Almost continuous Darboux functions and Reed's pointwise convergence criteria

by

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Abstract. This paper consists of two somewhat unrelated investigations. The well-known theorem (due to Kuratowski and Sierpiński) that every real Darboux function of Baire class 1 on $[0, 1]$ has a connected graph is strengthened in two ways: firstly, by weakening the hypothesis, and secondly, by strengthening the conclusion. An example is given to show that these cannot be accomplished simultaneously. In order to aid in the proof of the above results, a theorem is proved which, in a general topological setting, relates the pointwise convergence criteria of Baire to new criteria due to C. S. Reed.

1. Introduction. For real functions f with domain $I = [0, 1]$, consider the following definitions (no distinction is made between a function and its graph):

A: (almost continuous) every open subset of R^2 containing f contains a continuous function with domain I.

C: (connected) f is a connected subset of R^2.

D: (Darboux) if X is a connected subset of I, then $f(X)$ is a connected number set.

E: if $x \in [0, 1)$, the point $(x, f(x))$ is a limit point of $f|(x, 1]$, and if $x \in (0, 1]$, then $(x, f(x))$ is a limit point of $f|[0, x)$.

B_1, R_1, and J_1: f is the pointwise limit of a sequence of functions which are continuous, continuous on the right, and jump functions, respectively. g is a jump function if $g(0+)$, $g(1-)$, $g(x+)$, and $g(x-)$ exist for all $x \in (0, 1)$.

It is easy to see that A→C→D→E, but E$\not\to$D. It is also true that D$\not\to$C [7] and C$\not\to$A [3], [5], [10]. Reed, in [9], characterizes B_1, R_1, and J_1

and shows that $B_1 \to R_1 \to J_1$ but $J_1 \not\to R_1 \not\to B_1$. Kuratowski and Sierpiński have shown [8] that $DB_1 \to C$ ("D" can easily be replaced by "E"). It is the main purpose of this paper to strengthen this theorem in the following two ways:

THEOREM 1. $ER_1 \to C$.

THEOREM 2. $EB_1 \to A$.

It will also be shown that "B_1" cannot be replaced by "R_1" in Theorem 2 because $CR_1 \not\to A$. Also, "R_1" cannot be replaced by "J_1" in Theorem 1 because $DJ_1 \not\to C$. Theorem 2 supplies the answer to a question raised by Kellum in [6].

To aid in doing the above, it is convenient to use characterizations of properties R_1 and J_1 which are similar to Baire's condition for B_1. These characterizations will follow as immediate corollaries to the results of Reed and Theorem 0 of Section 2, which compares Baire's condition and Reed's condition in a general topological setting.

2. Baire's and Reed's convergence criteria in F_{II} spaces. Most of the topological properties and notations used here are as given in [4]. Consider the following two conditions for real valued functions f defined on a topological space (X, T):

(1) (Baire) if M is a perfect subset of X, then $f|M$ is continuous at some point of M,

(2) (Reed) if $\alpha > \beta$, $U \subset \{x|\ f(x) \geqslant \alpha\}$, and $V \subset \{x|\ f(x) \leqslant \beta\}$, then either $U \neq \mathrm{Cl}(V)$ or $V \neq \mathrm{Cl}(U)$.

Both of the conditions have been related to

(0) f is the pointwise limit of a sequence of continuous functions defined on X.

THEOREM 0. *If (X, T) is a topological space, then (1)→(2) for real valued functions defined on X, and (2)→(1) if and only if (X, T) is an F_{II} space.*

Proof. Suppose f is a real valued function defined on a topological space (X, T). Suppose f does not satisfy (2). Let α, β, U, and V be as in "not (2)". Let $M = \mathrm{Cl}(U)$. M is perfect, but $f|M$ has no point of continuity because U and V are both dense in M.

