{U}$ and the uniformity $\tilde{\mathfrak{U}}$ is induced by a base $\{\tilde{P}\colon P \in \mathfrak{U}\}$, where $\tilde{P} = \{\tilde{u}\colon u \in P\}$ and $\tilde{u} = \{\mathcal{F} \in \tilde{X}\colon u \in \mathcal{F}\}$. The uniform dense embedding i is defined by a condition $i(x) = \mathcal{F}$ iff $x = \lim \mathcal{F}$ (see [1]). The minimal Cauchy filters having empty limits form the remainder $\tilde{X} - i(X)$ of the completion of the space $(X, \mathfrak{U})$.

A uniformity $\mathfrak{U}$ on a set X induces a proximity relation $\delta(\mathfrak{U})$; for every $A, B \subset X$, $A\delta(\mathfrak{U})B$ iff for every $P \in \mathfrak{U}$, $\mathrm{st}(A, P) \cap B \neq \emptyset$. Let $p\mathfrak{U} = \{P \in \mathfrak{U}\colon P$ has a finite refinement belonging to $\mathfrak{U}\}$. The family $p\mathfrak{U}$ is the greatest totally bounded uniformity contained in $\mathfrak{U}$ and for every $A, B \subset X$

$$A\delta(\mathfrak{U})B \quad \text{iff} \quad A\delta(p\mathfrak{U})B\,.$$

Let us call $[\mathfrak{U}] = \{\mathfrak{V}\colon p\mathfrak{V} = p\mathfrak{U}\}$ the proximity class of the uniformity $\mathfrak{U}$.

Let P be a covering. A set D is said to be *P-discrete* iff for $d \neq d'$, $d, d' \in D$ we have $\mathrm{st}(d, P) \cap \mathrm{st}(d', P) = \emptyset$. A set D is said to be *uniformly* *discrete* iff there exists a $P \in \mathcal{U}$ such that D is P-discrete. A uniformity $\mathcal{U}$ has no base consisting of coverings of power less than $\mathfrak{m}$ iff there exists a uniformly discrete set of power $\mathfrak{m}$ ([3], pp. 24–25). We write $b\mathcal{U} \geqslant \mathfrak{m}$ iff there exists a uniformly discrete set of power $\mathfrak{m}$. If a space X is metrizable and the weight $X = \mathfrak{m}$, then there exists a uniformity with a countable base compatible with the topology on X, and for each such uniformity there exists a uniformly discrete set of power $\mathfrak{m}$, because such uniformity does not have a base consisting of coverings of power less than $\mathfrak{m}$.

Symbols $<$ and $\overset{*}{<}$ stand for refinement and star refinement.

The idea of the paper arises from Reed and Thron [7] and Reed [6].

2. In this paper we shall utilize a lemma proved by Pospíšil [5] (see also [4] and [2], Theorem 9.2):

LEMMA 1. *If D is an infinite set, then there exist* $\exp\exp|D|$ *ultrafilters on D having empty limits.*

LEMMA 2. *Let $\mathcal{U}$ be a uniformity on a set X with a uniformly discrete set $D \subset X$ of power $\mathfrak{m} \geqslant \aleph_0$. Then there exists a set π, $|\pi| = \exp\exp\mathfrak{m}$, of ultrafilters on D having empty limits in the topological space $(X, \mathcal{T}_{\mathcal{U}})$ and such that for every $\mathcal{F} \in \pi$ a filter $\overline{\mathcal{F}}$ generated by a base*

$$\{\mathrm{st}(A, P) \colon A \in \mathcal{F}, P \in p\mathcal{U}\}$$

is the minimal Cauchy filter in $p\mathcal{U}$, $\overline{\mathcal{F}}$ is not Cauchy filter in $\mathcal{U}$ and $\overline{\mathcal{F}} \neq \overline{\mathcal{F}}'$ iff $\mathcal{F} \neq \mathcal{F}'$.

