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A new definition of the circle by the use of bisectors

by

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Abstract. The subset $B(x, y) = \{q \in X: \varrho(x, q) = \varrho(y, q)\}$ in a metric space (X, ϱ) is called the *bisector* of a pair x, y. It is known that any connected metric space in which each bisector is a unique point, is topologically an interval of the real line R.

If each bisector consists of exactly two points, then X has DBP property.

The question whether every connected metric space with DBP is homeomorphic to the one-sphere S^1 is still open.

A metric space is segment-convex if for each pair p, r of its points it contains an arc joining p to r which is isometric to a line segment.

We show that any segment-convex metric space with DBP is isometric to a metric one-sphere with its natural geodesic metric.

1. Introduction. For any pair of distinct points x and y in a non-trivial metric space (X, ϱ) the subset $B(x, y) = \{q \in X|\ \varrho(x, q) = \varrho(y, q)\}$ will be called the *bisector* ([3], see also [1] where it is called the midset). If each bisector is a unique point, then X has [1] the *unique bisector property* (UBP). If each bisector consists of exactly two points, then X has the *double bisector property* (DBP).

It is known [1] that any connected metric space with UBP is homeomorphic to a subset of the real line R, and is therefore an interval.

The question whether every connected metric space with DBP is homeomorphic to the one-sphere S^1 is still open.

The aim of the present paper is the following result: If (X, ϱ) is a segment-convex metric space with DBP, then X is isometric to a metric one-sphere.

The proof will be based on the following three auxiliary propositions:

Let a_1 and a_2 be two distinct points of X, and let $B(a_1, a_2) = \{x_1, x_2\}$, then

1° $L_1 = \overline{x_1 a_1} \cup \overline{a_1 x_2}$ and $L_2 = \overline{x_1 a_2} \cup \overline{a_2 x_2}$ are two simple arcs joining x_1 to x_2, and $L_1 \cap L_2 = B(a_1, a_2)$.

2° More precisely, L_1 and L_2 are two metric segments joining x_1 to x_2.

3° $L_1 \cup L_2 = X$.

2. Definitions and notation. Let (X, ϱ) be a metric space and $p, q, r \in X$. A point q is *between* p and r (pqr) if $\varrho(p,q)+\varrho(q,r)=\varrho(p,r)$. A point m is a *center* of a pair x, y if $\varrho(x,m)=\varrho(m,y)=\frac{1}{2}\varrho(x,y)$. For any center of x, y we have xmy and $m \in B(x,y)$; the converse is not necessarily true.

A space X is *convex* [5] provided it contains for each pair of its distinct points p and r at least one point q, such that $p \neq q \neq r$ and pqr.

A *metric segment* between p and r in a metric space X is an arc $T(p,r)$ joining p to r which is isometric to a line segment of length $\varrho(p,r)$; if unique or fixed it will be denoted by $\overline{pr}$.

A metric space is *segment-convex* provided it contains for each pair of its points at least one metric segment between them.

The transitivity of the metric betweeness implies [3]

2.1. *If pqr and $p \neq r$ then for any $\overline{pq}$ and for any $\overline{qr}$ the union $\overline{pq} \cup \overline{qr}$ is a metric segment joining p to r.*

A metric space is *without ramifications* (WR-space) if pqr, pqs, and $p \neq q$ imply prs or psr.

It is known ([3] see also [4]) that

2.2. *In a metric WR-space if the intersection $\overline{pr} \cap \overline{ps}$ possesses at least two points and if $r \neq s$, then either $\overline{pr} \subset \overline{ps}$ or $\overline{ps} \subset \overline{pr}$.*

By a metric n-sphere (with the ordinary metric d) we will mean the set $S^n = \{p \mid p=(x_1, x_2, \ldots, x_{n+1}),\ x_i \in R,\ x_1^2+x_2^2+\ldots+x_{n+1}^2=r^2\}$, where for any $p=(x_1, x_2, \ldots, x_{n+1})$ and $q=(y_1, y_2, \ldots, y_{n+1})$ we set $d(p,q) = r\arccos(|x_1y_1+\ldots+x_{n+1}y_{n+1}|/r^2)$ and where the positive number r is the radius of the n-sphere. For $n=1$ the complex number notation will be used, i.e. $S^1=\{z \mid z=re^{it},\ t \in R\}$.

