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A characterization of locally connectedness by means of the set function T

by

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Abstract. In this paper the connective properties of the set function T are investigated. In particular, the images of closed sets under T are shown to contain closed connected subsets which are also in the image of T. These results are used to give a characterization of locally connectedness in unicoherent continua. This characterization generalizes a result of Kuratowski which concerned continua contractible with respect to S^1.

A continuum is a compact connected topological space. Throughout this paper X will denote a continuum. If $A \subset X$, then the interior of A in X will be denoted by $\mathrm{int}_X A$ and 2^X will denote the collection of all non-empty closed subsets of X. If $A \in 2^X$ and $p \in X - A$, then X is said to be *aposyndetic at p with respect to A* provided there is a subcontinuum M of X such that $p \in \mathrm{int}_X M \subset M \subset X - A$ [3]. The set function T is a mapping from 2^X into 2^X such that for each $A \in 2^X$, $T(A) = A \cup \{x \in X \mid X$ is not aposyndetic at x with respect to $A\}$.

For terms used but not defined herein, the reader is referred to [4] and [6].

It is easily seen that for each $A \in 2^X$, $T(A)$ is closed in X. In [1] it is shown that if A is connected, then $T(A)$ is also connected. In [5] Vought proved that if X is n-aposyndetic and A is a set consisting of $n+1$ points then $T(A)$ is connected. We shall extend these results concerning the connective properties of T.

The proof of the following lemma parallels that of Lemma 3.1 of [5].

LEMMA 1. *Suppose $S \in 2^X$, S is totally disconnected, $p \in T(S) - S$, and for each closed proper subset S' of S, $p \notin T(S')$. Then $T(S)$ is connected.*

Proof. Let S_0 be a non-empty subset of S which is both open and closed in S. Since $p \notin T(S - S_0)$, there is a subcontinuum H such that $p \in \mathrm{int}_X H \subset H \subset X - (S - S_0)$. Let $\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ be decreasing sequences of open sets such that for each positive integer n, $S - S_0 \subset U_n$, $S_0 \subset V_n$, $U_1 \cap \overline{V}_1 = U_1 \cap H = \overline{V}_1 \cap \{p\} = \emptyset$, and $S - S_0 = \bigcap_{n=1}^{\infty} U_n$ while $S_0 = \bigcap_{n=1}^{\infty} V_n$.

For each n, let C_n be the component of $X-U_n$ that contains H. Since p is not in the interior of the component of C_n-V_n in which it lies, let $\{O_n^j\}_{j=1}^{\infty}$ be a sequence of distinct components of C_n-V_n such that for each positive integer j, $O_n^j \cap \overline{V}_n \neq \emptyset$ and $p \in O_n = \lim_{j\to\infty} O_n^j$. Then for each n, O_n is a continuum, $O_n \subset C_n - V_n$, $p \in O_n$, and $O_n \cap \overline{V}_n \neq \emptyset$. Let $O = \lim_{n\to\infty} O_n$. Then O is a continuum containing p and $O \cap S_0 \neq \emptyset$.

Now $O \subset T(S)$. For if not, there is a $q \in O - T(S)$ and a subcontinuum K such that $q \in \mathrm{int}_X K \subset K \subset X - S$. It follows that there is a positive integer N_1 such that for $n > N_1$, $K \subset X - (U_n \cup V_n)$. Since $q \in (\mathrm{int}_X K) \cap O$, there is a N_2 such that for $n > N_2$, $(\mathrm{int}_X K) \cap O_n \neq \emptyset$. Let $m > N_1 + N_2$. Since $(\mathrm{int}_X K) \cap O_m \neq \emptyset$, there is a N_3 such that for $j > N_3$, $(\mathrm{int}_X K) \cap O_m^j \neq \emptyset$. Let $j > N_3$. Then $O_m^j \subset C_m$ and $K \cap C_m \neq \emptyset$. Thus $K \subset C_m - V_m$ so K is contained in some component of $C_m - V_m$. This contradicts the fact that for all $j > N_3$, $(\mathrm{int}_X K) \cap O_m^j \neq \emptyset$. This establishes that $O \subset T(S)$.

Now let $s \in S$. There is a sequence $\{S_n\}_{n=1}^{\infty}$ of subsets of S such that for each n, S_n is both open and closed in S, and $\{s\} = \bigcap_{n=1}^{\infty} S_n$. By the previous construction, for each n there is a continuum $A_n \subset T(S)$ such that $p \in A_n$ and $A_n \cap S_n \neq \emptyset$. Let $A_s = \lim_{n\to\infty} A_n$. Then A_s is a continuum, $A_s \subset T(S)$, and $\{p, s\} \subset A_s$.