Proof. Let $D \subset X$ be a uniformly discrete set in $\mathcal{U}$ of power $\mathfrak{m}$. From Lemma 1 it follows that there exist $\exp\exp\mathfrak{m}$ ultrafilters on D having empty limits on D. Let us denote the family of ultrafilters by π. For every $\mathcal{F} \in \pi$ let $\overline{\mathcal{F}}$ be a filter generated by a base $\{\mathrm{st}(A, P) \colon A \in \mathcal{F}, P \in p\mathcal{U}\}$.

Let $P \in \mathcal{U}$ be a covering such that D is P-discrete and let $\mathcal{F} \in \pi$. Since for every $A \in \mathcal{F}$, $|A| \geqslant \aleph_0$, there are no $u \in P$ such that $A \subset \mathrm{st}(u, P)$ for some $A \in \mathcal{F}$. Thus $\mathcal{F}$ is not a Cauchy filter in $\mathcal{U}$ and so $\overline{\mathcal{F}}$.

To show that $\overline{\mathcal{F}}$ is a minimal Cauchy filter in $p\mathcal{U}$ it suffices to show that a filter in X generated by $\mathcal{F}$ is a Cauchy filter in $p\mathcal{U}$. Let P be a finite covering belonging to $p\mathcal{U}$. Put $P_D = \{u \cap D \colon u \in P\}$. Since P_D is a finite covering of D and $\mathcal{F}$ is an ultrafilter on D, we have $u \cap D \in \mathcal{F}$ and in consequence $u \in \mathcal{F}$ for some $u \in P$.

If $\mathcal{F} \neq \mathcal{F}'$, then $A \cap A' = \emptyset$ for some $A \in \mathcal{F}$ and $A' \in \mathcal{F}'$. Let $P \in \mathcal{U}$ be such that $P \overset{*}{<} P_1$ and D is P_1-discrete. A covering

$$Q = \{\mathrm{st}(A, P), \bigcup\{u \in P \colon \mathrm{st}(u, P) \cap A \neq \emptyset\}, \mathrm{st}(A', P), \bigcup\{u \in P \colon \mathrm{st}(u, P) \cap A' \neq \emptyset\}, \bigcup\{u \in P \colon \mathrm{st}(u, P) \cap (A \cup A') = \emptyset\}\}$$

belongs to $p\mathcal{U}$ and $\mathrm{st}(A, Q) \cap \mathrm{st}(A', Q) = \emptyset$ because $\mathrm{st}(A, Q) \subset \mathrm{st}(A, P_1)$ and $\mathrm{st}(A', Q) \subset \mathrm{st}(A', P_1)$. Thus $\overline{\mathcal{F}} \neq \overline{\mathcal{F}}'$.

Since every element of a filter $\mathcal{F} \in \pi$ is closed in the space X we have $\lim \mathcal{F} = \emptyset$ in X and hence $\lim \overline{\mathcal{F}} = \emptyset$ in X. This completes the proof.

Notice that the Lemma can be deduced also from two facts; if a D is an infinite uniformly discrete set, then the closure of D is βD and $|\beta D| = \exp\exp|D|$.

THEOREM 1 (Reed). *Let $\mathcal{U}$ be a uniformity on a set X with $b\mathcal{U} \geqslant \mathfrak{m} \geqslant \aleph_0$. Then the proximity class $[\mathcal{U}]$ contains at least* $\exp\exp\mathfrak{m}$ *distinct uniformities.*

THEOREM 2. *Let $\mathcal{U}$ be a uniformity on a set X with $b\mathcal{U} \geqslant \mathfrak{m} \geqslant \aleph_0$. Then there exist at least* $\exp\exp\mathfrak{m}$ *uniformities $\mathcal{V}$ contained in $\mathcal{U}$ with $b\mathcal{V} \geqslant \mathfrak{m} \geqslant \aleph_0$ and belonging to distinct proximity classes.*

Remark. If a topological space X has a dense set of power $\mathfrak{m}$, then there exist no more than $\exp\exp\mathfrak{m}$ uniformities on X.