3. Preliminary lemmas. Let (X, ϱ) be a segment-convex metric space with DBP. Let a_1 and a_2 be two distinct points of X and let $B=B(a_1,a_2) = \{x_1, x_2\}$. Let $A_1=\{x \mid \varrho(x,a_1)<\varrho(x,a_2)\}$, $A_2=\{x \mid \varrho(x,a_1)>\varrho(x,a_2)\}$, $B_1=A_1 \cup B$, and $B_2=A_2 \cup B$. Then we have

3.1. *$X-B$ is disconnected.*

Indeed [1], $X-B=A_1 \cup A_2$ and A_1 and A_2 are mutually separated.

3.2. *If $z \in A_i$, $i=1,2$, and $\overline{a_iz}$ is any metric segment from a_i to z, then $\overline{a_iz} \subset A_i$.*

Proof. We may suppose that $i=1$; so let $z \in A_1$ and let $q \in \overline{a_1z}$, then $\varrho(z,q)+\varrho(q,a_1)=\varrho(z,a_1)<\varrho(z,a_2)$. If $q \notin A_1$, then $\varrho(q,a_1) \geqslant \varrho(q,a_2)$. Combining these we get $\varrho(z,a_2) \leqslant \varrho(z,q)+\varrho(q,a_2) \leqslant \varrho(z,g)+\varrho(q,a_1) = \varrho(z,a_1)$, a contradiction since $z \in A_1$.

3.3. *If $z \in B_i$, $i=1,2$ and $\overline{a_iz}$ is any metric segment from a_i to z, then $\overline{a_iz} \subset B_i$.*

From 3.2 and 3.3 we see

3.4. *The sets A_1, A_2, B_1, and B_2 are connected.*

3.5. *If $p \in A_1$ and $q \in A_2$ then for any fixed segment $\overline{pq}$, $\overline{pq} \cap B \neq \emptyset$.*

More precisely

3.6. *If $p \in A_1$, $m \in A_2$, $\varrho(m,x_1)=\varrho(m,x_2)$ and if $\varrho(p,x_1)<\varrho(p,x_2)$, then for any segment $\overline{pm}$, $\overline{pm} \cap B=\{x_1\}$.*

The proof follows from 3.5 and from the inequality

$$\varrho(p,m) \leqslant \varrho(p,x_1)+\varrho(x_1,m) < \varrho(p,x_2)+\varrho(x_1,m) = \varrho(p,x_2)+\varrho(x_2,m).$$

Similar results follow if $\varrho(p,x_2)<\varrho(p,x_1)$ or if, $p \in A_2$ and $m \in A_1$.

The following lemma will be used several times.

3.7. *A segment-convex metric space with* DBP *is a* WR-*space.*

Proof. Suppose this were not the case. Then there exist p, q, r, and s so that pqr, pqs, $p \neq q$, and neither prs nor psr. Let $\overline{pq}$, $\overline{qr}$, and $\overline{qs}$ be three fixed segments. By 2.1 $\overline{pq} \cup \overline{qr}$ is a fixed segment joining p to r and $\overline{pq} \cup \overline{qs}$ is a fixed segment joining p to s. We may suppose that $\varrho(q,r) \geqslant \varrho(q,s)>0$ and we can find on $\overline{qr}$ a point s_1 so that $\varrho(q,s_1)=\varrho(q,s)$. We have $s \neq s_1$ otherwise qsr and pqr implies psr. Now for each $x \in \overline{pq}$ we have $\varrho(s,x)=\varrho(s_1,x)$, contradicting DBP.

4. Auxiliary propositions. Let (X, ϱ) be a segment-convex metric space with DBP. Let a_1 and a_2 be two distinct points of X, and let $B(a_1,a_2)=\{x_1,x_2\}=B$. According to the lemmas of the preceding paragraph we have: $X-B=A_1 \cup A_2$, A_1 and A_2 are mutually separated connected sets; $X=B_1 \cup B_2$, $A_1 \subset B_1$ and $A_2 \subset B_2$, B_1 and B_2 are connected closed sets with $B_1 \cap B_2=B$; and, $a_1 \in A_1$ and $a_2 \in A_2$. Under the above assumptions:

4.1. *For any four fixed segments $\overline{a_ix_j}$, $i,j=1,2$, $L_1=\overline{a_1x_1} \cup \overline{a_1x_2}$ and $L_2=\overline{a_2x_1} \cup \overline{a_2x_2}$ are two simple arcs joining x_1 to x_2 and $L_1 \cap L_2=B$.*

