Take an open covering $\mathfrak{U}$ of X, whose elements meet only finite number of members of $\mathfrak{F}$ and a σ-discrete closed refinement $\mathfrak{G} = \bigcup_{m=1}^{\infty} \mathfrak{G}_m$ of $\mathfrak{U}$, where $\mathfrak{G}_m$ are discrete for $m = 1, 2, \ldots$ The sets $C_m = \bigcup \mathfrak{G}_m$ satisfy our assumptions.

Added in proof. Eric van Douwen has independently obtained our Theorem 2. His paper will appear in Indagations Mathematicae.

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Further results on the achromatic number

by

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Abstract. We investigate: (1) the effect on the achromatic number of removing points or lines, (2) exact values for the achromatic numbers of paths and cycles, and (3) general bounds for the achromatic number and similar but tighter bounds for the achromatic numbers of bipartite graphs.

A coloring of a graph ([1]) G is *complete* if for every two colors i and j there are adjacent points u and v, colored i and j respectively. The achromatic number $\psi = \psi(G)$ is the largest number n such that G has a complete coloring with n colors. The achromatic number was introduced in [4] as the largest order of the complete homomorphisms of G. Later results appeared in [1] and [3]. In this paper we investigate the effect on the achromatic number of removing points and lines from G, find the values $\psi(C_n)$ and $\psi(P_n)$, and develop bounds for the achromatic number of any graph and for the achromatic number of any bigraph.

THEOREM 1. *For any graph G and point $u \in G$,*

$$\psi(G) \geqslant \psi(G-u) \geqslant \psi(G)-1 .$$

Proof. If $\psi(G-u) = n$, then G has a complete n-coloring unless each of the n colors is assigned to some point adjacent to u, in which case G has a complete $(n+1)$-coloring. Thus $\psi(G) \geqslant \psi(G-u)$.

On the other hand, if $\psi(G) = n$, consider the coloring of $G-u$ induced by a complete n-coloring of G, in which u is assigned color i. If this coloring is not complete, there is some color j not adjacent to any point colored i. If all points of $G-u$ which are colored i are recolored j the result is a complete $(n-1)$-coloring, so that $\psi(G-u) \geqslant \psi(G)-1$.

COROLLARY. *If $\psi(G-u) = \psi(G)$ there is a complete $\psi(G)$-coloring of G which induces a complete $\psi(G)$-coloring of $G-u$.*

Proof. Suppose that $\psi(G-u) = \psi(G) = n$. If no complete n-coloring of G induces a complete n-coloring of $G-u$, then in every complete n-coloring of $G-u$ every color appears on some point adjacent to u, in which case $\psi(G) \geqslant 1+\psi(G-u)$ as shown above.

([1]) Definitions and notations are those of [2].

COROLLARY. *If H is an induced subgraph of G, then $\psi(G) \geqslant \psi(H)$.*

THEOREM 2. *For any graph G and line x,*

$$\psi(G)+1 \geqslant \psi(G-x) \geqslant \psi(G)-1 .$$

Proof. Let $x = uv$ and suppose that $\psi(G) = n$ and $\psi(G-x) = n+k > n+1$. Note that in any complete coloring of $G-x$ with more than n colors u and v must be assigned the same color. Color $G-x$ with $n+k$ colors, assigning color 1 to both u and v. If there is a color t such that no point adjacent to v is colored t, we first recolor v with t. This is a coloring, but may not be complete, as there may have been colors j_i such that the only points colored j_i which were adjacent to points colored 1 were in fact adjacent to v. Consider one such color j_1. Recolor all points colored 1 with j_1. Since u and v now have different colors, this is a complete coloring of G with $n+k-1 > n = \psi(G)$ colors.

On the other hand, if in the complete $(n+k)$-coloring of $G-x$ each of u and v is adjacent to a point of every other color, recolor v with a new color $n+k+1$. This yields a complete coloring except that colors 1 and $n+k+1$ are not adjacent, so that if we replace line x we get a complete coloring of G with $n+k+1 > n = \psi(G)$ colors.

As shown in [4], if $\psi(G) = n$ there is a homomorphism from G onto the complete graph K_n. Thus, for some line y of K_n there is a homomorphism of $G-x$ onto K_n-y, which in turn has a homomorphism onto K_{n-1}, defined by identifying the endpoints of y. The composition of these mappings is a homomorphism from $G-x$ onto K_{n-1}; thus $\psi(G-x) \geqslant n-1$.

For an example of a graph for which the upper bound is achieved consider the cycle C_4.

