Fixed points of certain symmetric product mappings of a metric manifold

by

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Abstract. Robert F. Brown proved a generalization of the Brouwer's fixed point theorem by making use of Bing's retraction theorem. J. W. Jaworowski extended Brown's result in a similar sense as the Lefschetz fixed point theorem extends the Brouwer fixed point theorem. In this paper we generalize J. W. Jaworowski's result to that of the compact symmetric product mappings of a metric manifold.

1. Introduction. Fixed point theorems of symmetric product mappings of a finite polyhedron were first studied by C. N. Maxwell [12]. The results obtained by Maxwell were generalized by S. Masih [10] to that of polyhedra (not necessarily finite) and metric ANR's.

This paper deals with symmetric product mappings of a manifold.

Robert F. Brown [3] proved a generalization of the Brouwer fixed point theorem by making use of Bing's retraction theorem [1]. A special case of Bing's retraction theorem was used by Henderson and Livesay [7] to prove a theorem which is now a special case of Brown's Theorem. J. W. Jaworowski [9] extended Brown's result in a similar sense as the Lefschetz fixed point theorem extends the Brouwer fixed point theorem.

In this paper, we generalize J. W. Jaworowski's [9] result to that of the compact symmetric product mappings of a metric manifold. Brown's result [3] extended to compact symmetric product mappings of a metric manifold becomes a special case of the result which we will be proving.

2. Preliminaries. Let X be a topological space and X^n, the nth cartesian product in the usual topology. Let G be any group of permutations of the letters $[1, \ldots, n]$. Then G can be considered as a group of homeomorphisms on X^n by defining, for $g \in G$ and $(x_1, \ldots, x_n) \in X^n$, $g(x_1, \ldots, x_n) = (x_{g(1)}, \ldots, x_{g(n)})$. The orbit space under this action with the identification topology will be denoted by X^n/G and called a *G-product of X*. Let $\eta\colon X^n \to X^n/G$ be the identification map. Then the map η is both open and closed for $\eta^{-1}\eta(A) = \bigcup_{g \in G} gA$ for any open (closed) $A \subseteq X^n$, is open (closed respectively) ($g \in G$ being a homeomorphism).

Let $f\colon X \to Y$ be a map. The map f defines a map $f^n\colon X^n \to Y^n$, where

$$f^n(x_1, x_2, \ldots, x_n) = \big(f(x_1), f(x_2), \ldots, f(x_n)\big) \quad \text{for} \quad (x_1, x_2, \ldots, x_n) \in X^n.$$

The map f^n commutes with the action of G on X^n and Y^n and hence there exist a map $\bar{f}\colon X^n/G \to Y^n/G$ such that the following diagram commutes

$$\begin{tikzcd} X^n \arrow[r, "f^n"] \arrow[d, "\eta^X"'] & Y^n \arrow[d, "\eta^Y"] \\ X^n/G \arrow[r, "\bar{f}"'] & Y^n/g \end{tikzcd}$$

A map $f\colon X \to X^n/G$ is called a *symmetric product map of the space X*. A point $x \in X$ is said to be a *fixed point of f* if x is a coordinate of $f(x)$, that is, if $(x_1, x_2, \ldots, x_n) \in X^n$ such that $\eta(x_1, x_2, \ldots, x_n) = f(x)$, then $x = x_i$ for some i, $1 \leqslant i \leqslant n$.

Let X be a metric space with metric d. Let d' be the metric on X^n defined by

$$d'(z, z') = \Big(\sum_{i=1}^{n} \{d(z_{i'}, z'_i)\}^2\Big)$$

where $z = (z_1, z_2, \ldots, z_n)$ and $z' = (z'_1, z'_2, \ldots, z'_n) \in X^n$.

A metric $\bar{d}$ is defined on X^n/G as follows [11]

$$\bar{d}\big(\eta(z), \eta(z')\big) = \operatorname{Inf}\{d(z, gz')|\ g \in G\} \quad \text{where} \quad z, z' \in X^n.$$

For the definition and some properties of the trace of a vector space homomorphism, see J. W. Jaworowski and Michael Powers [8].