Let $A = \bigcup_{s\in S} A_s$. Then A is connected and $S \subset A \subset T(S)$. For each $x \in T(S)$ let C_x be the component of x in $T(S)$. Then by Corollary 1.1 of [2], $C_x \cap S \neq \emptyset$. Thus $C_x \cap A \neq \emptyset$ and it follows that

$$T(S) = A \cup \left(\bigcup \{C_x \mid x \in T(S)\}\right)$$

is connected.

We now show that "totally disconnected" can be dropped from the hypothesis.

THEOREM 1. *Suppose $G \in 2^X$, $p \in T(G) - G$, and for any closed proper subset G' of G, $p \notin T(G')$. Then $T(G)$ is connected.*

Proof. If $T(G) = X$, then the theorem is established so assume $T(G)$ is a proper closed subset of X. Let Ω be an indexing set and for each $\alpha \in \Omega$ let G_α be a component of G such that if $\alpha \neq \beta$ then $G_\alpha \cap G_\beta = \emptyset$. Then $D = \{G_\alpha \mid \alpha \in \Omega\} \cup \{m \mid m \in X - G\}$ is an upper semi-continuous decomposition of X and the hyperspace X^* of this decomposition is a continuum.

Let π be the quotient map from X onto X^* and let $A^* = \{x^* \in X^* \mid$ there is an $\alpha \in \Omega$ such that $\pi^{-1}(x^*) = G_\alpha\}$. Then π is a monotone mapping and A^* is a closed totally disconnected subset of X^*. Thus by Lemma 1, $T(A^*)$ is connected in X^*. Since $\pi^{-1}[T(A^*)] = T(G)$ and π is monotone, it follows that $T(G)$ is connected.

The following is the key theorem of this paper concerning connective properties of the function T.

THEOREM 2. *If $A \in 2^X$ and $x \in T(A) - A$, then there is a $B \in 2^X$ such that* (1) $B \subset A$, (2) $x \in T(B)$, (3) *if $D \in 2^X$ and $D \subsetneq B$, then $x \notin T(D)$, and* (4) *$T(B)$ is a continuum.*

Proof. Let $B_1, B_2, \ldots, B_i, \ldots$ be a decreasing sequence in 2^X with the property that for each positive integer i, $x \in T(B_i)$. Let $B = \bigcap_{i=1}^{\infty} B_i$. Now if $x \notin T(B)$, there is a subcontinuum $F \subset X - B$ such that $p \in \mathrm{int}_X F_N$. Since F is compact, there is a positive integer N such that $F \subset \bigcup_{i=1}^{N} (X - B_i)$ which contradicts the fact that $x \in T(B_N)$. Thus $x \in T(B)$.

Thus the property that $x \in T(A)$ is an inducible property on A. Therefore properties (1), (2), and (3) follow immediately from the Brouwer Reduction Theorem [6]. Condition (4) follows from Theorem 1.

As a corollary we show that if $F \in 2^X$ then the components of $T(F)$ are also in the image of the mapping T.

COROLLARY 1. *If $F \in 2^X$ and K is a component of $T(F)$, then $K = T(K \cap F)$.*

Proof. Let $F \in 2^X$, K be a component of $T(F)$, and $k \in K$. If for each closed proper subset F' of F, $k \notin T(F')$, then by Theorem 1, $T(F)$ is connected. Thus $K = T(F) = T(K \cap F)$.

Suppose there is a closed proper subset G of F such that $k \in T(G)$. Then there is a $G' \in 2^X$ satisfying conditions (1)-(4) of Theorem 2. Since $T(G') \subset T(F)$ and $T(G') \cap K \neq \emptyset$, it follows that $T(G') \subset K$. Now $G' \subset T(G') \subset K$, thus $G' \subset K \cap F$ which implies that $T(G') \subset T(K \cap F)$. Hence $K \subset T(K \cap F)$.

Let C be a component of $T(K \cap F)$. By Corollary 1.1 of [2], $C \cap (K \cap F) \neq \emptyset$. Thus $C \cap K \neq \emptyset$ so $T(K \cap F) = K \cup \{C \mid C$ is a component of $T(K \cap F)\}$ is connected. Since $T(K \cap F) \subset T(F)$ and $T(K \cap F) \cap K \neq \emptyset$, then $T(K \cap F) \subset K$. Therefore $K = T(K \cap F)$.