Now, suppose (X, T) is an F_{II} space and f is a real valued function defined on X. Suppose f does not satisfy (1) and let M be a perfect subset of X such that $f|M$ is totally discontinuous. Let $g = f|M$. For each positive integer n, let

$$M_n = \{x \text{ in } M|\ \limsup_x g - \liminf_x g > 1/n\},$$

where

$$\limsup_x g = \inf\{\sup_{y \text{ in } D} [g(y)]|\ D \text{ is open in } M \text{ and contains } x\}.$$

and $\liminf_x g$ is defined similarly. Since M is 2-nd category in itself, there is a positive integer n such that M_n is 2-nd category in M. For each x in M_n, let t_x be a number between $\limsup_x g - 1/2n$ and $\liminf_x g + 1/2n$. Since M_n is 2-nd category in M, there exist numbers $\alpha > \beta$ such that $\alpha - \beta < 1/2n$ and $H = \{x \text{ in } M_n|\ \beta < t_x < \alpha\}$ is 2-nd category in M. Thus, there is some set Q open in M, in which H is dense. For each subset D of Q which is open in M, let u_D and v_D be elements of D such that $g(u_D) \geqslant \alpha > \beta \geqslant g(v_D)$. Let $U = \{u_D|\ D \subset Q \text{ is open in } M\}$ and $V = \{v_D|\ D \subset Q \text{ is open in } M\}$. Then $U \subset \mathrm{Cl}(V)$ and $V \subset \mathrm{Cl}(U)$ and f does not satisfy (2).

Now, suppose (X, T) is not an F_{II} space and let M be a perfect subset of X which is not 2-nd category in M. $M = N_1 \cup N_2 \cup \ldots$, where each N_i is nowhere dense in M. Let $H_1, H_2, \ldots$ be such that

$$H_n = \mathrm{Cl}(N_1 \cup N_2 \cup \ldots \cup N_n)$$

for each n. Each H_n is closed, nowhere dense in M, and a subset of H_{n+1}, and $M = H_1 \cup H_2 \cup \ldots$ Assume H_n is a proper subset of H_{n+1} for each n. Let f be the function defined on X such that $f(x) = n$ if x is in $H_{n+1} - H_n$ for some positive integer n, and $f(x) = 0$ if x is in $H_1 \cup (X - M)$. Since $f|M$ is unbounded on every set which is open in M, $f|M$ must be totally discontinuous, so f does not satisfy (1). But suppose α, β, U, and V are as in (2). If V contains a point of $X - M$, then it is not a limit point of U, which would be a subset of M. Suppose V contains no point of $X - M$. Then V is a subset of $H_1 \cup H_2 \cup \ldots \cup H_n$ for some n, and U contains a point of $H_{n+1} \cup H_{n+2} \cup \ldots$, which would not be a limit point of V. Thus f satisfies (2).

Remark 1. It is well known that the property of a space being an F_{II} space is intimately related to condition (1) for the real valued functions defined on X. Hausdorff [4], shows in Theorem V, p. 287, that if the (metric) space A is a G_{II} set, then every function of the "first class" (which is equivalent to (0) in metric spaces) is pointwise discontinuous (i.e. continuous at each element of a dense subset of A). He uses this to deduce in Theorem VI, p. 288, that if A is an F_{II} set and f is a function of the first class, then every non-empty closed set M contains a point of continuity of $f|M$. It is stated on p. 288 of [4] that "a flat converse to Theorem V, in the sense that pointwise discontinuous functions are of the first class as well, is out of the question, if only because of considerations of cardinality". Actually, the true converse of Theorem V, p. 287, does hold, as well as the converses of Theorem VIII, p. 289, and