COROLLARY. *If $x = uv$ is a line of G for which $\psi(G-x) = \psi(G)+1$ then $\psi(G-u) = \psi(G)$.*

Proof. Since $G-uv-u = G-u$, $\psi(G) \geqslant \psi(G-u) \geqslant \psi(G-uv)-1 = \psi(G)$.

COROLLARY. *If $\psi(G-uv) = \psi(G)-1$, then $\psi(G-u) = \psi(G)-1$.*

It is known that for any independent set S of points of a graph G, the chromatic number $\chi(G-S)$ satisfies $\chi(G) \geqslant \chi(G-S) \geqslant \chi(G)-1$. In contradiction to what might be expected from Theorem 1, a similar result does not always hold for the achromatic number. If S is a set of points all of which receive the same color in some complete ψ-coloring of G then indeed $\psi(G-S) \geqslant \psi(G)-1$. That the achromatic number can drop an arbitrary amount upon removal of an independent set of points will follow from the next set of results in which we determine the achromatic number of the cycle C_n.

LEMMA 1. *If $\psi(C_p) = n$ and $k \geqslant 2$, then $\psi(C_{p+k}) \geqslant n$.*

Proof. Consider the points of C_{p+k} labelled as $u_1, u_2, \ldots, u_{p+k}$, and assign to each point $u_1, \ldots, u_p$ the color it receives in a complete n-coloring of C_p. It then suffices to assign colors from $1, 2, \ldots, n$ to the points $u_{p+1}, \ldots, u_{p+k}$ so that the result is a valid complete coloring. Suppose u_1 is assigned color α and u_p is assigned color β. If k is even assign α to $u_{p+1}, u_{p+3}, \ldots, u_{p+k-1}$, and β to $u_{p+2}, u_{p+4}, \ldots, u_{p+k}$. If k is odd assign α to $u_{p+1}, u_{p+3}, \ldots, u_{p+k-2}$, β to $u_{p+2}, u_{p+4}, \ldots, u_{p+k-1}$, and any other color to u_{p+k}.

LEMMA 2. *If $\psi(C_p) = n$, then $p \geqslant n\left\{\frac{n-1}{2}\right\}$.*

Proof. Since each color must be adjacent to each of the other $(n-1)$-colors, and each point has degree 2, each color must be assigned to at least $\left\{\frac{n-1}{2}\right\}$ points, so that $p \geqslant n\left\{\frac{n-1}{2}\right\}$.

Note that if n is odd, then $n\left\{\frac{n-1}{2}\right\} = \frac{n(n-1)}{2}$, while if n is even, then $n\left\{\frac{n-1}{2}\right\} = \frac{n^2}{2} > \frac{n(n-1)}{2}$. Thus in either case we are also assured that there are enough lines for every pair of the n colors to be adjacent. Furthermore, if n is even, some adjacency between a pair of colors will occur twice.

LEMMA 3. *For $n > 2$ and even, let $p = \frac{1}{2}n^2$. Then $\psi(C_p) = \psi(C_{p+1}) = n$.*

Proof. For $n = 4$ the cycle C_8 has a complete 4-coloring determined by coloring the points, in cyclic order, with colors $1, 2, 3, 4, 1, 3, 2, 4$. We proceed by induction. Suppose that $\psi(C_p) = n$ for $p = \frac{1}{2}n^2$, and consider the case $p' = \frac{1}{2}(n+2)^2$. Note that $\frac{1}{2}(n+2)^2 - \frac{1}{2}n^2 = 2n+2$. Let C_p be colored with n colors, such that the adjacency between the colors assigned to points u_1 and u_p is repeated elsewhere in the cycle. Let u_1 be colored 1 and u_p be colored n. Obtain the cycle $C_{p'}$ by inserting $2n+2$ points, $u_{p+1}, \ldots, u_{p'}$ on the line $u_p u_1$. Choose two new colors α and β. Assign colors in the following manner, beginning at $u_{p'}$ and working down to u_{p+3}: $\alpha, 2, \beta, 3, \alpha, 4, \beta, \ldots, \alpha, n, \beta, 1$. Since $u_{p'}$ is adjacent to u_1 this adds all adjacencies between the colors $1, 2, \ldots, n$ and the colors α and β. Then, assign α to u_{p+2} and β to u_{p+1}, adding the adjacency between colors α and β.

That $\psi(C_{p+1}) = \psi(C_p)$ for $p = \frac{1}{2}n^2$ follows in a similar manner. There is a line uv such that $\psi(C_p - uv) = \psi(C_p)$, so that we can introduce a new point w, colored differently from u and v, and replace line uv by lines uw and wv.