It is well known that if a continuum fails to be locally connected at a point p, then there is a non-degenerate subcontinuum L containing p such that the continuum is not locally connected at any point of L [6]. In the following theorem we show a similar result for the image of the T mapping.

THEOREM 3. *Suppose $A \in 2^X$ and $x \in T(A) - A$. Then there is a non-degenerate subcontinuum N of X such that $x \in N \subset T(A)$.*

Proof. By Theorem 2, there is an $A' \in 2^X$ satisfying conditions (1)-(4). Let V be an open subset of X such that $x \in V \subset \overline{V} \subset X - A$, K be the component of x in $V \cap T(A')$, and let $N = \overline{K}$. Since $V \cap T(A')$ is an open subset of the continuum $T(A')$, then N intersects the boundary of V. Thus N is a non-degenerate subcontinuum contained in $T(A)$.

The continuum X is *unicoherent* provided that if H and K are proper subcontinua such that $X = H \cup K$, then $H \cap K$ is a continuum.

THEOREM 4. *Suppose X is a unicoherent continuum. X is locally connected if and only if for each $C \in 2^X$ which separates X between two points x and y, there is a component E of C which separates X between x and y.*

Proof. First suppose X is locally connected, $C \in 2^X$, and C separates X between x and y. Let A be the component of x in $X-C$ and B be the component of b in $X-\bar{A}$. Then both A and B are open in X, $X-(\bar{A} \cap \bar{B}) \subset (X-\bar{B}) \cup B$, $x \in X-\bar{B}$, and $y \in B$. Thus $\bar{A} \cap \bar{B}$ is a closed set which separates X between x and y.

Let Ω be an indexing set and for each $a \in \Omega$ let S_a be a component of $X-\bar{A}$ such that $S_a \cap B = \emptyset$. Then $\bar{A} \cup (\bigcup_{a \in \Omega} \bar{S}_a)$ is connected and $X = [\bar{A} \cup \overline{(\bigcup_{a\in\Omega} S_a)}] \cup \bar{B}$. Since X is unicoherent, $\bar{A} \cap \bar{B} = [\bar{A} \cup \overline{(\bigcup_{a\in\Omega} S_a)}] \cap \bar{B}$ is a continuum. Let E be the component of C which contains $\bar{A} \cap \bar{B}$. If x and y are in the same component of $X-E \subset X-(\bar{A} \cap \bar{B})$, then x and y are in the same component of $X-(\bar{A} \cap \bar{B})$. Since this is not the case, E separates X between x and y.

Now to prove the condition is sufficient, suppose X is not locally connected, hence not connected im kleinen at a point $p \in X$. There is a $A \in 2^X$ such that $p \in T(A)-A$. Let $B \in 2^X$ satisfying conditions (1)-(4) Theorem 2. Let $x \in B$ and O be an open set such that $p \in O \subset \bar{O} \subset X-B$. Then the boundary of O is a closed set which separates X between x and p so there is a component N of the boundary of O which separates X between p and x.

Let U and V be open sets, $U \cap V = \emptyset$, such that $X-N = U \cup V$, $p \in U$, and $x \in V$. Then $H = N \cup U$ and $K = N \cup V$ are continua and $X = H \cup K$. Since $x \in B \cap K$, then $B \cap K \neq \emptyset$. If $B \cap H = \emptyset$, then $p \notin T(B)$ which is contrary to condition (2) of Theorem 2. So assume $B \cap H \neq \emptyset$. Since $B \cap H$ and $B \cap K$ are non-empty closed proper subsets of B, it follows from (3) of Theorem 2 that there are continua L_1 and L_2 such that $p \in \operatorname{int}_X L_1 \subset L_1 \subset X-(B \cap H)$ and $p \in \operatorname{int}_X L_2 \subset L_2 \subset X-(B \cap K)$. Again if $L_1 \cap (B \cap K) = \emptyset$ or $L_2 \cap (B \cap H) = \emptyset$, it follows that $p \notin T(B)$. Assume that $L_1 \cap (B \cap K) \neq \emptyset \neq L_2 \cap (B \cap H)$. Let $X_1 = L_1 \cup K$ and $X_2 = L_2 \cup H$. Then X_1 and X_2 are continua and $X = X_1 \cup X_2$. Now the unicoherence of X implies that $X_1 \cap X_2$ is a continuum. Then $p \in \operatorname{int}_X(X_1 \cap X_2) \subset X_1 \cap X_2 \subset X-B$ implies that $p \notin T(B)$ which is contrary to condition 2 of Theorem 2. Therefore X is locally connected.

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