Theorem VI, p. 295, and partial converses of Theorem VI, p. 288, and the theorem on p. 296. These converses can be summarized as follows: *If* (X, T) *is any perfectly normal topological space which is not an* F_{II} *space (respectively, not a* G_{II} *space), then there exists a function* f *which satisfies* (0) *and a closed set* M *(respectively, an open set* M*) such that if* N *is dense in* M*, then* $f|N$ *is totally discontinuous.* To prove this, consider the function f constructed in the last paragraph of the proof of Theorem 0 (if M is open, then the sets $N_1, N_2, \ldots$ must be specially chosen so that $H_1 \cup H_2 \cup \ldots = M$). If N is dense in M and Q is open in N, then Q intersects $H_{n+1} - H_n$ for infinitely many n, so $f|Q$ is unbounded. Therefore $f|N$ is totally discontinuous. However, f does satisfy (0), for construct $f_1, f_2, \ldots$ as follows: Let $O_1, O_2, \ldots$ be a monotonic decreasing sequence of open sets with intersection M, and for each positive integer j let $O_{j1}, O_{j2}, O_{j3}, \ldots$ be a monotonic decreasing sequence of open sets with intersection H_j. For each n, let $K_n, H_{1n}, H_{2n}, \ldots, H_{nn}$ be the finite sequence of mutually exclusive closed sets defined by $K_n = X - O_n$, $H_{1n} = H_1$, and $H_{jn} = H_j \cap \cap (X - O_{j-1,n})$ for $j = 2, 3, \ldots, n$. Let f_n be a continuous function defined on X for each positive integer n such that $f_n(x) = 0$ if x is in K_n and $f_n(x) = j-1$ if x is in H_{jn}. Then $f_1, f_2, \ldots$ converges pointwise to f, and f satisfies (0).

Remark 2. Reed, in [9], gives characterizations of B_1, R_1, and J_1, respectively, which are surprisingly similar to each other. On the interval I, let T_1, T_2, and T_3 be the ordinary relative line topology, the relative topology generated by the basis of all left sects $[x, y)$, and the relative topology generated by the basis $\{b \mid b$ is an open interval minus a countable set$\}$, respectively. Reed shows that B_1, R_1, and J_1 are equivalent to (2) in (I, T_1), (2) in (I, T_2), and (2) in (I, T_3), respectively. It is not surprising that the characterizations for B_1 and R_1 are so similar, because B_1 and R_1 are just (0) in (I, T_1) and (0) in (I, T_2), respectively. However, it is surprising such a similar characterization for J_1 is possible, because J_1 is certainly not just (0) in (I, T_3). In fact, (0) in (I, T_3) is the same as B_1, because a function is T_3-continuous on all of I if and only if it is T_1-continuous on all of I. So, (0) and (2) are not equivalent in the space (I, T_3).

Notice that (I, T_2) is not metrizable, and (I, T_3) is not even 1st countable or normal, but *in all three spaces,* I *is an* F_{II} *set.* Notice also that a set is T_3-perfect if and only if it is T_1-perfect.

Corollary. *For real valued functions defined on* I, R_1 *is equivalent to "every* T_2*-perfect set* M *contains a point at which* $f|M$ *is* T_2*-continuous", and* J_1 *is equivalent to "every* T_1*-perfect set* M *contains a point at which* $f|M$ *is* T_3*-continuous".*

3. Proof of Theorems 1 and 2. To avoid confusion, the following notation will be used: A^i denotes the interior on the line of a number set A, and T^I denotes the interior in the plane of a planar set T.

Lemma. *If* f *satisfies* R_1*, then* f *is pointwise discontinuous.*

Proof. If f is totally discontinuous on some subinterval J of I, then there would be some open subinterval K of J and numbers $\alpha > \beta$ such that $U = \{x \mid f(x) \geqslant \alpha\}$ and $V = \{x \mid f(x) \leqslant \beta\}$ are both T_1-dense in K. Then U and V are also T_2-dense in K, and f does not satisfy Reed's condition for R_1.

