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THE UNIVERSITY OF IOWA
Iowa City, Iowa
SLIPPERY ROCK STATE COLLEGE
Slippery Rock, Pennsylvania

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Remarks on the absolute suspension

by

Andrzej Szymański (Katowice)

Abstract. There is proved that an n-dimensional compact metric space is n-dimensional sphere whenever each pair of distinct points is a pair of tops of some suspension representation and $n = 1, 2, 3$. This is a positive answer, for $n \leqslant 3$, on de Groot's conjecture.

A *suspension over* Y is a space SY formed from $Y\times[-1,1]$ by identifying $Y\times\{1\}$ and $Y\times\{-1\}$ to single points, called the *tops* of the suspension (the resulting set being equipped with the quotient topology).

A metrizable compact space will be said to be an *absolute suspension* if for each pair p, q of its distinct points it is a topologically suspension with tops p and q.

If X is the suspension over Y, then for $F\subset Y$, we can assume that F and SF are the subspaces of X.

Professor de Groot at the Prague Symposium 1971 asked whether an absolute suspension is homeomorphic to an n-sphere, whenever it is n-dimensional. We shall show that this conjecture is true in dimensions 1, 2 and 3.

Throughout the paper all the spaces will be assumed to be metric with the finite dimension in the sense of dim.

As was shown by de Groot in [4], Theorem 2, it suffices to show that the absolute suspension is a manifold in order to get the solution even for an arbitrary finite dimension. Thus showing that the absolute suspension in the dimensions 1, 2 and 3 is a manifold, is the most important step in the proof.

LEMMA 1 (Hurewicz; see Kuratowski [2], p. 311). *If Y is compact and* $\dim Z = 1$, *then* $\dim(Y\times Z) = \dim Y+1$.

LEMMA 2. *If X is compact and $X = SY$, then Y is compact.*

Proof. Since $Y\times[-\frac{1}{2},\frac{1}{2}]$ is a closed subset of compact space X, it is compact. Hence Y is compact.

LEMMA 3. *If Y is compact, then* $\dim SY = \dim Y+1$.

Proof. Clearly, the inductive dimension at each point of $Y\times(-1,1)$ is the same as that of the corresponding point in SY, hence it is at most $\dim Y+1$, in virtue of Lemma 1. The inductive dimension in the tops of SY is also at most $\dim Y+1$. Hence the inductive dimension of SY is at most $\dim Y+1$. Since the converse inequality is easy and the inductive dimension of SY is the same as $\dim SY$, the lemma is proved.

LEMMA 4. *If X and Y are non-degenerate continua, then no pair of points of $X\times Y$ disconnects $X\times Y$.*

The proof is easy.

COROLLARY 1. *If Y is a non-degenerate continuum, then no pair of points disconnects SY.*

LEMMA 5. *If X is an absolute suspension, then X is a locally connected continuum.*

The proof is obvious.

LEMMA 6. *If X is an absolute suspension and* $\dim X\geqslant 2$, *then Y is a locally connected continuum for each Y such that X is a suspension over Y.*

Proof. 1. The connectedness of Y.

Observe that among those Y for which $X = SY$ there is at least one Y which is connected (it is then a continuum in virtue of Lemma 2). In fact, otherwise, for each two distinct points p and q of X, the space X, being a suspension with tops p and q with a non-connected Y, would be disconnected by $\{p,q\}$. But if each pair of points disconnects a metrizable locally connected continuum, then, by a theorem of Moore (see [3], p. 188), it is S^1 topologically. In particular we have $\dim X = 1$ — a contradiction.

Now let Y_0 be a continuum such that $X = SY_0$, whose existence follows from the above part of the proof. Since $\dim X\geqslant 2$,. Y_0 is non-degenerate and, by Corollary 1, no pair of points disconnects X. Now, if $X = SY$, then Y is connected because otherwise X would be disconnected by tops of SY.

2. To prove the local connectedness of Y let us consider Y as a subspace of $X = SY$. Let $y\in Y$. Since X is locally connected by Lemma 5, there exist open and connected neighborhoods of y in X having arbitrarily small diameters and such that the tops of SY do not belong to those neighborhoods. The projection of $Y\times(-1,1)$ onto Y maps these neighborhoods onto neighborhoods of y in Y and the diameters are not greater than those of the corresponding neighborhoods in X.

LEMMA 7. *An absolute suspension is locally contractible.*

Proof. This follows from the fact that the space of the form SY is locally contractible at the tops.

In the sequel, following Borsuk's book [1], we distinguish between AR's and ANR's with respect to the class of all compact metric spaces and more wider notions of $AR(\mathfrak{M})$'s and $ANR(\mathfrak{M})$'s, absolute retracts and absolute neighborhood retracts with respect to all metric spaces. For compact metric spaces these notions coincide.

LEMMA 8. *If X is an absolute suspension, then* $X\in ANR$.

Proof. This follows, in virtue of Lemma 7, from the fact that each locally contractible compact space is an ANR whenever the dimension is finite (see [1], Corollary 10.4, p. 122).

LEMMA 9. *If X is an absolute suspension, then* $X\setminus\{p\}\in AR(\mathfrak{M})$ *for each $p\in X$.*

Proof. Since, by Lemma 8, $X\in ANR$, we have, by a theorem of Hanner ([1], Theorem 10.1, p. 96), $X\setminus\{p\}\in ANR(\mathfrak{M})$ for arbitrary p in X. Let $q\in X\setminus\{p\}$. Since X is an absolute suspension, p and q are the tops of SY for some Y such that $X = SY$. Clearly, $SY\setminus\{p\}$ is contractible to the point q. Hence $X\setminus\{p\}$ is contractible. But a contractible $ANR(\mathfrak{M})$ is $AR(\mathfrak{M})$ ([1], Theorem 9.1, p. 96).

