A non-Desarguesian space geometry

by

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Abstract. This paper gives a model of a space geometry in which all of Hilbert's axioms of incidence for space, together with two of his betweenness axioms hold but in which Desargues' Theorem is false. This model demonstrates that there is a substantial difference between the projective space incidence axioms' relationship to the remaining axioms and the affine or Euclidean space incidence axioms' relationship to the remaining axioms.

The theorem of Desargues for projective geometry can be stated as:

If $\triangle ABC$ and $\triangle A'B'C'$ *are two distinct triangles so that the lines determined by the vertices* A *and* A', B *and* B', *and* C *and* C' *have a common point, then the points of intersection of the lines determined by the sides* AB *and* $A'B'$, AC *and* $A'C'$, *and* BC *and* $B'C'$ *are collinear, and conversely.*

A very well known fact from projective geometry is that the axioms of incidence for space are both necessary and sufficient to prove Desargues' Theorem for projective geometry. In fact both Hilbert [1] and F. R. Moulton [3] have proved that unless Desargues' Theorem is a theorem of a given plane projective geometry, this plane geometry cannot be embedded in a space projective geometry.

A form of Desargues' Theorem for Euclidean geometry is given in Hilbert's *Foundations of Geometry* [2] as follows:

If two triangles are situated in a plane so that pairs of corresponding sides are parallel, then the lines joining the corresponding vertices pass through one and the same point or are parallel.

Conversely, if two triangles lie in a plane so that lines joining corresponding vertices pass through one point or are parallel, and further, if two pairs of corresponding sides of the triangle are parallel, then the third sides of the two triangles are parallel [2, p. 72].

Hilbert observes that it is well known that this theorem can be proved on the basis of his (i.e. Hilbert's) space axioms of incidence and betweenness together with an extension of Euclid's parallel axiom which postulates existence as well as uniqueness of parallels [2, p. 72].

Hilbert has proved that if his planar incidence axioms, his betweenness axioms, and the extended parallel axiom mentioned above are satisfied in a plane geometry, then the validity of Desargues' Theorem is a necessary and sufficient condition that this plane geometry can be embedded in a space geometry where his space incidence axioms, betweenness axioms and the above mentioned extended parallel axiom hold [2, p. 88].

The following is a model of a space geometry in which all of Hilbert's axioms of incidence for space, together with two of his betweenness axioms, hold. Yet, in this space geometry, Desargues' Theorem is false. This model demonstrates that there is a substantial difference between the projective space incidence axioms' relationship to the remaining axioms and the affine or Euclidean space incidence axioms' relationship to the remaining axioms.

In the subsequent discussion the letters w, x, y, z will always be used as real valued variables and the letters a, b, c will always be real valued constants.

DEFINITION. A *point* is an ordered triple (x, y, k) where k is 0 or 1.

DEFINITION. A *line* is a set of points of either of the following types:

$$\boldsymbol{l} = \{(x, y, z), (w, z, 1)\}$$

or

$$\boldsymbol{m} = \{(x, y, k)\colon [ax+c]f(a, b, y)+by = 0,\ k = 0 \text{ or } 1 \text{ constant}\}$$

where we define

$$f(s, t, y) = \begin{cases} 1 & \text{if} \quad s/t \geqslant 0 \text{ or } \quad t = 0\,, \\ 1 & \text{if} \quad s/t < 0 \text{ and } y \leqslant 0\,, \\ \frac{1}{2} & \text{if} \quad s/t < 0 \text{ and } y > 0\,. \end{cases}$$

f is a "kinking" function which takes ordinary lines with positive slope and "kinks" them at the X-axis. The lines of type $\boldsymbol{m}$ are essentially the lines used by F. R. Moulton in his model mentioned above. (Note: These "kinked" lines will allow us to construct counterexamples to Desargues' Theorem in the planes of type Φ described below.)

DEFINITION. A *plane* is a set of points of either of the two following types:

$$\Phi = \{(x, y, k)\colon k = 0 \text{ or } 1 \text{ constant}\}$$

or

$$\Psi = \{(x, y, j)\colon [ax+c(j)]f(a, b, y)+by = 0,\ c(j) \text{ a constant for each } j = 0 \text{ or } 1\}\,.$$

We observe that the two planes of type Φ consist of the points of the two Cartesian planes defined by the equations $z = 0$ and $z = 1$. These two planes are in fact two "Moulton" planes. By contrast, the planes of type Ψ are just certain pairs of "parallel" lines of type $\boldsymbol{m}$. In our model, incidence is defined in terms of set containment.

DEFINITION. A point (x, y, k) is *on a line* l or *on a plane* π means $(x, y, k) \in l$ or $(x, y, k) \in \pi$ respectively.

Note. In this paper *collinear* and *coplanar* will be used in the usual sense to mean *on the same line or plane*, respectively. Betweenness is defined as follows:

DEFINITION. If $P = (p, p', p'')$, $Q = (q, q', q'')$ and $R = (r, r', r'')$ are distinct points, then Q is *between* P and R if and only if P, Q, and R are collinear and one of the following is true.

(1) If p' and r' have opposite signs and $(r'-p')/(r-p) > 0$, then either $p' < q' < r'$ or $r' < q' < p'$ is true.