Proof of Theorems 1 and 2. Let D be a uniformly discrete set of power $\mathfrak{m}$ and let π be a family of ultrafilters on D having empty limits. From Lemma 1 it follows that $|\pi| = \exp\exp\mathfrak{m}$.

For every $\mathcal{F} \in \pi$ let $\mathcal{U}_{\mathcal{F}}$ be a uniformity induced by a subbase consisting of coverings belonging to $p\mathcal{U}$ or of coverings of the form

$$P_A = \{v \colon V = \mathrm{st}(A, P) \text{ or } v = u \in P \text{ if } u \cap A = \emptyset\},$$

where $A \in \mathcal{F}$ and $P \in \mathcal{U}$. To verify that the subbase is well defined it suffices to show that for every P_A there exists a $P' \in \mathcal{U}$ such that $P'_A \overset{*}{<} P_A$. Let us take $P' \overset{*}{<} P$, $P' \in \mathcal{U}$.

Let $u' \in P'_A$. Then $u' = \mathrm{st}(A, P')$ or $u' \in P'$. If $u' = \mathrm{st}(A, P')$, then

$$\begin{aligned}\mathrm{st}(u', P'_A) &= \mathrm{st}[\mathrm{st}(A, P'), P'_A] \\ &= \bigcup\{\mathrm{st}(u, P') \colon \mathrm{st}(u, P') \cap A \neq \emptyset, u \in P'\} \subset \mathrm{st}(A, P) \in P_A,\end{aligned}$$

If $u' \in P'_A$ and $u' \in P'$, then $u' \cap \mathrm{st}(A, P') \neq \emptyset$ or $u' \cap \mathrm{st}(A, P') = \emptyset$; if $u' \cap \mathrm{st}(A, P') \neq \emptyset$, then $\mathrm{st}(u', P') \cap A \neq \emptyset$ and $\mathrm{st}(u', P'_A) = \mathrm{st}(A, P') \cup \mathrm{st}(u', P') \subset \mathrm{st}(A, P) \in P_A$; if $u' \cap \mathrm{st}(A, P') = \emptyset$, then $\mathrm{st}(u', P'_A) = \mathrm{st}(u', P') \in v \in P < P_A$. Thus the subbase is well defined.

Notice that $p\mathcal{U}_{\mathcal{F}} = p\mathcal{U}$ because $p\mathcal{U} \subset \mathcal{U}_{\mathcal{F}} \subset \mathcal{U}$.

Now we show that if $\mathcal{F} \neq \mathcal{F}'$ then $\mathcal{U}_{\mathcal{F}} \neq \mathcal{U}_{\mathcal{F}'}$. There exist infinite sets $A \in \mathcal{F}$ and $A' \in \mathcal{F}'$ which are subsets of D such that $A \cap A' = \emptyset$. From the construction of the uniformities $\mathcal{U}_{\mathcal{F}}$ and $\mathcal{U}_{\mathcal{F}'}$ it follows that A is uniformly discrete in $\mathcal{U}_{\mathcal{F}'}$ and A is not uniformly discrete in $\mathcal{U}_{\mathcal{F}}$. Hence $\mathcal{U}_{\mathcal{F}} \neq \mathcal{U}_{\mathcal{F}'}$.

For every two filters $\mathcal{F} \neq \mathcal{F}'$, $\mathcal{F}, \mathcal{F}' \in \pi$, let us define a uniformity $\mathfrak{U}_{\mathcal{F},\mathcal{F}'}$ induced by a subbase consisting of coverings of the form

$$P_{A,A'} = \{v\colon\ v = \mathrm{st}(A \cup A', P) \text{ or } v = u \in P \text{ if } u \cap (A \cup A') = \emptyset\},$$

where $A \in \mathcal{F}$, $A' \in \mathcal{F}'$ and $P \in \mathfrak{U}$.