LEMMA 10. *If X is an absolute suspension and* $\dim X\geqslant 2$, *then X is unicoherent.*

Proof. Let us assume that X is not unicoherent. Then X, being a locally connected continuum, contains, according to Borsuk's theorem (see [3], p. 437), a simple closed curve S which is a retract of X. Since $\dim X\geqslant 2$, there exists a point p in $X\setminus S$. By Lemma 9, $X\setminus\{p\}\in AR(\mathfrak{M})$. The curve S, being a retract of X, is a retract of $X\setminus\{p\}$. This means that S is an absolute retract — a contradiction.

LEMMA 11. *If X is an absolute suspension and $X = SY$, then* $Y\in ANR$.

Proof. Let p and q be the tops of SY. Then $X\setminus\{p,q\} = Y\times(-1,1)$, topologically. Since $X\setminus\{p,q\}$ is an open subset of ANR-space X, in virtue of Lemma 8, we have $Y\times(-1,1)\in ANR(\mathfrak{M})$, by Hanner's theorem loco cit. Thus Y, being a factor of $ANR(\mathfrak{M})$-space $Y\times(-1,1)$, is $ANR(\mathfrak{M})$ ([1], Theorem 7.2, p. 92). Then $Y\in ANR$, being compact in virtue of Lemma 2.

LEMMA 12. *If X is an absolute suspension and* $\dim X\geqslant 2$, *then no arc disconnects X.*

Proof. Suppose that there exists an arc $L\subset X$ which disconnects X. The arc L contains a closed subset F which irreducibly disconnects X (see [3], Theorem 3, p. 250). By Lemmas 5 and 10, X is unicoherent and therefore ([3], Theorem 3, p. 437) F is a continuum. Hence, by Lemma 6 and Corollary 1, F is an arc. Let b be an inner point of F. Let A

be a component of $X \setminus F$. Let B denote the union of all components of $X \setminus F$ different from A. We have ([3], Theorem 1, p. 249) $\mathrm{Fr}\, C_1 = \mathrm{Fr}\, C_2 = F$ for each two different components of $X \setminus F$ and therefore $\mathrm{Fr}\, A = \mathrm{Fr}\, B = F$ because F is irreducible. Since $b \in F$, $\mathrm{cl}\, C \setminus \{b\}$ is connected for an arbitrary component C of $X \setminus F$. Since $(\mathrm{cl}\, C_1 \setminus \{b\}) \cap (\mathrm{cl}\, C_2 \setminus \{b\}) = F \setminus \{b\}$ for arbitrary two different components C_1 and C_2 of $X \setminus F$, $\mathrm{cl}\, B \setminus \{b\}$ is connected. Hence $X \setminus \{b\}$ is not unicoherent, being a union of the above-mentioned closed (in $X \setminus \{b\}$) and connected sets $\mathrm{cl}\, A \setminus \{b\}$ and $\mathrm{cl}\, B \setminus \{b\}$, whose intersection is not connected. But $X \setminus \{b\}$, being contractible and being an $\mathrm{ANR}(\mathfrak{M})$ in virtue of Lemma 9, is unicoherent (see [3], Theorem 2, p. 435) — a contradiction.

LEMMA 13. *If X is an n-dimensional absolute suspension, then each $(n-1)$-dimensional sphere S^{n-1} contained in X disconnects X.*

Proof. There exists a point $p \in X \setminus S^{n-1}$. We infer that S^{n-1} is contractible in $X \setminus \{p\}$ because, by Lemma 9, $X \setminus \{p\}$ is contractible, being $\mathrm{AR}(\mathfrak{M})$. Hence, by Theorem 16.1, [1], p. 191, S^{n-1}, being cyclic in dimension $n-1$ must disconnect X, X being an n-dimensional homogeneous ANR and $X \setminus \{p\}$ being a proper subset of X.

LEMMA 14 (Borsuk [1], Theorem 15.1, p. 191). *An n-dimensional connected* ANR *is a manifold whenever it is homogeneous and contains topologically a Euclidean n-ball.*

THEOREM. *If $n = 1$, 2 and 3, then the n-dimensional absolute suspension is an n-dimensional sphere, topologically.*

Proof. If $n = 1$ and 2, the conclusion follows from Lemmas 8 and 14 and Theorem 2 of de Groot, cited at the begining.

To prove the conclusion for $n = 3$ let us note that, by Lemmas 3, 6 and 12, Y is a locally connected continuum without cut points and $\dim Y = 2$ for each Y such that SY is a 3-dimensional absolute suspension. Let us see that each one-dimensional sphere S^1 contained in Y disconnects Y. Otherwise SS^1, the suspension over S^1 (with the suspension structure inherited from SY), does not disconnect SY, since $SY \setminus SS^1 = (Y \setminus S^1) \times (-1, 1)$, and this contradicts the fact that $SS^1 = S^2$ disconnects SY in virtue of Lemma 13. Hence, by Young's [5] characterization of two-dimensional manifolds (as two-dimensional locally connected continua without cut points and such that "small" one-dimensional spheres disconnect them) we infer that Y is a two-dimensional manifold. In particular, Y contains a two-dimensional Euclidean ball and therefore SY, being a suspension over Y, contains a three-dimensional Euclidean ball. Hence, by Lemma 14, SY is a three-dimensional manifold. According to the de Groot reduction, SY is a three-dimensional sphere, topologically.

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SILESIAN UNIVERSITY, Katowice

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