Proof of Theorem 1 ($R_1E \to C$). The outline of the proof that $B_1D \to C$ which appears in [2] will be followed. Suppose f satisfies R_1 and E but not C. Let Q_1 and Q_2 be mutually exclusive open sets such that $f \subset Q_1 \cup Q_2$. Let $A = \{x \mid (x, f(x))$ is in $Q_1\}$ and let $B = \{x \mid (x, f(x))$ is in $Q_2\}$, and let K be the boundary of A (= the boundary of B). K is perfect because f satisfies E. Since f is pointwise discontinuous, $A^i \cup B^i$ is dense in I.

Now, let K' be the set of all points of K which are T_2-limit points of K. Let x be a point of K' and $[x, y)$ be a left sect containing x. There is an element a of K' between x and y. Suppose a is in A. Then there is an element b' of B between a and x. If b' is not itself in K', then it would be the left end of or an element of some component W of B^i. Then the right end b of W would be an element of $K' \cap B$ between x and y. Then, an element of $K' \cap A$ could be found between x and b, so in any case, $[x, y)$ intersects both $K' \cap A$ and $K' \cap B$. Thus $f|K'$ cannot be T_2-continuous at any element x of K', and f does not satisfy R_1.

Example 1. It is not possible to strengthen Theorem 1 by replacing "R_1" by "J_1" in Theorem 1. For consider the function f constructed in Example 2 of [1]. f satisfies D but not C and is continuous on the complement of the "middle third" Cantor set M. f is also constant on $M' \cap \cap [0, 1/2]$ and on $M' \cap [1/2, 1]$, where M' is the set of all points of M which are not end points of M. f also satisfies J_1 because otherwise there would be a T_1-perfect set N such that $f|N$ is not T_3-continuous at any point. Then N would have to be a subset of the Cantor set M. Let x be an element of $M' \cap N$, assume $x < 1/2$. Then $f|N$ is T_3-continuous at x because $Q = M' \cap N \cap [0, 1/2]$ is T_3-open in N and $f|Q$ is constant.

Proof of Theorem 2 ($B_1E \to A$). Suppose f satisfies property E and property B_1 but not property A. Let Q be an open subset of R^2 containing f such that no continuous function with domain I lies in Q. Let $Q_1 = \{P$ in $R^2 \mid P$ belongs to a continuous function which lies in Q and has domain an interval containing $0\}$. Q_1 is open, for suppose P is an element of Q_1. Let T be a vertical rectangle (includes the interior) with center P such that $T^I \subset Q$. P belongs to some continuous function g which lies in Q and has domain an interval containing 0. Assume P has abscissa > 0. Then there is a vertical rectangle U with center P such that $U^I \subset T^I$ and g intersects the left edge of U. It follows that $U^I \subset Q_1$.

Now, let $Q_2 = Q - \mathrm{Cl}(Q_1)$. Let $A = \{x|\ (x, f(x))$ is in $Q_1\}$, $B = \{x|\ (x, f(x))$ is in $Q_2\}$, and $C = I - (A \cup B)$.

Every element of C must be a limit point of A from the right and a limit point of B from the left, for suppose x is an element of C and S is an open interval containing x. Let T be a vertical rectangle with center $(x, f(x))$ such that $T^I \subset Q$ and T has X-projection lying in S. T^I contains a point $(z, f(z))$ such that $z > x$. Let U be a vertical rectangle with center $(x, f(x))$ such that $U^I \subset T^I$ and the X-projection of U does not contain z. U^I intersects Q_1, so it follows that $(z, f(z))$ is also in Q_1 and z is in $A \cap S$. T^I also contains a point $(y, f(y))$ such that $y < x$. y is not in A or else x would be, too. If y were in C, there would be a point of Q_1 in T^I with abscissa between y and x, which would mean that x is in A. Therefore, y is an element of $B \cap S$.