3. Admissible maps, admissible spaces and μ-Λ-spaces. In the following sections, the nature of a homology theory under consideration is important only to the extent that the homology groups be vector spaces, that they agree with the usual homology groups with rational coefficients for compact polyhedra and compact ANR's, and they constitute a functor H_* satisfying the homotopy axiom and the dimension axiom for the category $\mathfrak{J}$ of topological spaces and continuous maps. Thus H_* may be the singular homology, the Cech homology or any other functor satisfying the above requirements. The homomorphism induced by a map $f\colon X \to Y$ will be denoted, as usual, by $f_{*_n}\colon H_n(X) \to H_n(Y)$.

In this section we recall the concepts of admissible category μ_G^n, admissible spaces μ-Λ-spaces, define the concept of admissible maps and quote or prove some factorization theorems.

Let us recall the definition of a compact map.

3.1. Definition. Let X and Y be two spaces. A map $f\colon X \to Y$ is said to be a *compact map* if $f(X)$ is contained in a compact subset of Y.

3.2. Lemma (see Masih [10]). *Let X be any space. A map $f\colon X \to X^n/G$ is a compact map iff there exist a compact subset K of X such that $f(X) \subseteq K^n/G$.*

For the definitions of a Lefschetz map and a Λ-space, see Jaworowski and Powers [8], 2.5.

Let us recall some definitions:

Let $\Gamma, \Gamma_G\colon \mathfrak{J} \to \mathfrak{J}$ be two covariant functors from $\mathfrak{J}$ to $\mathfrak{J}$, the category of topological spaces defined as follows:

For an object X of $\mathfrak{J}$, let

$$\Gamma(X) = X^n \quad \text{and} \quad \Gamma_G(X) = X^n/G$$

and for map $f\colon X \to Y$, let $\Gamma(f) = f^n$ and $\Gamma_G(f) = \bar{f}$. Then Γ and Γ_G are homotopy preserving functors (see [12]).

Consider the functors $H_*, H_*\Gamma, H_*\Gamma_G\colon P \to V$ from the category of compact polyhedron to vector spaces. Let $\eta_*\colon H_*\Gamma \Rightarrow H_*\Gamma_G$ and $\pi_*\colon H_*\Gamma \Rightarrow H_*$ be defined as follows.

For an object X of $\mathfrak{J}$, let

$$\eta_*^X = H_*(\eta^X)\colon H_*(X^n) \to H_*(X^n/G)$$

and

$$\pi_*(X) = \sum_{i=1}^{n} \pi_{i*}^X\colon H_*(X^n) \to H_*(X)$$

where π_i^X denotes the projection of X^n onto its ith factor.

If X is a compact polyhedron then Maxwell [11] defined a natural homomorphism $\mu_*^X\colon H_*(X^n/G) \to H_*(X)$ such that $\mu_*^X \eta_* = \pi_*$. S. Masih [10] extended this homomorphism called Maxwell homomorphism to the category of ANR's with morphisms as compact maps by means of the concept of admissible category.

We recall the definition of an admissible category denoted by μ_G^n.

3.5. Definition. A full subcategory $\mathcal{K}$ of the category $\mathfrak{J}$, the category of all topological spaces is said to be *admissible* if

(1) $\mathcal{CP}$ is subcategory of $\mathcal{K}$,

(2) the natural transformation $\mu_*\colon H_*\Gamma_G \Rightarrow H_*$ on $\mathcal{CP}$ has an extension $\mu\colon H_*\Gamma_G \Rightarrow H_*$ to the category $\mathcal{K}$ such that $\mu\eta_* = \pi_*$ holds on $\mathcal{K}$, where η_* and π_* are natural transformations induced by the identification and the projections respectively.