Proof. We focus our attention on L_1 and first show that $\overline{a_1x_1} \cap \overline{a_1x_2}=\{a_1\}$. If not then, by 3.7 and 2.2, either $\overline{a_1x_1} \subset \overline{a_1x_2}$ or $\overline{a_1x_2} \subset \overline{a_1x_1}$. In the first case we would have $x_2x_1a_1$. Then, because $B=\{x_1,x_2\}$, we would get $x_2x_1a_2$. So by 3.7, either $x_2a_1a_2$ or $x_2a_2a_1$, a contradiction, as $a_1 \neq a_2$ and $x_2 \in B$. Assuming $\overline{a_1x_2} \subset \overline{a_1x_1}$ we get a similar contradiction. So L_1 is a simple arc joining x_1 to x_2. In the same way we can show that L_2 is a simple arc joining x_1 to x_2. Finally, by 3.3, we have $L_1 \cap L_2 \subset B_1 \cap B_2=B$, hence $L_1 \cap L_2=B$.

4.2. *Both arcs L_1 and L_2 are metric segments.*

Proof. Consider a continuous real valued function F defined in X by $F(z)=\varrho(z,x_1)-\varrho(z,x_2)$. Then $F(x_1)<0$ and $F(x_2)>0$, so there exist two points m_1 and m_2 such that

$$(1) \qquad m_i \in L_i \quad \text{and} \quad F(m_i)=0, \quad i=1,2.$$

Evidently, $m_1 \neq m_2$ and $m_1, m_2 \in B(x_1, x_2)$. Therefore, by DBP, the points m_1 and m_2 are uniquely determined by (1). We know that $L_1 = \overline{a_1x_1} \cup \overline{a_1x_2}$ is a simple arc, $L_1 \subset B_1$, $m_1 \in L_1$, and $\varrho(m_1, x_1) = \varrho(m_1, x_2)$. Let p_1 and p_2 be two points such that $p_1 \in \overline{a_1x_1}$, $p_2 \in \overline{a_1x_2}$, $F(p_1) < 0$, and $F(p_2) > 0$. Let $\overline{p_1p_2}$ be a fixed segment from p_1 to p_2 and let s be a point on $\overline{p_1p_2}$ such that $F(s) = 0$. We shall show that $s = m_1$. Suppose not, that is, suppose $s = m_2$. Let $\overline{p_1p_2} = \overline{p_1m_2} \cup \overline{m_2p_2}$. Applying 3.6 to $\overline{p_1m_2}$ and to $\overline{p_2m_2}$ we get $x_1 \in \overline{p_1m_2}$ and $x_2 \in \overline{p_2m_2}$, therefore

$$\varrho(p_1, m_2) = \varrho(p_1, x_1) + \varrho(x_1, m_2) \quad \text{and} \quad \varrho(p_2, m_2) = \varrho(p_2, x_2) + \varrho(x_2, m_2).$$

So there exist four segments such that $\overline{p_1m_2} = \overline{p_1x_1} \cup \overline{x_1m_2}$ and $\overline{p_2m_2} = \overline{p_2x_2} \cup \overline{x_2m_2}$. Since

$$\begin{aligned}\varrho(p_1, x_1) + \varrho(x_1, x_2) + \varrho(x_2, p_2) \geqslant \varrho(p_1, p_2) &= \varrho(p_1, m_2) + \varrho(m_2, p_2)\\ &= \varrho(p_1, x_1) + \varrho(x_1, m_2) + \varrho(m_2, x_2) + \varrho(x_2, p_2)\\ &\geqslant \varrho(p_1, x_1) + \varrho(x_1, x_2) + \varrho(x_2, p_2),\end{aligned}$$

we have $x_1m_2x_2$. Thus, by 2.1, the union $\overline{x_1m_2} \cup \overline{m_2x_2}$ is a segment from x_1 to x_2.

