Fixed points of certain symmetric product mappings of a metric manifold

by

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Abstract. Robert F. Brown proved a generalization of the Brouwer’s fixed point theorem by making use of Bing’s retraction theorem. J. W. Jaworowski extended Brown’s result in a similar sense as the Lefschetz fixed point theorem extends the Brouwer fixed point theorem. In this paper we generalize J. W. Jaworowski’s result to that of the compact symmetric product mappings of a metric manifold.

1. Introduction. Fixed point theorems of symmetric product mappings of a finite polyhedron were first studied by C. N. Maxwell [12]. The results obtained by Maxwell were generalized by S. Maiah [10] to that of polyhedra (not necessarily finite) and metric ANR’s.

This paper deals with symmetric product mappings of a manifold.

Robert F. Brown [3] proved a generalization of the Brouwer fixed point theorem by making use of Bing’s retraction theorem [1]. A special case of Bing’s retraction theorem was used by Henderson and Livezey [7] to prove a theorem which is now a special case of Brown’s Theorem.

J. W. Jaworowski [9] extended Brown’s result in a similar sense as the Lefschetz fixed point theorem extends the Brouwer fixed point theorem.

In this paper, we generalize J. W. Jaworowski’s [9] result to that of the compact symmetric product mappings of a metric manifold. Brown’s result [3] extended to compact symmetric product mappings of a metric manifold becomes a special case of the result which we will be proving.

2. Preliminaries. Let $X$ be a topological space and $X^n$, the $n$th cartesian product in the usual topology. Let $\mathcal{G}$ be any group of permutations of the letters $[1, \ldots, n]$. Then $\mathcal{G}$ can be considered as a group of homeomorphisms on $X^n$ by defining, for $g \in \mathcal{G}$ and $(x_1, \ldots, x_n) \in X^n$, $g(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. The orbit space under this action with the identification topology will be denoted by $X^n/\mathcal{G}$ and called a $\mathcal{G}$-product of $X$.

Let $\eta: X^n \to X^n/\mathcal{G}$ be the identification map. Then the map $\eta$ is both open and closed for $\eta^{-1}(A) = \bigcup_{g \in \mathcal{G}} gA$ for any open (closed) $A \subseteq X^n$, is open (closed respectively) $(g \in \mathcal{G}$ being a homeomorphism).
3.2. Lemma (see Masi [10]). Let \( X \) be any space. A map \( f: X \to X^\# \) is a compact map if and only if there exists a compact subset \( K \) of \( X \) such that \( f(K) \subseteq X^\# \).

For the definitions of a Lefschetz map and a \( A \)-space, see Jaworski and Powers [8], 2.5.

Let us recall some definitions:

Let \( \Gamma, \Gamma_0 : \mathcal{J} \to \mathcal{J} \) be two covariant functors from \( \mathcal{J} \) to \( \mathcal{J} \), the category of topological spaces defined as follows:

For an object \( X \) of \( \mathcal{J} \), let

\[
\Gamma(X) = X^\# \quad \text{and} \quad \Gamma_0(X) = X^\# / G
\]

and for map \( f: X \to Y \), let \( \Gamma(f) = f^\# \) and \( \Gamma_0(f) = \tilde{f} \). Then \( \Gamma \) and \( \Gamma_0 \) are homotopy preserving functors (see [12]).

Consider the functors \( H_\ast, H_\ast \Gamma, H_\ast \Gamma_0: P \to \mathcal{V} \) from the category of compact polyhedra to vector spaces. Let \( \eta_\ast: H_\ast \Gamma \to H_\ast \Gamma_0 \) and \( \eta_\ast: H_\ast \Gamma \to H_\ast \) be defined as follows.

For an object \( X \) of \( \mathcal{J} \), let

\[
\eta^X_\ast = H_\ast(\eta^X_\ast) : H_\ast(X^\#) \to H_\ast(X^\# / G)
\]

and

\[
\eta^X_\ast = H_\ast(X^\# / G),
\]

where \( \pi^X_\ast \) denotes the projection of \( X^\# \) onto its \( \ast \)th factor.