3.4. Definition. Let $X \in \mu_G^n$ and $f\colon X \to X^n/G$ be a continuous map. Then f is said to be a *μ-map* if μf_* is a Lefschetz endomorphism (that is, of finite type roughly). Then the Lefschetz number, $\Lambda(f)$ is defined by

$$\Lambda(f) = L(\mu f_*) = \sum_{i=0}^{\infty} (-1)^n \operatorname{tr}\big((\mu f_*)_n\big).$$

3.5. Definition. Let $X \in \mu_G^n$ and $f\colon X \to X^n/G$ be a continuous map. Then f is said to be a *μ-Lefschetz map* if f is a μ-map and $\Lambda(f) \neq 0$ implies f has a fixed point.

3.6. Definition. Let $X \in \mu_G^n$. Then X is said to be a *μ-space* if every compact map $f\colon X \to X^n/G$ is a μ-map.

3.7. Definition. Let X be a μ-space. Then X is said to be a *μ-Λ-space* if each compact, μ-map $f\colon X \to X^n/G$ is a μ-Lefschetz map.

For simplicity, when writing the induced homomorphisms the dimension subscript will be omitted.

3.8. Theorem. *Let $f\colon X \to X^n/G$ be a map, $X \in \mu_G^n$ and suppose there exist a μ-Λ-space Y and maps $h\colon Y \to X$ and $g\colon X \to Y^n/G$ such that $f = \bar{h} \circ g$ where $\bar{h}\colon Y^n/G \to X^n/G$ is the induced map and either* (a) *g is compact; or* (b) *Y is Hausdorff and h is compact. Then*

(i) *f is a μ-map.*

(ii) *$\Lambda(f) = \Lambda(g \circ h)$ where $g \circ h\colon Y \to Y^n/G$ and*

(iii) *f is a μ-Leftschetz map.*

Proof. Consider the following diagram

$$\begin{tikzcd} & Y^n/G \arrow[d, "\bar{h}"] \\ X \arrow[ur, "g"] \arrow[r, "f"'] & X^n/G \end{tikzcd}$$

Consider $g \circ h\colon Y \to Y^n/G$ Y, a μ-Λ-space implies $g \circ h$ is defined and $\Lambda(g \circ h) \neq 0$ implies $g \circ h$ has a fixed point. Let $y_0 \in Y$ be a fixed point of $g \circ h$, that is, $y_0 \in g \circ h(y_0)$.

Consider the following diagram

$$\begin{tikzcd} H_*(Y) \arrow[r, "g_*h_*"] \arrow[d, "h_*"'] & H_*(Y^n/G) \arrow[r, "\mu^Y"] \arrow[d, "\bar{h}_*"] & H_*(Y) \arrow[d, "h_*"] \\ H_*(X) \arrow[ur, "g_*"] \arrow[r, "f_*"'] & H_*(X^n/G) \arrow[r, "\mu^X"'] & H_*(X) \end{tikzcd}$$

It commutes because of definitions and naturality of μ. Since Y is a μ-Λ-space, $\mathrm{tr}(\mu^Y g_* h_*)$ is defined. Therefore,

$$\mathrm{tr}(\mu^Y g_* h_*) = \mathrm{tr}(h_* \mu^Y g_*) = \mathrm{tr}(\mu^X \bar{h}_* g_*) = \mathrm{tr}(\mu^X f_*) .$$

Hence, $\mathrm{tr}(\mu^X f_*)$ is defined and therefore f is a μ-map and $\Lambda(f) = \Lambda(g \circ h)$.

Suppose $\Lambda(f) \neq 0$. This implies $\Lambda(g \circ h) \neq 0$ and hence there exist a y_0, a fixed point of $g \circ h$.

We claim that the point $h(y_0)$ is a fixed point of f.

Consider $fh(y_0) = \bar{h} \circ g \circ h(y_0)$. Since $y_0 \in g \circ h(y_0)$, $h(y_0) \in \bar{h} \circ g \circ h(y_0) = fh(y_0)$. Therefore $h(y_0)$ is a fixed point of f.

Many results derived by S. Masih from factorization Theorem 3.10 in [10] can also be proved by using our factorization Theorem 3.8. So we do not state or prove those results.