Let $P \in \mathfrak{U}$ be a covering such that D is P-discrete, $|D| = \mathfrak{m} \geqslant \aleph_0$. There exist three disjoint sets A_1, A_2, A_3 such that $|A_1| = |A_2| = |A_3| = \mathfrak{m}$ and $D = A_1 \cup A_2 \cup A_3$. Then, at most, two sets A_{i_1} and A_{i_2} from A_1, A_2, A_3 belong to $\mathcal{F}, \mathcal{F}'$ and hence the set A_{i_3} is $P_{A_{i_1}A_{i_2}}$-discrete, $i_3 \neq i_1, i_2$. Hence $b\mathfrak{U}_{\mathcal{F},\mathcal{F}'} \geqslant \mathfrak{m}$.

Let $\mathcal{F}_1 \neq \mathcal{F}_1'$, $\mathcal{F}_1, \mathcal{F}_1' \in \pi$ be two filters such that $\mathcal{F}_1 \neq \mathcal{F}, \mathcal{F}'$ and $\mathcal{F}_1' \neq \mathcal{F}, \mathcal{F}'$. Notice that the filters $\overline{\mathcal{F}}_1, \overline{\mathcal{F}}_1'$ have distinct minimal Cauchy filters in $p\mathfrak{U}_{\mathcal{F},\mathcal{F}'}$ and they have a common minimal Cauchy filter in $p\mathfrak{U}_{\mathcal{F}_1,\mathcal{F}_1'}$. Hence $p\mathfrak{U}_{\mathcal{F},\mathcal{F}'} \neq p\mathfrak{U}_{\mathcal{F}_1,\mathcal{F}_1'}$.

The uniformity $\mathfrak{U}_{\mathcal{F},\mathcal{F}'}$ is compatible with the topology induced by $\mathfrak{U}$. Let W be an open neighbourhood of a point $x \in X$. It may be assumed that $W = \mathrm{st}(x, P)$, $P \in \mathfrak{U}$. Since the filters $\mathcal{F}$ and $\mathcal{F}'$ have empty limits, there exists a $P' \in \mathfrak{U}$ such that $\mathrm{st}(x, P') \cap A = \mathrm{st}(x, P') \cap A' = \emptyset$ for some $A \in \mathcal{F}$ and $A' \in \mathcal{F}'$. Let $P'' \in \mathfrak{U}$ be such that D is P''-discrete and $P'' \overset{**}{<} P$, $P'' \overset{**}{<} P'$. Then $\mathrm{st}(x, P''_{A,A'}) \subset \mathrm{st}(x, P_{A,A'}) = W$. This completes the proof.

Proof of the remark. Let A be a dense subset of X of power $\mathfrak{m}$. There are no more than $\exp\exp\mathfrak{m}$ families of subsets of $A \times A$ containing the diagonal of $A \times A$; thus there are no more uniformities on X than $\exp\exp\mathfrak{m}$.

THEOREM 3. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be uniformities on sets X and Y, respectively. If $b\mathfrak{U} \geqslant \mathfrak{m} \geqslant \aleph_0$ and $b\mathfrak{V} \geqslant \mathfrak{m}$, then there exist at least* $\exp\exp\mathfrak{m}$ *uniformities $\mathfrak{U}_{\mathcal{F}}$ belonging to $[\mathfrak{U}]$ and there exist at least* $\exp\exp\mathfrak{m}$ *uniformities $\mathfrak{V}_{\mathcal{F}}$ belonging to $[\mathfrak{V}]$ such that if $\mathcal{F} \neq \mathcal{F}'$, then $p(\mathfrak{U}_{\mathcal{F}} \times \mathfrak{V}_{\mathcal{F}}) \neq p(\mathfrak{U}_{\mathcal{F}'} \times \mathfrak{V}_{\mathcal{F}'})$.*