If \( X \) is a compact polyhedron then Maxwell [11] defined a natural homomorphism \( \mu_\ast: H_\ast(X^\# / G) \to H_\ast(X) \) such that \( \mu^{\ast}_\ast = \pi \). S. Masi [10] extended this homomorphism called Maxwell homomorphism to the category of \( A \)-maps with morphisms as compact maps by means of the concept of admissible category.

We recall the definition of an admissible category denoted by \( \rho^X_\ast \).

3.5. Definition. A full subcategory \( \mathcal{K} \) of the category \( \mathcal{J} \), the category of all topological spaces is said to be admissible if

1. \( \mathcal{K} \) is a subcategory of \( \mathcal{J} \),
2. the natural transformation \( \mu_\ast: H_\ast \Gamma_0 \to H_\ast \) on \( \mathcal{K} \) has an extension \( \mu: H_\ast \Gamma_0 \to H_\ast \) to the category \( \mathcal{J} \) such that \( \mu^{\ast}_\ast = \pi \) holds on \( \mathcal{K} \), where \( \eta_\ast \) and \( \pi_\ast \) are natural transformations induced by the identification and the projections respectively.

3.4. Definition. Let \( X \in \mathcal{K} \) and \( f: X \to X^\# / G \) be a continuous map. Then \( f \) is said to be a \( \mu \) map if \( \eta^{\ast}_\ast \) is a Lefschetz endomorphism (that is, of finite type roughly). Then the Lefschetz number, \( \Lambda(f) \) is defined by

\[
\Lambda(f) = \lim_{n \to \infty} (-1)^n \text{tr}((\eta^{\ast}_\ast)^n).
\]
3.5. Definition. Let \( X \in \mathcal{P}_G \) and \( f : X \times X^m / G \) be a continuous map. Then \( f \) is said to be a \( \mu \)-Lefschetz map if \( f \) is a \( \mu \)-map and \( \Delta(f) \neq 0 \) implies \( f \) has a fixed point.

3.6. Definition. Let \( X \in \mathcal{P}_G \). Then \( X \) is said to be a \( \mu \)-space if every compact map \( f : X \times X^m / G \) is a \( \mu \)-map.

3.7. Definition. Let \( X \) be a \( \mu \)-space. Then \( X \) is said to be a \( \mu \)-space if every compact, \( \mu \)-map \( f : X \times X^m / G \) is a \( \mu \)-Lefschetz map.

For simplicity, when writing the induced homomorphisms the dimension subscript will be omitted.

3.8. Theorem. Let \( f : X \times X^m / G \) be a map, \( X \in \mathcal{P}_G \) and suppose there exist a \( \mu \)-space \( Y \) and maps \( h : Y \to X \) and \( g : X \to X^m / G \) such that \( f = h \circ g \) where \( h : Y^m / G \times Y^m / G \) is the induced map and either (a) \( g \) is compact; or (b) \( X \) is Hausdorff and \( h \) is compact. Then

(i) \( f \) is a \( \mu \)-map.

(ii) \( \Delta(f) = \Delta(g \circ h) \) where \( h : Y \to X^m / G \) and \( g : X \to (X^m / G \) such that \( f = h \circ g \).

(iii) \( f \) is a \( \mu \)-Lefschetz map.

Proof. Consider the following diagram

\[
\begin{array}{ccc}
Y^m / G & \xrightarrow{\mu} & X^m / G \\
\downarrow{f} & & \downarrow{g} \\
X & \xrightarrow{h} & X^m / G
\end{array}
\]

Consider \( g \circ h : Y \to X \) and maps \( h : Y \to X \) and \( g : X \to X^m / G \) such that \( f = h \circ g \). Assume \( g \circ h \neq 0 \) implies \( g \circ h \) has a fixed point. Let \( y_0 \in Y \) be a fixed point of \( g \circ h \), that is, \( y_0 \in g \circ h(y_0) \).