We now claim that $x_1a_2x_2$. If $a_2 = m_2$ we are done, as $x_1m_2x_2$. Therefore assume $a_2 \neq m_2$. We will show that $a_2 \in \overline{x_1m_2} \cup \overline{m_2x_2}$. Since $a_2 \neq m_2$ either $\varrho(a_2, x_2) > \varrho(a_2, x_1)$ or $\varrho(a_2, x_2) < \varrho(a_2, x_1)$. Assume the former. We claim that $\varrho(a_2, x_2) < \varrho(x_1, x_2)$, otherwise there exists a point $x_1' \in \overline{a_2x_2}$ such that $\varrho(x_1, x_2) = \varrho(x_1', x_2)$. But $m_2 \in \overline{x_1m_2} \cup \overline{m_2x_2}$ and $m_2 \in \overline{a_2x_2}$ which implies $x_2m_2x_1$, $x_2m_2x_1'$, and $x_2 \neq m_2$. Now since the space is WR we have $x_2x_1x_1'$ or $x_2x_1'x_1$ either of which implies that $\varrho(x_1, x_1') = 0$ or $x_1 = x_1'$. Therefore $a_2 \neq x_1'$, otherwise $x_1 = a_2$ contrary to assumption. Now $x_1 \in \overline{a_2x_2}$ implies $x_2x_1a_2$ but $\varrho(x_1, a_2) = \varrho(x_1, a_1)$ so $x_2x_1a_1$ and $x_2 \neq x_1$. Now, since the space is WR, we get that $x_2a_1a_2$ or $x_2a_2a_1$ either of which implies that $\varrho(a_2, a_1) = 0$ or $a_1 = a_2$, contrary to assumption. We have established that $\varrho(a_2, x_2) < \varrho(x_1, x_2)$ which gives us the existence of an $a \in \overline{x_1m_2} \cup \overline{m_2x_2}$ such that $\varrho(a, x_2) = \varrho(a_2, x_2)$. Now $x_2m_2a_2$, x_2m_2a, and $x_2 \neq m_2$ by the WR property implies that x_2aa_2 or x_2a_2a which implies that $\varrho(a, a_2) = 0$ or $a = a_2$. But now $a_2 \in \overline{x_1m_2} \cup \overline{m_2x_2}$ which is a metric segment between x_1 and x_2. Therefore $x_1a_2x_2$.

Thus, by 2.1, L_2 is a segment between x_1 and x_2. We have $\varrho(x_1, x_2) = \varrho(x_1, a_2) + \varrho(a_2, x_2)$. Now we get a contradiction.

$$\begin{aligned}\varrho(p_1, p_2) &= \varrho(p_1, m_2) + \varrho(m_2, p_2)\\ &= \varrho(p_1, x_1) + \varrho(x_1, m_2) + \varrho(m_2, x_2) + \varrho(x_2, p_2)\\ &= \varrho(p_1, x_1) + \varrho(x_1, x_2) + \varrho(x_2, p_2)\\ &= \varrho(p_1, x_1) + \varrho(x_1, a_2) + \varrho(a_2, x_2) + \varrho(x_2, p_2)\\ &= \varrho(p_1, x_1) + \varrho(x_1, a_1) + \varrho(a_1, x_2) + \varrho(x_2, p_2)\\ &> \varrho(p_1, a_1) + \varrho(a_1, p_2) \geqslant \varrho(p_1, p_2).\end{aligned}$$

The strict inequality follows since $\varrho(p_1, x_1) + \varrho(x_1, a_1) > \varrho(a_1, p_1)$. If this were not the case then

$$\varrho(p_1, x_1) + \varrho(x_1, a_1) = \varrho(a_1, p_1) \quad \text{and} \quad \varrho(a_1, p_1) + \varrho(p_1, x_1) = \varrho(a_1, x_1)$$

which implies $\varrho(p_1, x_1) = 0$ contrary to assumption. We have proven that for any p_1 and p_2 on L_1 sufficiently near respectively to x_1 and to x_2, and for any fixed segment $\overline{p_1p_2}$, $m_1 \in \overline{p_1p_2}$. Now, take p_1 and p_2 converging respectively to x_1 and to x_2, we have

$$\begin{aligned}\varrho(x_1, x_2) &\leqslant \varrho(x_1, m_1) + \varrho(m_1, x_2) = 2\varrho(m_1, x_1)\\ &\leqslant \varrho(m_1, p_1) + \varrho(p_1, x_1) + \varrho(m_1, p_2) + \varrho(p_2, x_2)\\ &= \varrho(p_1, x_1) + \varrho(p_1, p_2) + \varrho(p_2, x_2)\\ &\leqslant \varrho(x_1, x_2) + 2\big(\varrho(p_1, x_1) + \varrho(p_2, x_2)\big).\end{aligned}$$

Hence m_1 is a center of the pair x_1 and x_2. Now since $x_1m_1x_2$ we have a situation symmetric with one which we met earlier in the proof. Using that technique it is easy to show that $x_1a_1x_2$. Then applying 2.1 we have that L_1 is a segment joining x_1 to x_2.