(2) If p' and r' have the same sign or $(r'-p')/(r-p) < 0$, then there is a real number t so that $0 < t < 1$ and $q = pt+r(1-t)$ and $q' = p't+ +r'(1-t)$.

At first this definition seems objectionable in that no account seems to be taken of the third coordinates of the points. However, recall that the lines of type $\boldsymbol{l}$ have exactly two points so only lines of type $\boldsymbol{m}$ can ever contain three distinct points. By the definition of type $\boldsymbol{m}$-lines, one will always have $p'' = q'' = r''$ and we see that no special account need be taken of the third coordinates.

For the convenience of the reader, the incidence axioms (group I), and betweenness axioms (group II), are listed below. These are from the tenth edition of Hilbert's *Foundations of Geometry* [2].

I,1. For every two points A, B there exists a line a that contains each of the points A, B.

I,2. For every two points A, B there exists no more than one line that contains each of the points A, B.

I,3. There exist at least two points on a line. There exist at least three points that do not lie on a line.

I,4. For any three points A, B, C that do not lie on the same line there exists a plane α that contains each of the points A, B, C. For every plane there exists a point which it contains.

I,5. For any three points A, B, C that do not lie on one and the same line there exists no more than one plane that contains each of the three points A, B, C.

I,6. If two points A, B of a line a lie in a plane α, then every point of a lies in the plane α.

I,7. If two planes α, β have a point A in common, then they have at least one more point B in common.

I,8. There exist at least four points which do not lie in a plane.

II,1. If a point B lies between a point A and a point C, then the points A, B, C are three distinct points of a line, and B then also lies between C and A.

II,2. For two points A and C, there always exists at least one point B on the line AC such that C lies between A and B.

II,3. Of any three points on a line there exists no more than one that lies between the other two.

II,4. Let A, B, C be three points that do not lie on a line and let a be a line in the plane ABC which does not meet any of the points A, B, C. If the line a passes through a point of the segment AB, it also passes through a point of the segment AC, or through a point of the segment BC.

The verification that this model satisfies the axioms I,1-8 and II,1, 3 is essentially a textbook sort of exercise with the details mostly being elementary though tedious. One of the more tedious jobs is to get a general equation of the unique line incident upon two points $A = (a', a'', k)$ and $B = (b', b'', k)$ where k is 0 or 1. One such equation is:

$$[ax+c]f(a, b, y)+by = 0$$

where

$$a = a''g(b'')-b''g(a''),$$
$$b = (b'-a')g(b'')g(a''),$$
$$c = a'b''g(a'')-a''b'g(b'')$$

with

$$g(y) = f(a''-b'', b'-a', y).$$

To show that A and B satisfy this equation in the desired way is straightforward except for one step. For example, when one substitutes A into the equation one gets to the following equation by algebra:

$$(aa'+c)f(a, b, a'')+ba'' = a''f(a''-b'', b'a', b'')(a'-b')[f(a, b, a'')-f(a''-b'', b'-a', a'')].$$

When $b' \neq a'$ one must carefully examine the quotients

$$\frac{a}{b} = \frac{a''g(b'')-b''g(a'')}{(b'-a')g(b'')g(a'')} \quad \text{and} \quad \frac{a''-b''}{b'-a'}$$

and the corresponding values of f to get the result that

$$f(a, b, a'')-f(a''-b'', b'-a', a'') = 0.$$

This will verify that A is on this line.

Similar sorts of arithmetic will lead to verification of all the axioms I,1-8.

Axioms II,1, 3 also require some tedious arithmetic to verify in the case of points on lines of type $\boldsymbol{m}$. For the two-point lines (i.e. lines of type $\boldsymbol{l}$) both axioms hold vacuously.

Even though the specific verification is rather tedious, once one gains a "picture" of the points, lines, and planes, one can readily see that the model does satisfy the stated axioms. Further, axioms II,2,4 both fail to hold. This is readily seen if one considers lines of type $\boldsymbol{l}$.

What has been constructed is a model of a space geometry. Two of its planes are "Moulton" planes and these are well known to contain many examples showing Desargues' Theorem to be false [cf. 2, pp. 74–75]. Hence, our space geometry is a non-Desarguesian space geometry in which all of Axioms I,1-8 and Axioms II,1,3 hold, but Axioms II,2,4 fail to hold.

It is thus clear that, unlike projective geometry, in Euclidean geometry one needs more than just the space axioms of incidence to prove the appropriate formulation of Desargues' Theorem. Recall that in projective geometry any plane geometry which can be embedded in a space geometry has Desargues' Theorem as a valid theorem. This provides useful information concerning the relationship between plane and space geometries. Specifically, the failure of Desargues' Theorem in a plane projective geometry will guarantee that this plane geometry cannot be embedded in a space projective geometry. The model given above demonstrates that such an easy test is not available in a Euclidean or affine geometry.

References

[1] D. Hilbert, *Foundations of Geometry*, Chapter V, § 23, Open Court 1902, tr. by Townsend.

[2] — *Foundations of Geometry*, revised and enlarged by Dr. Paul Bernays, tr. by Leo Unger, Open Court 1972.

[3] F. R. Moulton, *A simple non-Desarguesian plane geometry*, Trans. Amer. Math. Soc. April, 1902.

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