Proof. Let $D \subset X$ be a uniformly discrete set in $\mathfrak{U}$ and let $C \subset Y$ be a uniformly discrete set in $\mathfrak{V}$ such that $|D| = |C| = \mathfrak{m}$. Let $\varphi\colon D \to C$ be a 1-1 map. Let π_D and π_C be families of all the ultrafilters on D and C having empty limits. There exists a 1-1 map $\psi\colon \pi_D \to \pi_C$ defined by $\psi(\mathcal{F}) = \{\varphi(A)\colon A \in \mathcal{F}\}$. Let $\mathfrak{U}_{\mathcal{F}}$ and $\mathfrak{V}_{\mathcal{F}} = \mathfrak{V}_{\psi(\mathcal{F})}$ be uniformities defined as in the proof of Theorems 2 and 3. Assume that $\mathcal{F} \neq \mathcal{F}'$ and $\mathcal{F}, \mathcal{F}' \in \pi$. There exists a set $A \in \mathcal{F}$ such that $A \cap A' = \emptyset$ for some $A' \in \mathcal{F}'$. The set A' is uniformly discrete in $\mathfrak{U}_{\mathcal{F}}$ but not in $\mathfrak{U}_{\mathcal{F}'}$.

Let us consider the sets

$$B_1 = \{(a, \varphi(a))\colon\ a \in A\},$$

$$B_2 = \{(a', \varphi(a))\colon\ a \neq a',\ a', a \in A\}.$$

Let $\delta_{\mathcal{F}}$ and $\delta_{\mathcal{F}'}$ mean the proximity relations induced by $\mathfrak{U}_{\mathcal{F}} \times \mathfrak{V}_{\mathcal{F}}$ and $\mathfrak{U}_{\mathcal{F}'} \times \mathfrak{V}_{\mathcal{F}'}$, respectively. Let $P \in \mathfrak{U}$, $Q \in \mathfrak{V}$ be such that D is P-discrete and C is Q-discrete. Then $\mathrm{st}(B_1, P_{A'} \times Q_{\varphi(A')}) \cap B_2 = \emptyset$ (the covering P_A is defined in the proof of Theorems 2 and 3), because A is $P_{A'}$-discrete and $\varphi(A)$ is $Q_{\varphi(A')}$-discrete. Thus $B_1 \operatorname{non}\delta_{\mathcal{F}'} B_2$. But $B_1 \delta_{\mathcal{F}} B_2$ holds. In fact, since every basic covering $P' \in \mathfrak{U}_{\mathcal{F}}$ is a form $P' = P^1_{A_1} \wedge \ldots \wedge P^m_{A_m} \wedge \wedge Q_1 \wedge \ldots \wedge Q_m$, where $P^i \in \mathfrak{U}$, $A_i \in \mathcal{F}$ and $Q_j \in p\mathfrak{U}$, there exists a $u_0 \in P'$ which contains at least $\aleph_0$ elements of A. For the same reasons for every $Q \in \mathfrak{V}_{\mathcal{F}}$ there exists a $v_0 \in Q$ which contains at least $\aleph_0$ elements of $\varphi(u_0 \cap A) \subset C$. Hence there exist some $a \neq a'$ such that $a, a' \in \varphi^{-1}[v_0 \cap \varphi(u_0 \cap A)]$. The points $(a, \varphi(a))$ and $(a', \varphi(a))$ belong to $u_0 \times v_0$. Thus $\mathrm{st}(B_1, P \times Q) \cap \cap B_2 \neq \emptyset$, for every $P \in \mathfrak{U}_{\mathcal{F}}$, $Q \in \mathfrak{V}_{\mathcal{F}}$. Hence $B_1 \delta_{\mathcal{F}} B_2$.

COROLLARY. *If the spaces X and Y are not pseudocompact, then there exist at least 2^c compactifications of the product $X \times Y$ finer than $\beta X \times \beta Y$. If, in addition, X and Y are metric with* weight X = weight $Y = \mathfrak{m}$, *$\mathfrak{m}$ is regular, then the set of the compactifications of $X \times Y$ finer than $\beta X \times \beta Y$ is equal to* $\exp\exp\mathfrak{m}$.