4.3. $X = L_1 \cup L_2$.

Proof. Let x be an arbitrary point of X. If $x = m_1$ or $x = m_2$ then $x \in L_1 \cup L_2$. Thus assume without loss of generality that $\varrho(x, x_1) < \varrho(x, x_2)$. Now either $\varrho(x, x_2) > \varrho(x_1, x_2)$ or $\varrho(x, x_2) < \varrho(x_1, x_2)$.

If $\varrho(x, x_2) > \varrho(x_1, x_2)$, choose $\varepsilon > 0$ so that $\varrho(x, x_1) < \varepsilon < \varrho(x, x_2)$. Then there exists $p_1 \in L_1$ and $p_2 \in L_2$ so that $\varrho(x, p_1) = \varrho(x, p_2)$. Now $x_2p_1x_1$ and $x_2p_2x_1$ and we may write $L_1 = \overline{x_2p_1} \cup \overline{p_1x_1}$ and $L_2 = \overline{x_2p_2} \cup \overline{p_2x_1}$. Notice that the arcs $\overline{x_2p_1} \cup \overline{x_2p_2}$ and $\overline{p_1x_1} \cup \overline{p_2x_1}$ are disjoint except for p_1 and p_2 and therefore must contain distinct points from $B(p_1, p_2)$. Since by DBP, there exist exactly two such points, x is one of them and must, lie in either L_1 or L_2.

Suppose $\varrho(x, x_2) < \varrho(x_1, x_2)$. Since $\varrho(x, x_1) < \varrho(x, x_2)$ either $m_2 \in \overline{x_2x}$ or $m_1 \in \overline{x_2x}$. Assume the former. Now there exists a point $x' \in L_2$ so that $\varrho(x, x_2) = \varrho(x', x_2)$. But then x_2m_2x' and x_2m_2x. Now the WR property yields that x_2xx' or $x_2x'x$. Either case gives $\varrho(x', x) = 0$ or $x = x'$. That is, either case gives $x \in L_2$.

5. In a segment-convex metric space (X, ϱ) with DBP, if a_1 and a_2 are two distinct points, $B(a_1, a_2) = \{x_1, x_2\}$ and if $B(x_1, x_2) = \{m_1, m_2\}$, then m_1 and m_2 are the centers of a pair x_1 and x_2, and X is the union of two metric segments

$$L_1(x_1, x_2) = \overline{x_1m_1} \cup \overline{m_1x_2} \quad \text{and} \quad L_2(x_1, x_2) = \overline{x_1m_2} \cup \overline{m_2x_2}$$

intersecting only at their end points.

THE MAIN THEOREM. *If (X, ϱ) is a segment-convex metric space with DBP, then X is isometric to a metric one-sphere S^1 with the ordinary geodesic metric.*

Proof. Assume the notation of 5 and let C denote a circle with the radius $r = \varrho(x_1, x_2)/\pi$; $C = \{z \mid z = re^{it},\ t \in R\}$.

Put $C(0, \pi) = \{z \in C \mid 0 \leqslant t \leqslant \pi\}$ and $C(\pi, 2\pi) = \{z \in C \mid \pi \leqslant t \leqslant 2\pi\}$. Let a_1: $C(0, \pi) \to X$ be the isometry sending respectively re^{i0}, $re^{(i\pi/2)}$, and $re^{i\pi}$ onto x_1, m_1, and x_2; let a_2: $C(\pi, 2\pi) \to X$ be the isometry sending respectively $re^{i\pi}$, $re^{i(3\pi/2)}$, and $re^{i2\pi}$ onto x_2, m_2, and x_1; let a: $C \to X$ be defined by

$$a(z) = \begin{cases} a_1(z) & \text{if} \quad z \in C(0, \pi), \\ a_2(z) & \text{if} \quad z \in C(\pi, 2\pi). \end{cases}$$

It is clear that a is a bijection and that the partial functions $a/C(0, \pi) = a_1$ and $a/C(\pi, 2\pi) = a_2$ are isometries.

Consider now the pair of distinct points x_1 and x_2. We have $B(x_1, x_2) = \{m_1, m_2\}$ and $B(m_1, m_2) = \{x_1, x_2\}$. Applying 5, we find that x_1 and x_2 are centers of the pair m_1 and m_2 and that $L_1(m_1, m_2) = \overline{m_1x_1} \cup \overline{x_1m_2}$ and $L_2(m_1, m_2) = \overline{m_1x_2} \cup \overline{x_2m_1}$ are two metric segments, hence $\varrho(m_1, m_2) = \varrho(x_1, x_2) = \pi r$.