LEMMA 4. *For $n \geqslant 3$ odd, let $p = \frac{1}{2}n(n-1)$. Then $\psi(C_p) = n$, but $\psi(C_{p+1}) = n-1$.*

Proof. Again, we proceed by induction. If $n=3$, then $\frac{1}{2}n(n-1)=3$, and $\psi(C_3)=3$. Suppose that for $p=\frac{1}{2}n(n-1)$, $\psi(C_p)=n$. Let $p'=\frac{1}{2}(n+2)(n+1)$, and note that $p'-p=2n+1$. Given a complete n-coloring of C_p with u_1 colored 1 and u_p colored n, insert $2n+1$ points $u_{p'}$, $u_{p'-1}, \ldots, u_{p+1}$, and color them $n, \alpha, 1, \beta, 2, \alpha, 3, \beta, \ldots, \beta, n-1, \alpha, \beta$. This replaces the $1-n$ adjacency, and adds the $\alpha-\beta$ adjacency as well as all adjacencies between α, β and colors $1, 2, \ldots, n$.

Suppose that for $p=\frac{n(n-1)}{2}$, $\psi(C_{p+1})=n$. Since $p+1=n\left\{\frac{n-1}{2}\right\}+1$, $n-1$ colors appear $\left\{\frac{n-1}{2}\right\}$ times, and one color, say α, appears $1+\left\{\frac{n-1}{2}\right\}$ times. Then α must repeat an adjacency to some other color, but there is no color available which could be adjacent to α. Thus $\psi(C_p)=n-1$.

To summarize,

THEOREM 3. *If $p=n\left\{\frac{n-1}{2}\right\}$, $n\geqslant 3$, then $\psi(C_p)=n$. For p between $n\left\{\frac{n-1}{2}\right\}$ and $(n+1)\left\{\frac{n}{2}\right\}$, $\psi(C_p)=n$ unless n is odd and $p=n\left\{\frac{n-1}{2}\right\}+1$, in which case $\psi(C_p)=n-1$.*

COROLLARY. *For every k there is a graph G and an independent set S of points of G such that $\psi(G)-\psi(G-S)\geqslant k$.*

Proof. Let $n\geqslant k+1$ be even. Then $p=\frac{1}{2}n^2$ is even, so that C_p is bipartite. The points $u_1, u_3, \ldots, u_{p-1}$ form an independent set S such that C_p-S is totally disconnected. Thus $\psi(C_p)-\psi(C_p-S)=n-1\geqslant k$.

COROLLARY. *For every k there is a graph G such that $\psi(G)-\chi(G)\geqslant k$.*

Let P_p be the path with p points.

LEMMA 5. *If $r>s$, then $\psi(P_r)\geqslant\psi(P_s)$.*

Proof. It is sufficient to show that $\psi(P_{r+1})\geqslant\psi(P_r)$. Let u_1 be an endpoint of P_{s+1} and let u_3 be at distance 2 from u_1. Then, identifying u_1 with u_3 defines a homomorphism from P_{s+1} to P_s, so $\psi(P_{s+1})\geqslant\psi(P_s)$.

THEOREM 4. *For $n\geqslant 3$ let $p=n\left\{\frac{n-1}{2}\right\}$. If n is even, $\psi(P_{p-1})<\psi(P_p)=n$. If n is odd, $\psi(P_p)<\psi(P_{p+1})=n$.*

Proof. We first examine the equalities. If n is even, then $\psi(C_p)=n$ and some adjacency is repeated. Thus, since $P_p=C_p-x$, $\psi(P_p)=n$. If n is odd, we obtain a complete n-coloring of P_{p+1} from one for C_p by removing some point u, adjacent to points v_1 and v_2, and then adding new points u_1 and u_2 and lines u_1v_1 and u_2v_2. By assigning to u_1 and u_2 the color assigned to u, we obtain a complete n-coloring.

For the inequality in the even case, we note that a complete n-coloring of P_{p-1} gives rise to one for C_{p-1} if the endpoints of the path are

colored differently, and one for C_{p-2} if they are colored the same. Each of these would violate Theorem 3. When n is odd, P_p has only $\frac{1}{2}n(n-1)-1$ lines, whereas at least $\frac{1}{2}n(n-1)$ would be required if the graph were to have a complete n-coloring.

Let H be a subgraph of $K_{m,n}$; we adopt the notational convention that $m\geqslant n$. Denote by $\tilde{H}$ the relative complement $K_{m,n}-H$. If the bipartition of $V=V(K_{m,n})$ is defined by $V=V_1|V_2$, where $|V_1|=m$, then for $i=1,2$ let τ_i be the variable which has the value 1 if there is at least one point of V_i which is adjacent to every point of V_{3-i}, and 0 otherwise. Finally, let $k(H)$ denote the number of non-trivial connected components of H.