$A^i \cup B^i$ is dense in I, for suppose S is an open subset of I. Since f is pointwise discontinuous, f is continuous at some element x of S. Consider a vertical rectangle T with center $(x, f(x))$ such that $T^I \subset Q$, the X-projection (a, b) of T^I lies in S, and for every y in (a, b), $(y, f(y))$ is in T^I. Then, if x is in A, the open interval (x, b) is a subset of $A^i \cap S$, and if x is in B or C, the open interval (a, x) is a subset of $B^i \cap S$.

Let A^0 be the collection of all components of A^i and B^0 be the collection of all components of B^i. It is important to observe that if a is in A^0, then the right end of a is in A, and if b is in B^0, then the left end of b is in B.

Let K denote $I - (A^i \cup B^i)$. K does not have to be perfect (some points of C may be isolated), but it will be shown that K contains a Cantor set N in which A and B are both dense. Zero is in A; if zero is neither an element of nor the left end of an element of $A^0 \cup B^0$, let $\alpha =$ zero; otherwise zero would necessarily be an element of or the left end of an element a of A^0. Then let α be the right end of a. In either case $0 \leqslant \alpha < 1$, and α is an element of $A \cap K$ which is the left end of no element of $A^0 \cup B^0$. Similarly, a number β can be determined such that $0 \leqslant \alpha < \beta \leqslant 1$ and β is an element of $B \cap K$ which is the right end of no element of $A^0 \cup B^0$. Since there are elements of B to the right of (and close to) α, and there are elements of A to the left of (and close to) β, numbers α' and β' can be determined such that $\alpha < \beta' < \alpha' < \beta$, β' is an element of $B \cap K$ which is the right end of no element of $A^0 \cup B^0$, and α' is an element of $A \cap K$ which is the left end of no element of $A^0 \cup B^0$. Continuing in this fashion, a sequence

$$[\alpha, \beta], [\alpha, \beta'] \cup [\alpha', \beta], [\alpha, \beta''] \cup [\alpha'', \beta'] \cup [\alpha', \beta'''] \cup [\alpha''', \beta], \ldots$$

of sets is determined which closes down on a Cantor set N in which A and B are both dense. $f|N$ could have no point of continuity, so f does not satisfy B_1.

EXAMPLE 2. It is not possible to combine Theorem 1 and Theorem 2 by replacing "B_1" by "R_1" in Theorem 2 because $\mathrm{R}_1\mathrm{C} \not\to \mathrm{A}$, as is shown by the following example (which is constructed in a manner similar to that used in [5]). Let g be an increasing real function defined on I such that $g(0) = 1/3$, $g(1) = 2/3$, g is continuous from the right, but g is discontinuous at each rational number in $(0, 1)$. Let $M = \mathrm{Cl}(g)$. Let K be the middle third Cantor set, and let $t_1, t_2, \ldots$ be a listing of the components of $I - K$. For each component $t = t_n = (a, b)$, let g_t be a continuous function with domain $[a, b]$ and range I which does not intersect M and is such that $g_t(b) = 0$ and $0 < g_t(a) - g(a) < 1/n$ (this last requirement is included only to make f a G_δ set). Now, let f be the function such that $f(1) = 1$, $f(x) = g_t(x)$ if x is in $\mathrm{Cl}(t)$, for some component t of $I - K$, and $f(x) = 0$ if x is any other element of I. It is easy to see that f satisfies C but not A. f also satisfies R_1, for suppose there is a T_2-perfect set J such that $f|J$ is nowhere T_2-continuous. Then J must be a subset of the set K' of all elements of K which are T_2-limit points of K. But $f|K'$ is constant. Therefore, f must satisfy R_1.

Remark 3. Notice that the function of Example 1 and the function of Example 2 are both G_δ subsets of R^2. This rules out the possibility that $\mathrm{J}_1 G_\delta \mathrm{D} \to \mathrm{R}_1$ or $\mathrm{R}_1 G_\delta \mathrm{C} \to \mathrm{B}_1$.

Remark 4. Theorem 2 was conjectured to the author by B. D. Garrett.

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