Proof. If X and Y are not pseudocompact then the finest uniformities $\mathfrak{U}$ on X and $\mathfrak{V}$ on Y compatible with the topologies are not totally bounded. This means that $b\mathfrak{U} \geqslant \aleph_0$ and $b\mathfrak{V} \geqslant \aleph_0$. The completion of $(X, p\mathfrak{U})$ leads to βX, the completion of $(Y, p\mathfrak{V})$ to βY and the completion of $p\mathfrak{U} \times p\mathfrak{V}$ to $\beta X \times \beta Y$. By Theorem 3, there exist at least 2^c uniformities $\mathfrak{U}_{\mathcal{F}} \times \mathfrak{V}_{\mathcal{F}}$ on $X \times Y$ such that all the uniformities $p(\mathfrak{U}_{\mathcal{F}} \times \mathfrak{V}_{\mathcal{F}})$ are different from one another and finer than $p\mathfrak{U} \times p\mathfrak{V}$. The compactifications of $X \times Y$ corresponding to $p(\mathfrak{U}_{\mathcal{F}} \times \mathfrak{V}_{\mathcal{F}})$ are different from one another and each of them majorizes $\beta X \times \beta Y$.

If X and Y are metric, then there are no more than $\exp\exp\mathfrak{m}$ compactifications of $X \times Y$, because $X \times Y$ has a dense set of power $\mathfrak{m}$. And there exist at least $\exp\exp\mathfrak{m}$ compactifications of $X \times Y$ finer than $\beta X \times \beta Y$ because the finest uniformities on X and Y have uniformly discrete sets of power $\mathfrak{m}$.

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Ideals in subalgebras of $C(X)$

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Abstract. For a k-space X let $C(X)$ denote the continuous real or complex-valued functions on X; consider a uniformly closed subalgebra A of $C(X)$. If $E = \{x \in X: |f(x)| = \sup_{t\in X}|f(t)|\}$ for some $f \in A$, and if $p \in E$ is isolated in the boundary of E or if X is first countable, then $C(N_\infty)$ ($N_\infty =$ the one point compactification of the natural numbers) is a homomorphic image of A so that 1) the maximal ideal $I_p = \{f \in A: f(p) = 0\}$ contains 2^c mutually disjoint infinite chains of prime ideals of A; 2) I_p is not countably generated; 3) if A has a peak point nonisolated in X, then A has a finitely generated ideal which is not principal and krull $\dim A = \infty$.

Under further restrictions on X and A, countably generated ideals and chains of ideals of A are discussed. Applications to generalizations of the disc algebra are considered.

Let $C(X)$ denote the algebra of complex-valued continuous functions on a space X. The relationship between X and $C(X)$ has been studied for a long time; it is known that if X is nontrivial in almost any sense, $C(X)$ has an intricate ideal structure with an abundance of prime ideals in particular. For fixed X what aspects of this ideal structure do various subalgebras A of $C(X)$ share? We discover, roughly speaking, that when A is uniformly closed and X has a modicum of compact parts, the algebra of all continuous functions on the one point compactification N_∞ of the natural numbers is a homomorphic image of A and from known properties of $C(N_\infty)$, we deduce that A and $C(X)$ share many qualitative aspects (2.1 ff). For example both will contain chains of prime ideals of arbitrary length.

Even more can be said about a class of subalgebras (§ 1) which generalize to noncompact spaces the familiar notion of uniform algebra [15]. Here a Silov boundary can be introduced and as with $C(X)$ a closed countably generated ideal has a hull which meets this boundary in an open-closed set (3.6). Under further restrictions on X, chains of arbitrary ideals are also discussed (§ 4). Our results generalize theorems in [10], [11] and [13], and bear on the problem of characterizing $C(X)$ among its subalgebras discussed in [5] and [18].

The concept of peak point (§ 1) serves as our motif. It is such points, together with uniform closure, which breed the intricacies which dis-