THEOREM 5. *If G is a spanning subgraph of $K_{m,n}$, then $n+1\geqslant\psi(G)\geqslant\tau_1+\tau_2+k(\tilde{G})$.*

Proof. In a ψ-coloring of G at most one color can appear only in V_1, and at most one color can appear only in V_2. Let r be the number of colors appearing only in V_1. Then there can be at most $n+r$ colors; this takes its maximum when $r=1$.

For the lower bound, let $k=k(\tilde{G})$ and let the non-trivial components of $\tilde{G}$ be $H_1, H_2, \ldots, H_k$. Find a maximal complete bipartite subgraph $B_i\subset H_i$ and color all of its points with color i; note that B_i may just be the line $K_2=K_{1,1}$. For $i\neq j$, let $u_i\in B_i\cap V_1$ and $v_j\in B_j\cap V_2$. Since u_i and v_j are in different components of $\tilde{G}$ they are adjacent in G. Thus, this coloring yields all adjacencies among the colors $1, 2, \ldots, k$. Since the subgraph induced by the points colored $1, \ldots, k$ has a complete k-coloring, it follows from Theorem 1 that G has achromatic number at least k. If $\tau_i=1$ and $\tau_{3-i}=0$, then there exists at least one point in V_i which is adjacent to every point of V_{3-i}. We color this point with the color $k+1$ and again apply Theorem 1 to see that G has achromatic number at least $k+1$. Finally, if $\tau_1=\tau_2=1$, then yet another application of Theorem 1 shows that G has achromatic number at least $k+2$.

For our last result, we let $\tau(G)$ be the number of points of degree $p-1$.

THEOREM 6. *For a graph G with p points, $p\geqslant\psi(G)\geqslant k(\bar{G})+\tau(G)$.*

Proof. The upper bound is obvious. Let $k=k(\bar{G})$ and let the non-trivial components of $\bar{G}$ be $H_1, H_2, \ldots, H_k$. Choose a clique $B_i\subseteq H_i$ and color the points of B_i with color i. As in Theorem 5, we are guaranteed that if $1\leqslant i<j\leqslant k$, colors i and j will be adjacent and there will be no $i-i$ adjacency. The $\tau(G)$ points of degree $p-1$ are isolated points of $\bar{G}$ and hence have not yet been colored. We color these points with $\tau(G)$ additional colors. The subgraph induced by all the colored points has a complete $k(\bar{G})+\tau(G)$ coloring. Thus, by Theorem 1, the graph G has achromatic number at least $k(\bar{G})+\tau(G)$.

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The undecidability of the existence of a non-separable normal Moore space satisfying the countable chain condition

by

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Abstract. It is shown that Martin's Axiom plus the negation of the continuum hypothesis implies the existence of a non-separable normal Moore space satisfying the countable chain condition. The consistency and independence of the existence of such spaces follows.

In [22], M. E. Rudin constructed a non-separable Moore space (1) satisfying the contable chain condition (2). The importance of her example lay in showing how far removed Moore spaces can be from metrizable spaces. Any such example cannot be locally metrizable, cannot have a dense metrizable subspace, and cannot be completed (3), indeed cannot be densely embedded in a Moore space satisfying the Baire category theorem [1, Theorem 3.31]. In [17], Pixley and Roy constructed a much simpler example, which in addition is metacompact(4). In this note we construct a subspace of their space with the same properties, which, moreover, is normal, if Martin's Axiom [16], [31] plus the negation of the continuum hypothesis is assumed. These assumptions are consistent with the usual axioms of set theory, e.g. Zermelo–Fraenkel, including the Axiom of Choice [25]. Some such assumption is necessary, since in [26], the second author established the consistency of the assumption that every countable chain condition normal Moore space is metrizable, and hence separable.

(1) A Moore space is a regular Hausdorff space X having a sequence of open covers $\{G_n\}_{n<\omega}$ such that for each $x \in X$ and U open containing x, there is an n such that $\bigcup \{g \in G_n\colon x \in g\} \subset U$.

(2) I.e. every collection of disjoint open sets is countable.

(3) There are several different notions of completeness and completability for Moore spaces. The reader is referred to [1] for details.

(4) The Proceedings of the 1971 Prague Topological Symposium have just reached the second author, who is probably responsible for A. V. Arhangel'skiĭ's incorrect discussion [5] of the Pixley–Roy example. No special set-theoretic assumptions are needed to construct their (completely regular) space.