Consider the following diagram

\[
\begin{array}{ccc}
H(Y) & \xrightarrow{\mu} & H(X^m / G) \\
\downarrow{h} & & \downarrow{g} \\
H(X) & \xrightarrow{\mu} & H(X^m / G)
\end{array}
\]

It commutes because of definitions and naturality of \( \mu \). Since \( Y \) is a \( \mu \)-space, \( \text{tr}(\mu^T g \cdot h) \) is defined. Therefore,

\[
\text{tr}(\mu^T g \cdot h) = \text{tr}((h \cdot \mu^T g \cdot h) = \text{tr}(\mu^T f).
\]

Hence, \( \text{tr}(\mu^T f) \) is defined and therefore \( f \) is a \( \mu \)-map and \( \Delta(f) = \Delta(g \circ h) \). Suppose \( \Delta(f) \neq 0 \), this implies \( \Delta(g \circ h) \neq 0 \) and hence there exist a \( y_0 \), a fixed point of \( g \circ h \).

We claim that the point \( h(y_0) \) is a fixed point of \( f \).

Consider \( f(h(y_0)) = f \circ h(y_0) \). Since \( y_0 \in g \circ h(y_0) \), \( h(y_0) \in f \circ h(y_0) \). Therefore \( h(y_0) \) is a fixed point of \( f \).

Many results derived by S. Maehara from factorization Theorem 3.10 in [10] can also be proved by using our factorization Theorem 3.8. So we do not state or prove those results.

3.9. Corollary. Let \( f : X \times X^m / G \) be a compact map with \( X \) belonging to \( \mathcal{P}_G \) and suppose \( f \) can be factored through a \( \mu \)-space or a symmetric \( G \)-product of a \( \mu \)-space, then \( f \) is a \( \mu \)-Lefschetz map.

Proof. Follows from Theorem 3.8 and Theorem 3.3.10 of Maehara [10].

4. Symmetric product maps of subsets of a metric manifold. In this section we will prove a fixed point theorem for a certain symmetric product of a metric manifold similar to that of J. W. Jaworowski [9].

4.1 Theorem. A metrizable manifold (with or without boundary) is a \( \mu \)-space.

Proof. If \( M \) is metrizable then it is a local ANR, and hence an ANR by [5]. Consequently, it is a \( \mu \)-space by the result of 3.3.7 [10].

4.2. Theorem. Let \( M \) be a metric \( m \)-manifold (with or without boundary); let \( X \) be a \( (m+2) \)-connected ANR imbedded as a closed subset of \( M \) and let \( U \) be a component of \( M - X \) whose closure is not compact. Let \( f : (M - U, X) \to (M^m / G, (M - U)^m / G) \) be a compact map, and let \( f^* : X \to (M - U)^m / G \) denote the map defined by the restriction of \( f \). Then there exist \( \mu \)-Lefschetz maps \( \nu : M \to M^m / G \) and \( \omega : M - U \to (M - U)^m / G \) such that \( \omega \) is an extension of \( f^* \) and \( \nu \) is an extension of \( f \). Let \( \lambda \) be the fixed point set of \( f \). Then \( \lambda \) is a \( \mu \)-space and \( \nu \) is a fixed point.

Proof. By Bing's Retraction Theorem [1], there exist a retraction \( X \to U \) which extends to a retraction \( r : M \to M - U \). This induces a retraction \( f^* : M^m / G \to (M - U)^m / G \) defined by, for every \( \{x_1, \ldots, x_m\} \subseteq M^m / G \);

\[
\overline{r}(x_1, \ldots, x_m) = (r(x_1), \ldots, r(x_m)).
\]

Since \( M \) is a manifold, by Proposition 4.1, \( M \) is a \( \mu \)-space. Since \( M \) is metric, it is a local ANR and hence an ANR by [9] or see [5]. Consequently, since \( M - U \) is a retract of \( M \), \( M - U \) is a metric ANR and hence a \( \mu \)-space.