3.9. Corollary. *Let $f\colon X \to X^n/G$ be a compact map with X belonging to μ_G^n and suppose f can be factored through a μ-Λ-space or a symmetric G-product of a μ-Λ-space, then f is a μ-Lefschetz map.*

Proof. Follows from Theorem 3.8 and Theorem 3.3.10 of Masih [10].

4. Symmetric product maps of subsets of a metric manifold. In this section we will prove a fixed point theorem for a certain symmetric product of a metric manifold similar to that of J. W. Jaworowski [9].

4.1 Theorem. *A metrizable manifold (with or without boundary) is a μ-Λ-space.*

Proof. If M is metrizable then it is a local ANR, and hence an ANR by [5]. Consequently, it is a μ-Λ-space by Masih's result 3.3.17 [10].

4.2. Theorem. *Let M be a metric m-manifold (with or without boundary); let X be a $(m-2)$-connected* ANR *imbedded as a closed subset of M and let U be a component of $M-X$ whose closure is not compact. Let $f\colon (M-U, X) \to \big(M^n/G, (M-U)^n/G\big)$ be a compact map, and let $f'\colon X \to (M-U)^n/G$ denote the map defined by the restriction of f. Then there exist μ-Lefschetz maps $v\colon M \to M^n/G$ and $u\colon M-U \to (M-U)^n/G$ such that u is an extension of f' and v is an extension of f, $\Lambda(u) = \Lambda(v)$ and $F_v \subseteq F_u \cap F_f$ (where F_f denotes the fixed point set of f). In particular $\Lambda(v) \neq 0$ implies f has a fixed point.*

Proof. By Bing's Retraction Theorem [1], there exist a retraction $X \cup U \to X$ which extends to a retraction $r\colon M \to M-U$ [9]. This induces a retraction $\bar{r}\colon M^n/G \to (M-U)^n/G$ defined by, for every $[x_1, \ldots, x_n] \in M^n/G$

$$\bar{r}[x_1, \ldots, x_n] = [r(x_1), \ldots, r(x_n)] .$$

Since M is a metric manifold, by Proposition 4.1, M is a μ-Λ-space. Since M is metric, it is a local ANR and hence an ANR by [9] or see [5]. Consequently, since $M-U$ is a retract of M, $M-U$ is a metric ANR and hence a μ-Λ-space.

Consider the following diagram

$$\begin{tikzcd} M \arrow[r, "f \circ r"] \arrow[d, "r"'] & M^n/G \arrow[d, "\bar{r}"] \\ M-U \arrow[ur, "f"] \arrow[r, "\bar{r} \circ f"'] & (M-U^n/G \end{tikzcd}$$

Set $v = f \circ r\colon M \to M^n/G$ and $u = \bar{r} \circ f\colon M - U \to (M-U)^n/G$. The sets M, $M-U$ being μ-Λ-spaces, by Corollary 3.9 $\Lambda(u) = \Lambda(v)$ and u, v are μ-Lefschetz maps.

Suppose that $x_0 \in F_v$, that is, $x_0 \in f \circ rx_0$. Then $y_0 = rx_0 \in F_u$ for $uy_0 = \bar{r}fy_0 = \bar{r}frx_0$. But $x_0 \in frx_0$, this implies

$$rx_0 = y_0 \in \bar{r}frx_0 = \bar{r}fy_0 .$$

If we suppose that $x_0 \in U$, $y_0 = rx_0 \in X$ and hence $fy_0 \in (M-U)^n/G$, since $f(X) \subseteq (M-U)^n/G$. Since $x_0 \in frx_0 = fy_0 \in (M-U)^n/G$, $x_0 \in M-U$, a contradiction.

Hence $x_0 \in M-U$ and hence $rx_0 = x_0$. Therefore, $y_0 = rx_0 = x_0 \in F_u$, $x_0 \in frx_0 = fx_0$. Hence $x_0 \in F_f$. Hence $F_v \subseteq F_u \cap F_f$.