Let

$$C(3\pi/2, \pi/2) = \{z \in C \mid 3\pi/2 \leqslant t < 2\pi \ \text{ or } \ 0 \leqslant t \leqslant \pi/2\}$$

and

$$C(\pi/2, 3\pi/2) = \{z \in C \mid \pi/2 \leqslant t \leqslant 3\pi/2\},$$

and let β_1: $C(3\pi/2, \pi/2) \to X$ be the isometry sending respectively $re^{i(3\pi/2)}$, re^{i0}, and $re^{i\pi/2}$ onto m_2, x_1, and m_1, let β_2: $C(\pi/2, 3\pi/2) \to X$ be the isometry sending respectively $re^{i\pi/2}$, $re^{i\pi}$, and $re^{i(3\pi/2)}$ onto m_1, x_2, and m_2; and let β: $C \to X$ be defined by

$$\beta(z) = \begin{cases} \beta_1(z) & \text{if} \quad z \in C(3\pi/2, \pi/2), \\ \beta_2(z) & \text{if} \quad z \in C(\pi/2, 3\pi/2). \end{cases}$$

By an argument analogous to that for a, we can show that β is a bijection and that $\beta/C(3\pi/2, \pi/2) = \beta_1$ and $\beta/C(\pi/2, 3\pi/2) = \beta_2$ are isometries. Moreover, $a = \beta$, and thus $\Gamma = a = \beta$ is a bijection of C onto X whose four partial functions $\Gamma/C(0, \pi)$, $\Gamma/C(\pi, 2\pi)$, $\Gamma/C(\pi/2, 3\pi/2)$, and $\Gamma/C(3\pi/2, \pi/2)$ are isometries.

Now let $z = re^{it}$ be an arbitrary point of C, let $\mu = re^{i(t+\pi/2)}$, and let $z = \Gamma(z)$ and $n = \Gamma(\mu)$. Since Γ is an isometry on any of the four half circles and since z and μ are included in at least one of them, we have $\varrho(z, n) = (\pi/2)r$. So, if $z_1 = re^{it}$, $\mu_1 = re^{i(t+\pi/2)}$, $z_2 = re^{i(t+\pi)}$, and $\mu_2 = re^{i(t+3\pi/2)}$, and if z_1, n_1, z_2, and n_2 are their corresponding images, $\varrho(z_1, n_1) = \varrho(n_1, z_2) = \varrho(z_2, n_2) = \varrho(n_2, z_1) = (\pi/2)r$. Consequently, we have $B(n_1, n_2) = \{z_1, z_2\}$ and $B(z_1, z_2) = \{n_1, n_2\}$. Applying 5, we get $L_1(z_1, z_2) = \overline{z_1n_1} \cup \overline{n_1z_2}$ and $L_2(z_1, z_2) = \overline{z_1n_2} \cup \overline{n_2z_2}$ are two metric segments of length πr intersecting only at their endpoints and whose union is X.

Let

$$C(t, t+\pi) = \{z \in C \mid z = re^{i\tau},\ t \leqslant \tau \leqslant t+\pi\}$$

and

$$C(t+\pi, t+2\pi) = \{z \in C \mid z = re^{i\tau},\ t+\pi \leqslant \tau \leqslant t+2\pi\}.$$

Let γ_1: $C(t, t+\pi) \to X$ be the isometry sending respectively z_1, μ_1, and z_2 onto z_1, n_1, and z_2, and let γ_2: $C(t+\pi, t+2\pi) \to X$ be the isometry sending respectively z_2, μ_2, and z_1 onto z_2, n_2, z_1, and let γ: $C \to X$ be defined by

$$\gamma(z) = \begin{cases} \gamma_1(z) & \text{if} \quad z \in C(t, t+\pi), \\ \gamma_2(z) & \text{if} \quad z \in C(t+\pi, t+2\pi). \end{cases}$$

It is clear that $\gamma = \Gamma$, so $\Gamma/C(t, t+\pi)$ and $\Gamma/C(t+\pi, t+2\pi)$ are two isometries. This completes the proof, since for any pair of points $z, z' \in C$, $z = re^{it}$, both points z and z' belong to one of the two half-circles $C(t, t+\pi)$, $C(t+\pi, t+2\pi)$, where Γ is an isometry.

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