Consider the diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{r} & M^m / G \\
\downarrow{f} & & \downarrow{g} \\
M - U & \xrightarrow{r^*} & (M - U)^m / G
\end{array}
\]

Consider the diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{r} & M^m / G \\
\downarrow{f} & & \downarrow{g} \\
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M - U & \xrightarrow{r^*} & (M - U)^m / G
\end{array}
\]
Weights of denumerable topological spaces

by

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Abstract. Let \((X, \mathcal{U})\) be a denumerable topological space— for \([\mathcal{U}] \subset \mathcal{U}_n\) we show that the weight of \((X, \mathcal{U})\) equals \([\mathcal{U}]\); and \([\mathcal{U}] \supset \mathcal{U}_n\) implies the weight of \((X, \mathcal{U})\) is greater than or equal to \(\mathcal{U}_n\), unless \(\mathcal{U}\) has the power of the continuum.

In this paper we will examine the possible weights of topological spaces \((X, \mathcal{U})\) where \(X\) is denumerable, answering a question of P. Erdős enroute. Throughout we will use lower-case German letters as well as alephs to denote cardinal numbers and lower-case Greek letters to denote ordinals, the letter \(\omega\) being reserved for the first infinite ordinal. The transfinite sequence \(\mathcal{U}_0, \mathcal{U}_1, \ldots\) denotes the cardinals indexed by ordinals and ordered by size. For an ordinal \(\alpha\), \(\alpha\) denotes the least ordinal such that \(\mathcal{U}_\alpha\) has \(\alpha\) predecessors (or members). The cardinal \(2^\omega\) is also denoted by \(\mathcal{U}\). \([\mathcal{U}]\) denotes the cardinal of the set \(\mathcal{U}\).

If \((X, \mathcal{T})\) is a topological space we will let \(w(X, \mathcal{T})\) be the weight of \((X, \mathcal{T})\), that is, the least cardinality of a base of \((X, \mathcal{T})\). (The reader is referred to Comfort's excellent survey article [2] for \(\omega < \mathcal{U}\) define

\[\mathcal{U}_n = w(X, \mathcal{T}) : |X| = \mathcal{U}_n, [\mathcal{U}] = n\].

Clearly \(\mathcal{U}_n\) implies \(I \subset \mathcal{U}\). First we prove \(\mathcal{U}_n\) is convex for \(n\) infinite.

**Theorem 1.** If \(n\) is infinite, \(I \subset \mathcal{U}_n\) and \(I \subset \mathcal{U}_n\), then \(m \subset \mathcal{U}_n\).

**Proof.** From the hypothesis of Theorem 1 we see that \(I \subset \mathcal{U}_n\) implies \(I\) is infinite. Let \((X, \mathcal{T}_n)\) be a denumerable topological space such that \([\mathcal{T}_n] = n\). Let \(X_1\) be a denumerable set disjoint from \(X_2\). Sierpiński [3] shows that it is possible to find a family \(\mathcal{I}\) of subsets of \(X_1\) such that \((I) A \in \mathcal{I}\) implies \(|A \cap B| \leq \mathcal{U}_n\), and \((ii) [\mathcal{I}] = m\). Let \(S = \{A \in \mathcal{I} : X_1 - A \in \mathcal{I}\}\), and let \(\mathcal{T}_n\) denote the topology on \(X_1\) generated by \(S\). It is not difficult to show that \(\mathcal{I}\) is a finite intersection of members of \(S\) intersected with a cofinite subset of \(X_1\). Hence \([\mathcal{T}_n] = \mathcal{U}_m\). From this we also see that \(w(X, \mathcal{T}_n) = \mathcal{U}_m\). Now let \(X = X_1 \cup X_2\) and let the topology \(\mathcal{T}\) on \(X\) be \(\{A_1 \cup A_2 : A_1 \subset \mathcal{T}_1, A_2 \subset \mathcal{T}_2\}\).