In particular, $\Lambda(V) \neq 0$ implies f has a fixed point.

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Weights of denumerable topological spaces

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Abstract. Let $(X, \mathfrak{T})$ be a denumerable topological space — for $|\mathfrak{T}| \leqslant \aleph_\omega$ we show that the weight of $(X, \mathfrak{T})$ equals $|\mathfrak{T}|$; and $|\mathfrak{T}| > \aleph_\omega$ implies the weight of $(X, \mathfrak{T})$ is greater than or equal to $\aleph_\omega$, unless $\mathfrak{T}$ has the power of the continuum.

In this paper we will examine the possible weights of topological spaces $(X, \mathfrak{T})$ where X is denumerable, answering a question of P. Erdös enroute. Throughout we will use lower-case German letters as well as alephs to denote cardinal numbers and lower-case Greek letters to denote ordinals, the letter ω being reserved for the first infinite ordinal. The transfinite sequence $\aleph_0, \aleph_1, \ldots$ denotes the cardinals indexed by ordinals and ordered by size. For α an ordinal, ω_α denotes the least ordinal such that ω_α has $\aleph_\alpha$ predecessors (or members). The cardinal $2^{\aleph_0}$ is also denoted by $\mathfrak{c}$. $|A|$ denotes the cardinal of the set A.

If $(X, \mathfrak{T})$ is a topological space we will let $w(X, \mathfrak{T})$ be the *weight* of $(X, \mathfrak{T})$, that is, the least cardinality of a base of $(X, \mathfrak{T})$. (The reader is referred to Comfort's excellent survey article [2].) For $\mathfrak{n} \leqslant \mathfrak{c}$ define

$$\mathfrak{W}_\mathfrak{n} = \{w(X, \mathfrak{T})\colon\ |X| = \aleph_0,\ |\mathfrak{T}| = \mathfrak{n}\} .$$

Clearly $\mathfrak{l} \in \mathfrak{W}_\mathfrak{n}$ implies $\mathfrak{l} \leqslant \mathfrak{n}$. First we prove $\mathfrak{W}_\mathfrak{n}$ is convex for $\mathfrak{n}$ infinite.

THEOREM 1. *If $\mathfrak{n}$ is infinite, $\mathfrak{l} \in \mathfrak{W}_\mathfrak{n}$ and $\mathfrak{l} \leqslant \mathfrak{m} \leqslant \mathfrak{n}$, then $\mathfrak{m} \in \mathfrak{W}_\mathfrak{n}$.*

Proof. From the hypothesis of Theorem 1 we see that $\mathfrak{l} \in \mathfrak{W}_\mathfrak{n}$ implies $\mathfrak{l}$ is infinite. Let $(X_0, \mathfrak{T}_0)$ be a denumerable topological space such that $|\mathfrak{T}_0| = \mathfrak{n}$, $w(X_0, \mathfrak{T}_0) = \mathfrak{l}$. Let X_1 be a denumerable set disjoint from X_0. Sierpiński [3] shows that it is possible to find a family $\mathcal{F}$ of subsets of X_1 such that (i) $A, B \in \mathcal{F}$ implies $|A \cap B| < \aleph_0$, and (ii) $|\mathcal{F}| = \mathfrak{m}$. Let $\mathcal{S} = \{A \subseteq X_1\colon X_1 - A \in \mathcal{F}\}$, and let $\mathfrak{T}_1$ denote the topology on X_1 generated by $\mathcal{S}$. It is not difficult to show that $T \in \mathfrak{T}_1$ implies T is a finite intersection of members of $\mathcal{S}$ intersected with a cofinite subset of X_1. Hence $|\mathfrak{T}_1| = |\mathcal{S}| = \mathfrak{m}$. From this we also see that $w(X_1, \mathfrak{T}_1) = \mathfrak{m}$. Now let $X = X_0 \cup X_1$ and let the topology $\mathfrak{T}$ on X be $\{A_0 \cup A_1\colon A_0 \in \mathfrak{T}_0, A_1 \in \mathfrak{T}_1\}$.